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# ON THE THEORY OF TUNNEL AND ABOVE-BARRIER IONIZATION OF ATOMS AND IONS IN THE FIELD OF LASER RADIATION

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The energy and angular distributions of photoelectrons in the tunnel ionization of atoms or ions by laser radiation are calculated. The results obtained are valid for all values of the Keldysh parameter $\gamma$. The cases of linear, circular and elliptic polarizations of light are considered. Using the adiabatic approximation, the probability of above-barrier ionization of hydrogen atoms by strong low-frequency radiation is calculated. Several remarks on the quantum tunnel interference and on the contribution of other authors to the problems considered are made.

## Туннельная и надбарьерная ионизация атомов и ионов в поле лазерного излучения.

В.С.Попов

Рассчитаны энергетические и угловые распределения фотоэлектронов при туннельной ионизации атома или иона под действием мощного лазерного излучения (для всех значений параметра Келдыша $\gamma$). Рассмотрены случаи линейной, циркулярной и эллиптической поляризации электромагнитной волны. Вычислена вероятность надбарьерной ионизации атомов водорода в низкочастотном ($\gamma \ll 1$) лазерном поле. Приведены формулы для импульсного спектра электронов при ионизации атомного уровня переменным электрическим полем общего вида (в случае линейной поляризации). Рассмотрен эффект туннельной интерференции в энергетическом спектре фотоэлектронов.

Fig - 8, ref. - 31 names

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The theoretical investigation of ionization of atoms and ions by laser radiation had been initiated in 1960's [1-5]. The results of subsequent investigations in this field are reviewed[1)] in Refs.[6,7]. The character of the ionization process depends essentially on the adiabaticity parameter $\gamma$ introduced by Keldysh [1]: at $\gamma \ll 1$ there is tunneling while at $\gamma \gg 1$ multiphoton ionization.

Formulae for the ionization probability (including the pre-exponential factor) which are valid for arbitrary values of $\gamma$ are obtained in Refs.[3-5]. The goal of the present letter is to study the energy and angular spectra of photoelectrons and their variation with the $\gamma$ parameter increasing for different polarizations of radiation. In addition, we have calculated the probability of the above-barrier ionization of H-atoms in the strong low-frequency ($\gamma \ll 1$) electromagnetic field. Owing to progress in the laser physics, these quantities have now become an object of experimental investigations (see, for example, the review [7] and the literature therein).

We use the atomic units $\hbar = e = m = 1$, $F_a = m^2e^5/\hbar^4 = 5.14 \cdot 10^9\ V \cdot cm^{-1}$ is the atomic unit of electric field strength, $\epsilon = F/\kappa^3 F_a$ is the reduced electric field ($\epsilon \equiv F$ for the ground state of a hydrogen atom), $E_i = \kappa^2/2$ is the ionization potential and $K_0 = \kappa^2/2\omega$ is the multiphoton parameter. Hereafter we everywhere imply that $\epsilon \ll 1$ and $K_0 \gg 1$, which provides applicability of the quasiclassical approximation in the problem considered.

**1.** For linear polarization the momentum spectrum of photoelectrons has the form [2)]

$$w(\mathbf{p}) = w(0)\ exp\left\{-\frac{1}{\omega}[\ c_1(\gamma)p_{\|}^2 + c_2(\gamma)p_{\perp}^2\ ]\right\}, \tag{1}$$

where $c_1 = Arsh\ \gamma - \gamma(1+\gamma^2)^{-1/2}$, $c_2 = Arsh\ \gamma$, $\gamma = \omega/\omega_t = \omega\kappa/F$, $\omega$ and $F$ - are the laser radiation frequency and the electric field strength, $\omega_t = F/\kappa$ is the tunneling frequency of electron in the field $F$ and $\mathbf{p} = (p_{\|}, p_{\perp})$ is the momentum of the outgoing electron, $p_{\|}$ being the momentum projection along the field $F(t) = F\ cos\ \omega t$, while $p_{\perp}$ is normal to it.

Formula (1) requires almost no discussion. From it immediately follows the energy-angular distribution of electrons

$$w(E,\theta) \propto exp\left\{-\frac{2\gamma}{\sqrt{1+\gamma^2}} \cdot \frac{E}{\omega} sin^2\theta\right\}, \tag{1'}$$

where $p_{\perp} = p\ sin\theta$, $E = p^2/2$. At $\gamma \ll 1$ this distribution has a sharp maximum towards direction of the electric field and $p_{\perp} \sim \sqrt{F/\kappa} \ll \kappa$, $\theta \lesssim \gamma$,

while $p_{\|} \sim \gamma^{-1} p_{\perp}$ may even exceed $\kappa$ (a characteristic momentum of electron in the bound state). On the other hand, at $\gamma \gg 1$ one has $p_{\|} \sim p_{\perp} \sim (\omega/ln\ 2\gamma)^{1/2} \ll \kappa$.

For the mean transverse and longitudinal (with respect to the direction of the electric field) electron momenta we get:

$$\langle p_{\perp}^2 \rangle^{1/2} / \langle p_{\|}^2 \rangle^{1/2} = \left(1 - \frac{\gamma}{\sqrt{1+\gamma^2} Arsh\gamma}\right)^{1/2} = \begin{cases} 0.577\ \gamma, & \gamma \ll 1, \\ 1 - (2\ ln\gamma)^{-1}, & \gamma \gg 1 \end{cases}$$

**2.** For circular polarization the probability of $n$-photon ionization of atomic s-level is [4]

$$w_n = \kappa^2 C_{\kappa}^2\ R(t,\gamma)\ exp\{-4\nu_c\ \varphi(t,\gamma)\}, \tag{2}$$

where

$$\varphi(t,\gamma) = \frac{Arth\ u - u}{1+t}, \quad u = \sqrt{\frac{t^2+\gamma^2}{1+\gamma^2}}, \tag{3}$$

(we have introduced instead of $n$ an auxiliary variable $t = (2\nu_c/n) - 1$, such that
$-1 < t \leq 1$), $\nu_c = K_0(1+\gamma^{-2})$ is the photoionization threshold for linearly polarized radiation and $C_{\kappa}$ is the asymptotic coefficient of the atomic wave function at infinity [3)].

The function $\varphi(t,\gamma)$ has a minimum at the point $t = t_0(\gamma)$ corresponding to the maximum ionization probability and determined from equation $t = 1 - u/Arth\ u$ (see Fig.1), this point corresponding to the number of absorbed photons $n_0(\gamma) = 2\nu_c/(1+t_0)$. Note that at $\gamma \ll 1$

$$t_0 = \frac{1}{3}\gamma^2 - \frac{28}{135}\gamma^4 + ..., \quad \varphi(t_0,\gamma) = \frac{1}{3}\gamma^3 - \frac{16}{45}\gamma^5 + ..., \quad n_0 = \frac{F^2}{\omega^3}\left(1 + \frac{2}{3}\gamma^2\right), \tag{4}$$

while at $\gamma \gg 1$

$$t_0 = 1 - \frac{1}{ln(\gamma\sqrt{ln\ \gamma})} + ..., \quad \varphi(t_0,\gamma) = \frac{1}{2} ln\ \gamma - \frac{1}{4} + ..., \quad n_0 = \left(1 + \frac{1}{2\ ln\gamma}\right) K_0 \tag{5}$$

Near the maximum, $n \approx n_0$, the $n$-distribution , i.e., the electron energy spectrum, is Gaussian [4)]:

$$w_n = w_{max} \cdot exp\{-c_3(n-n_0)^2/n_0\} = w_{max} \cdot exp\{-(E_n - E_{max})^2/2\Delta^2\}, \quad (6)$$

where $E_n = (n - \nu_c)\ \omega$,

$$c_3(\gamma) = \frac{\gamma^2(1+t_0^2) + 2t_0^2}{(1-t_0)^2\sqrt{(1+\gamma^2)(t_0^2+\gamma^2)}} = \begin{cases} \gamma(1+\frac{1}{3}\gamma^2), & \gamma \ll 1 \\ 2(ln\gamma)^2, & \gamma \gg 1 \end{cases} \quad (6')$$

$$\Delta = [\omega E_i(1+\gamma^{-2})/(1+t_0)\ c_3(\gamma)]^{1/2}, \quad (6'')$$

$\Delta \sim \gamma^{-3/2}\sqrt{\omega E_i}$ for $\gamma \ll 1$, and $\Delta = \sqrt{\omega E_i}/2\ ln\gamma$ at $\gamma \gg 1$. The most probable energy of photoelectrons is

$$E_{max} = (n_0 - \nu_c)\omega = \frac{F^2}{2\omega^2}(1+\gamma^2)\frac{1-t_0}{1+t_0} \approx \begin{cases} F^2/2\omega^2 = \gamma^{-2}E_i, & \gamma \ll 1 \\ E_i/2\ ln\gamma, & \gamma \gg 1 \end{cases} \quad (7)$$

The numerical calculation according to the above formulae gives the curves in Figs.2 and 3. The most probable number $n_0$ of absorbed photons for $\gamma \ll 1$ is twice as large as the photoionization threshold $\nu_c$ , approaching it, though slowly, at $\gamma \to \infty$:

$$(n_0 - \nu_c)/\nu_c = \begin{cases} 1 - \frac{2}{3}\gamma^2, & \gamma \ll 1, \\ (2\ ln\gamma)^{-1}, & \gamma \gg 1, \end{cases} \quad (8)$$

see also the curve in Fig.2. The coefficient $c_3(\gamma)$ in Eq.(6) increases monotonically with $\gamma$ (Fig.3) and

$$w_n = w_{max} \cdot exp\left\{-\frac{2(ln\gamma)^2}{K_0}(n-n_0)^2\right\}, \quad \gamma \gg 1 \quad (9)$$

Since

$$\frac{\Delta n}{\nu_c} \sim \begin{cases} \omega/\sqrt{\kappa F} = \gamma\sqrt{\epsilon}, & \gamma \ll 1, \\ (\sqrt{K_0}\ ln\gamma)^{-1}, & \gamma \gg 1, \end{cases}$$

the distribution over $n$ has the form of a comparatively narrow peak [3,4] (in units of $\nu_c$ or $n_0$). We note that for $\gamma \ll 1$ the distribution $w_n$ (6) is much

wider than the Poisson distribution and, conversely, for $\gamma \gg 1$ it is narrower ($\Delta n = \sqrt{n_0}$ for $\gamma \approx 0.468$).

The distribution of photoelectrons over the angle $\psi$ between the emergence direction of photoelectrons and the polarization plane of laser radiation is also of interest. It is determined by Eq.(64) of Ref.[4], into which one should substitute the values of $n = n_0$ and of the momentum $p_{max} = \sqrt{2\omega(n_0 - \nu_c)}$, corresponding to the maximum ionization probability,

$$w(\psi) \propto [J_{n_0}(n_0\zeta)]^2, \quad \zeta = \sqrt{\frac{1-t_0^2}{1+\gamma^2}}\, cos\ \psi, \tag{10}$$

where $J_{n_0}(z)$ - is the Bessel function. Using for it at $n_0 \gg 1$ the Langer asymptotics [10], we find (see Appendix A for details):

$$w(\psi) = u_0\eta^{-1}exp\{-n_0\ s(\psi,\gamma)\}\ w(0), \tag{11}$$

where

$$s = 2\,[Arth\ \eta - \eta - (Arth\ u_0 - u_0)\,], \quad \eta = \sqrt{1-\zeta^2}, \quad u_0 = \sqrt{(t_0^2+\gamma^2)/(1+\gamma^2)}, \tag{12}$$

variable $\eta$ changing within $u_0$ (at $\psi = 0$) up to 1 (at $\psi = \pi/2$). The angular distribution width is $\Delta\psi \sim \gamma\sqrt{\epsilon}$ for $\gamma \ll 1$ and $\Delta\psi \sim K_0^{-1/2}$ for $\gamma \gg 1$, so $\Delta\psi \ll 1$. As is shown in Fig.4, the function $s(\psi,\gamma)$ varies rather slowly with the parameter $\gamma$ increasing. That is why the variation of $w(\psi)$ when going over from the adiabatic region to the case of $\gamma \gtrsim 1$ is mainly determined by variation of $n_0(\gamma)$, i.e., the most probable number of absorbed photons.

In the adiabatic region Eq.(9) simplifies:

$$w(\psi) = (1+\chi^2)^{-1/2}exp\Big\{-\frac{2\kappa^3}{3F}[(1+\chi^2)^{3/2}-1]\Big\}, \quad \gamma \ll 1, \tag{13}$$

where $\chi = \psi/\gamma$. For very small angles we get the Gaussian distribution [7,11-13]:

$$w(\psi) \approx exp\Big(-\frac{F\kappa}{\omega^2}\psi^2\Big), \qquad \psi \ll \gamma \ll 1 \tag{13'}$$

However, the number of electrons decreases then more rapidly:

$$w(\psi) \approx \gamma\psi^{-1}exp\Big(-\frac{2F^2}{3\omega^3}\ \psi^3\Big), \qquad \gamma \lesssim \psi \ll 1, \tag{13''}$$

and most of the photoelectrons always emerge near the plane of polarization of light.

As $\psi \to 0$ (and for any value of $\gamma$) we have $s(\psi,\gamma) = u_0\psi^2 + O(\psi^4)$. For this reason, in the small angles region

$$w(\psi) = const \cdot exp\left\{-c_4(\gamma)\frac{F\kappa}{\omega^2}\psi^2\right\}, \tag{14}$$

$c_4 = \sqrt{(1+\gamma^{-2})(t_0^2+\gamma^2)}/(1+t_0)$. If $\gamma \ll 1$, then $c_4 = 1 + \frac{2}{9}\gamma^2$ and the expression (14) goes over into (13′). On the other hand, for $\gamma \gg 1$ the coefficient $c_4 = \gamma/2$ and $w(\psi) \approx exp(-K_0\psi^2)$.

With $\gamma$ increasing the angular distribution noticeably broadens, but even for $\gamma \gg 1$ it still remains rather narrow (Fig.5). In the two limiting cases we have

$$s(\psi,\gamma) = \begin{cases} 2\ [\ Arth(sin\ \psi) - sin\ \psi\ ], & \gamma \to 0, \\ -2\ ln\ cos\ \psi, & \gamma \to \infty \end{cases} \tag{15}$$

which determines the limiting curves in Fig.4.

One can find the photoelectron distribution over the components of the momentum $p_\perp$ (in the polarization plane) and $p_z$ (along the direction of propagation of the electromagnetic wave):

$$w(p_\perp,p_z) \propto exp\left\{-\frac{1}{\omega}\left[\frac{u_0}{1-t_0}\ p_z^2 + (1-t_0)c_3(\gamma)(p_\perp - p_{max})^2\right]\right\} \tag{16}$$

(we omit here the pre-exponential factor). Hence

$$w(p_\perp,p_z) \propto exp\left\{-\frac{\kappa}{F}\left[p_z^2 + (p_\perp - \frac{F}{\omega})^2\right]\right\}, \quad \gamma \ll 1 \tag{16′}$$

(see also [12,13]), and for $\gamma \gg 1$ , up to logarithmic accuracy

$$w(p_\perp,p_z) \propto exp\{-\frac{\Lambda}{\omega}[\ p_z^2 + 2(p_\perp - p_{max})^2\ ]\}, \quad \Lambda = ln((\gamma\sqrt{ln\gamma}) \gg 1, \tag{16″}$$

where $p_{max} = \sqrt{2F_{max}}$ is the most probable electron momentum, see Eq.(7).

**3.** For the case of a monochromatic wave with elliptic polarization

$$\mathbf{F}(t) = \mathbf{e}_x F\ cos\ \omega t + \mathbf{e}_y\ \xi F\ sin\ \omega t, \quad -1 \le \xi \le 1 \tag{17}$$

($\xi$ – ellipticity of light) we give here the energy spectrum of photoelectrons $E_n = (n-\nu)\omega$ in two limiting cases, $\gamma \ll 1$ and $\gamma \gg 1$. In the first case

$$w_n = \frac{2}{\sqrt{\pi}} A\ \epsilon^{3/2}\gamma^2\ a(\gamma n')\ exp\left\{-\frac{2}{3}(1-\xi^2)\gamma^3 n'\right\}, \quad n > n_0, \tag{18}$$

where $n' = n - n_0$, $n_0 = (1+3\xi^2)F^2/4\omega^3 = \nu(1+3\xi^2)/(1+\xi^2)$,

$$A = \kappa^2\ C_\kappa^2\ exp\left\{-\frac{2}{3\epsilon}\left[1-\frac{1}{10}\left(1-\frac{\xi^2}{3}\right)\gamma^2\right]\right\}, \tag{19}$$

$a(x) = e^{-x}I_0(x)$ is a slowly varying function: $a(x) = 1 - x + \frac{3}{4}x^2 + ...$ at $x \to 0$, $a(x) \approx (2\pi x)^{-1/2}$ at $x \gg 1$, and $I_0(x)$ is the modified Bessel function. The main contribution to the total ionization rate $W$ comes from the region $n' \gtrsim \gamma^{-1}$, in which

$$w_n \approx \frac{A}{2\pi K_0^{3/2}} \cdot \frac{exp(-\alpha\ n')}{\sqrt{n'}}, \qquad \alpha = \frac{2}{3}(1-\xi^2)\gamma^3 \tag{20}$$

For the mean number of absorbed photons $\langle n\rangle$ and its dispersion $\Delta n$ from Eqs.(18) and (B.4) we obtain

$$\langle n\rangle = n_0 + \langle n'\rangle = \frac{F^2}{4\omega^3}\left(1+3\xi^2+\frac{3\epsilon}{1-\xi^2}\right), \quad \Delta n = \frac{3\epsilon}{2^{3/2}(1-\xi^2)} \cdot \frac{F^2}{\omega^3} \tag{21}$$

if $1-\xi^2 \gg \epsilon$. On the other hand, in the case of circular polarization ($\xi = \pm 1$)

$$\langle n\rangle = n_0 \approx F^2/\omega^3, \quad \Delta n = \sqrt{\frac{F^3}{2\ \omega^4\kappa}} = 0.71\gamma\sqrt{\epsilon} \cdot \frac{F^2}{\omega^3} \tag{21'}$$

($F^2/\omega^3 = 1/\epsilon\gamma^3 \gg 1$). In the transitional region close to the circular polarization ($1-\xi^2 \sim \epsilon \ll 1$), the width of the distribution $w_n$ sharply decreases, and Eq.(18) goes over into (6).

Summation of (18) or (20) over $n$ gives [5)]

$$W(F,\omega) = \sum_n w_n = \sqrt{\frac{3\epsilon^3}{\pi(1-\xi^2)}}\ A, \quad 1-\xi^2 \gg \epsilon \tag{22}$$

(in the case of $s$-level), in accordance with the result of the adiabatic approximation [4,14].

On the other hand, for $\gamma \gg 1$ the Gaussian distribution holds,

$$w_n = w_{max} \cdot exp\{-(n-n_0)^2/2\mu^2 n_0\}, \quad n > \nu \tag{23}$$

with dispersion $\Delta n = \mu\sqrt{n_0}$ and

$$\mu = \frac{\sqrt{\xi^2[\,(1-\xi^2)\tau_0 + \xi^2]}}{(1-\xi^2)\tau_0^2 + 2\xi^2\tau_0}, \quad n_0 = K_0\Big(1 + \frac{\xi^2}{(1-\xi^2)\tau_0^2 + 2\xi^2\tau_0}\Big),$$

$$\nu = K_0\Big(1 + \frac{1+\xi^2}{2\gamma^2}\Big) \tag{23'}$$

Here $\tau_0 = \tau_0(\gamma, \xi)$ is determined by the equation

$$e^{\tau_0}\Big[1 - \xi^2\Big(1 - \frac{1}{\tau_0}\Big)^2\Big]^{1/2} = 2\gamma, \tag{24}$$

so $\tau_0 = ln(\,2\gamma/\sqrt{1-\xi^2}\,)$, $\mu = \tau_0^{-3/2}\sqrt{\xi^2/(1-\xi^2)}$ for $\gamma \gg 1$ and $1 - \xi^2 \gg \epsilon$. For $\xi = \pm 1$ (circular polarization) one has $\mu = 1/2\tau_0$, $\tau_0 = ln(\gamma\sqrt{2\,ln\gamma})$, and Eq.(23) coincides with (6). Since $\mu \ll 1$, the distribution $w_n$ (23) is much narrower than the Poisson distribution with $< n >= n_0$. The ratio $\rho(\gamma, \xi) = n_0/\nu$ for some values of $\xi$ is shown in Fig.6.

**4.** The above-barrier ionization of atoms holds in strong fields, $F \sim (0.2 \div 0.5)F_a$, when the level energy is above the top of the potential barrier (along the direction of the electron emission). The ionization probability per a unit time (or the ionization rate of a bound state) for the case of a low-frequency laser field is calculated in the adiabatic approximation:

$$w_a = \frac{2}{T}\int_{-T/4}^{T/4} w_{st}(F(t))\,dt = \frac{1}{2\pi}\int_0^{2\pi} w_{st}(F(\phi))d\phi, \quad \gamma \ll 1, \tag{25}$$

where $\phi = \omega t$ is the phase of the field, $T = 2\pi/\omega$ - its period, and $w_{st}(F)$ is the value of the ionization probability, or of the Stark resonance width, for the static electric field $F$.

We use the values $w_{st}(F)$ for the ground state of a hydrogen atom, obtained by summing the perturbation theory series over powers of $F$ and $1/n$-expansion [15,16]. Note, that in the region $F \leq 0.15$ these calculations agree quite well with the results of other authors done by independent numerical methods – see, f.e., Refs.[17-19]. For the case of elliptically polarized wave (16) we have $F(\phi) = F\sqrt{cos^2\phi + \xi^2 sin^2\phi}$, and $w_a(F, \xi)$ can be readily found from Eq.(25) numerically.

The results of the calculations are given in Fig.7 from which it is seen that the $F$-dependence of $w_a$ in the above-barrier region is surprisingly close to the linear one that holds also for excited states of $H$-atom [15], as well as for

$He$-atom [20]. This approximate linearity of the Stark widths in strong field was explained in [21] using $1/n$-expansion, which, as is known, in quantum mechanical problems has not only high accuracy for large quantum numbers, but continues rather frequently even to the region $n \sim 1$.

In the subbarrier region from (25) it follows the relation between adiabatic and static ionization rates:

$$w_a(F,\xi) = |\xi|^{-1}\, a\Big((1-\xi^2)/6\xi^2\epsilon\Big) \cdot w_{st}(F), \qquad \gamma \ll 1, \tag{26}$$

which is asymptotically exact at $F \to 0$ and any ellipticity $\xi$. Here $a(x)$ is the same function as in Eq.(18). The comparison of Eq.(26) with the results of numerical calculations shows that this simple formula has an accuracy of $\sim 5 \div 10\%$ in the region $0.2 \lesssim F \lesssim 0.5$ that is of interest for above-threshold ionization (note, however, that it is important to employ here exact values of the static ionization probability $w_{st}$).

If $1-\xi^2 \gg \epsilon$, Eq.(26) is equivalent to (22). At $\xi = \pm 1$ (i.e., for left or right circular polarizations) Eq.(26) gives: $w_a(F,\pm 1) = w_{st}(F)$, which is evident, since $|\,\mathbf{F}(t)\,| = F = const$ in this case. In the transitional region, $1-\xi^2 \sim \epsilon \ll 1$, the $F$-dependence of the pre-exponential factor in $w_a$ is more complicated than the power one.

**5.** There is also a specific effect in the energy spectrum of photoelectrons, which manifests itself in tunneling ionization of an atomic state by linearly polarized laser radiation. Its origin is explained by the interference of transition amplitudes (from initial bound state to final states of continuum) corresponding to two saddle points in the complex plane of "time" $t$ within the same period of the field $F(t) = F\, cos\, \omega t$. If the longitudinal electron momemtum $p_{\|} \neq 0$, some phase shift of the above mentioned amplitudes arises. As a result, an additional factor in the electron momentum spectrum appears [4]:

$$w(\mathbf{p}_n) \to w(\mathbf{p}_n) \cdot [\, 1 + (-1)^n cos\phi_n \,], \tag{27}$$

where, in the case of $n$-photon ionization,

$$\phi_n = 2K_0\, \frac{p_{\|}}{\omega}\Big[\, \frac{2\gamma}{1+\sqrt{1+\gamma^2}} + \frac{\gamma(1-\gamma^2)}{3(1+\gamma^2)^{3/2}} \cdot \frac{p_{\|}^2}{\kappa^2} \,\Big], \tag{28}$$

$\nu = K_0(1+2\gamma^2)/2\gamma^2$ is the ionization threshold for linearly polarized radiation, $p_n = \sqrt{2\omega(n-\nu)}$, while $w(\mathbf{p})$ is the same function as in Eq.(1). In particular, for the adiabatic case

$$\phi_n = \frac{2}{F}\left(E_{\rm i} p_{\|} + \frac{1}{6} p_{\|}^3\right), \qquad \gamma \ll 1 \tag{28'}$$

In the quasiclassical region the phase $\phi_n$ is large, $\phi_n \gg 1$, since $K_0 \gg 1$ or $1/\epsilon \gg 1$. Therefore after integration of (28) over angles of emission of photoelectron the contribution of the second term in Eq.(27) becomes small. So, this term ($\propto cos\ \phi_n$) gives negligible contribution to the total ionization rate, but it leads to rapid oscillations of the ionization probabilities $w_n$, i.e. in the energy spectrum of photoelectrons $w(E_n)$. In the recent papers [22,23] such effect was called the "quantum tunnel interference".

Note that from Eqs.(27) and (28) the threshold behaviour [3] of the probabilities $w_n$ immediately follows: $w_n \propto (n-\nu)^{1/2}$ at $n \to \nu$ and $n$ odd, while $w_n \propto (n-\nu)^{3/2}$ for $n$ even.

**6.** In conclusion, we would like to make some remarks concerning the review by N.B.Delone and V.P.Krainov entitled "Tunneling and barrier-suppression ionization of atoms and ions in a laser radiation field" [7].

a) The principal formulae of Secs.4.1 and 4.2 of the review [7] for the energy and angular distributions of photoelectrons at tunnel ionization of atoms had been obtained in Refs.[3,4] long before papers [11-13] appeared. In addition, the problem of atomic level ionization by strong laser radiation had been solved in Refs.[3,4,14] for arbitrary values of the Keldysh parameter $\gamma$, not only for the adiabatic region $\gamma \ll 1$ (low frequency, large intensity of electric field), which is considered in Refs.[7,11-13]. For example, the electron momentum spectrum in the case of linear polarization of radiation had the form (1). At $\gamma \ll 1$ we obtain: $c_1 = \frac{1}{3}\gamma^3 + ...$, $c_2 = \gamma - \frac{1}{6}\gamma^3 + ...$, $\gamma = \omega\sqrt{2E_{\rm i}}/F$, and Eq.(1) directly gives

$$w(\mathbf{p}) = w(0) \cdot exp\left\{-\left[\frac{\omega^2 (2E_{\rm i})^{3/2}}{3F^3} p_{\|}^2 + \frac{(2E_{\rm i})^{1/2}}{F} p_{\perp}^2\right]\right\}, \tag{29}$$

that completely coincides with Eq.(10) in Ref.[7]. From this formula it follows Eqs.(12)-(17) of Ref.[7], which determine the widths of the energy and angular distributions of photoelectrons, etc.

Similarly, the results of Sec.4.2 of Ref.[7] for the case of circular polarization of low-frequency radiation can be readily obtained from a more general Eq.(2) valid for all values of $\gamma$. If $\gamma \ll 1$, then $u \approx (t^2+\gamma^2)^{1/2} = \gamma(1+z^2)^{1/2} \ll 1$, $Arth\ u - u \approx \frac{1}{3}u^3$ and

$$\varphi(t,\gamma) = \frac{\gamma^3(1+z^2)^{3/2}}{3(1+\gamma z)} = \frac{1}{3}\gamma^3\left[1 - \gamma z + \left(\frac{3}{2} + \gamma^2\right) z^2 + O(\gamma^4)\right],$$

where $z = t/\gamma \sim \gamma$. Thereby, determination of the maximum ($n = n_0$) ionization probability reduces in this case ($\gamma \ll 1$) to finding the minimum of the square trinomial, $z = z_m$, that immediately yields

$$z_m = \frac{\gamma}{3} + O(\gamma^3), \quad n_0 = 2\nu_c\Big(1 - \frac{\gamma^2}{3}\Big) = \frac{F^2}{\omega^3}\Big(1 + \frac{2}{3}\gamma^2\Big),$$
$$w_n = w_{max} \cdot exp\left\{-\frac{\omega^4\kappa}{F^3}(n - n_0)^2\right\}, \quad w(E) \propto \; exp\left\{-\frac{\omega^2\kappa}{F^3}(E - E_{max})^2\right\}, \tag{30}$$

where $\kappa = \sqrt{2E_i}$ and $E_{max} = (n_0 - \nu_c)\omega = \frac{F^2}{2\omega^2}\Big(1 + \frac{1}{3}\gamma^2\Big)$. These expressions coincide with Eq.(19) in Ref.[7], with (33) in Ref.[12] and with Eqs.(8),(11) of Ref.[13] obtained by a special method under condition $\gamma \ll 1$.

Finally, the angular distribution of photoelectrons is determined by Eq.(10) first derived in Ref.[4]. If $\gamma \ll 1$, then

$$p_{max} = F\omega^{-1}(1 + \frac{1}{6}\gamma^2), \quad \zeta = \frac{Fp_{max}}{n_0\,\omega^2} cos\;\psi = 1 - \frac{1}{2}(\gamma^2 + \psi^2) + ..., \quad \psi \ll 1.$$

Taking account the known asymptotics [10,24] for the Bessel function $J_{n_0}(n_0\zeta)$ at
$n_0 \approx F^2/\omega^3 \gg 1$, one obtains Eq.(12) which coincides with Eq.(23) in Ref.[7] for small angles, $\psi \ll \gamma \ll 1$.

Thus, all the results of Secs.4.1 and 4.2 of the review [7] easily follow from more general formulae of papers [3,4], if one puts in them $\gamma \ll 1$ (that corresponds to the low-frequency intense laser radiation). This fact, however, is noted neither in the reviews [6,7] nor in the other papers of these authors.

b) When considering the Coulomb correction (see Sec.5.1 of Ref.[7] which is based on the paper [13]), the authors restricted themselves to the case of stationary electric field, for which this correction had been known for a long time [25,26]. We note, however, that in Ref.[5] this correction is calculated not only for $\gamma \ll 1$ but for a considerably wider range of the values of parameter $\gamma$. And here the method used in [7,13] does not in fact differ from that proposed earlier in Ref.[5] (namely, the account of Coulomb interaction using quasiclassical perturbation theory).

It is rather surprising that the revelant references are absent in Ref.[7], the more so as papers [3-5,14] were well known to the authors of the review [7] and in due time were discussed in detail by the authors of the above papers with N.B.Delone.

c) When discussing Eq.(6) of Ref.[7] obtained in [4], the authors of Ref.[7] made an error. This formula [6)]

$$W_{lm}(F) = (2l+1)\frac{(l+\mid m\mid)!}{2^{|m|}(\mid m\mid)!(l-\mid m\mid)!}\,\kappa^2 C_{\kappa l}^2\, 2^{2n^*-|m|}\,\epsilon^{|m|+1-2n^*}\, exp(-2/3\epsilon) \tag{31}$$

is true not for the hydrogen-atom states (as is well known [25], for the latter, in the presence of external electric field $F$ the orbital moment $l$ is not conserved), but for the $lm$-state of an arbitrary atom without an accidental Coulomb degeneracy. That is why to apply this formula to an arbitrary atom there is no need to generalize it using the quantum defect method (contrary to the statement of Refs.[7,30]). Furthermore, the authors of [7,12,30] use the expression for the asymptotic coefficient $C_{\kappa l}$ of atomic wave function at infinity proposed by Hartree even in 1927:

$$C_{\kappa l}^2 = \frac{2^{2n_*-2}}{n^*(n^*+l)!\,(n^*-l-1)!}, \qquad n^* > l, \tag{32}$$

(Eq.(7.6) in Ref.[27], see also [28,29]). After replacing factorials according to the Stirling formula, $x! \equiv \Gamma(x+1) \approx \sqrt{2\pi x}(x/e)^x$, they obtain Eqs.(7)-(9) of the review [7], which are called by the authors of [7] as "ADK formulae".

Naturally, it is difficult to reject one or another name, but one should still note that the original contribution of the authors of the "ADK-theory" (Ammosov, Delone, and Krainov) reduces to applying of the Stirling formula to the coefficients ($F$-independent) obtained earlier in [4,27]. Thus, for the case of the circular polarization the ionization rate of $s$-level of a neutral atom is

$$W(F) = N\kappa^2\epsilon^{1-2n^*}\, exp\Big[-\frac{2}{3\epsilon}\Big(1-\frac{1}{15}\gamma^2\Big)\Big], \quad \gamma \ll 1, \tag{33}$$

where $\epsilon = F/\kappa^3$ is the reduced electric field strength, $n^* = 1/\kappa$ and $N = 2^{2n_*}C_\kappa^2$ is a dimensionless numerical coefficient.

The "ADK-formula" can be hence obtained by substituting

$$N = \Big[\frac{2^{2n*-1}}{\Gamma(n^*+1)}\Big] \to N_{ADK} = \frac{1}{8\pi n^*}\Big(\frac{4e}{n^*}\Big)^{2n^*} \tag{34}$$

The $F$-dependence remains the same as in Refs.[3,4], the effective quantum number $n^*$ being of order of one. As for $s$-electrons in neutral ($Z = 1$) atoms, the values of $n^*$ vary from 0.744 for $He$ up to 1.869 for $Cs$ , while for

singly charged positive ions $Z = 2$ and $n^* = 2/\kappa$ varies from $0.848(Li^+)$ to $2.221(Sr^+)$. The use of the Stirling approximation in this region has neither sufficient grounds nor any numerical advantages, and only systematically overestimates absolute values of the ionization probability. In particular, for a hydrogen atom (the ground state) it is proposed [7,13] to replace the coefficient $N = 4$ in the well-known formula [25] $W(F) = NF^{-1}\, exp(-2/3F)$ which is asymptotically exact within the weak fields, by $N_{ADK} = 2e^2/\pi \approx 4.70$ which can hardly be considered as a good approximation (the error is 18%; for the case of $He$-atom it amounts to 25%, etc).

It is worth mentioning a peculiar method used by the authors when citing Refs.[3-5,14]: even if these papers are mentioned somewhere, this is done not in connection with their principal content but for some or another secondary problem, the reader being as a rule referred for details to papers by Delone and Krainov. Some examples of such kind can be easily found in Refs.[7,12,13], in the monography [6] and in other papers of these authors.

I would like to hope that the above remarks will be useful to readers and will help to restore the truth.

**7.** Thus, we have analysed the form of the energy and angular spectra of photoelectrons in tunnel ionization throughout the whole variation range of the Keldysh parameter, $0 < \gamma < \infty$, for linear and circular polarizations of electromagnetic radiation. The general case of elliptic polarization also can be considered for arbitrary $\gamma$, but the expressions obtained are rather complicated [14] and require a separate discussion. The total ionization rate of hydrogen atoms in the above-barrier region is calculated (for $\gamma \ll 1$ and any $\xi$), and formulae for the quantum tunnel interference in the energy spectrum of electrons are presented.

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**Footnotes**

1) Unfortunately, it should be said that the authors of Refs.[6] and [7] considered the contributions of papers [3-5] quite unobjectively. For details see Ref.[8] and Sect.6 below.
2) See Eqs.(44),(53) in Ref.[4].
3) These coefficients are often encountered in atomic and nuclear physics. Tables of their numerical values for $s$-states of neutral atoms and positive ions can be found in Ref.[9]. For instance, $C_\kappa = 1$ for the ground state of a hydrogen atom. The pre-exponential factor $R$ is also calculated in [4], but there it is of no need for us here.
4) For the case of $\gamma \ll 1$ this result had been obtained in Ref.[3].
5) See Eqs.(B.1),(B.2) in Appendix B.
6) See Eq.(4) in Ref.[4]. Here $n^* = Z/\sqrt{2E_i}$ is the effective principal quantum number [27-29], which is calculated according to the experimental energy of atomic state ( $Z$ is the charge of atomic or ionic core). Thus, for the valent $s$-electron in $Rb$ atom Hartree [27] gives $n^* = 1.805$, that is very close to the modern value $n^* = 1.80477$. For the (non-degenerate) ground state of a hydrogen atom $n^* = 1$, $C_\kappa = 1$ and Eq.(31) gives the ionization rate $W_{00} = 4F^{-1}exp(-2/3F)$, in accordance with Ref.[25].

## Appendix A

For the angular distribution of electrons Eq.(10) contains the Bessel function at large values of argument and index. The Langer uniform asymptotic expansion is known for it:

$$J_n(n\zeta) = \frac{1}{\pi}\left[\frac{h(\sqrt{1-\zeta^2})}{\sqrt{1-\zeta^2}}\right]^{1/2} K_{1/3}(\, nh(\sqrt{1-\zeta^2}\,) + O(n^{-4/3}), \qquad \text{(A.1)}$$

see [10,24] (it should be noted that formulae 7.13.3(34), (25)-(27) in the Russian edition of the reference-book [24] contain some misprints). Here

$$h(x) = Arth\ x - x = \begin{cases} \frac{1}{3}x^3 + \frac{1}{5}x^5 + ..., & x \to 0 \\ \frac{1}{2}\, ln\frac{1}{1-x} - c + ..., & x \to 1, \end{cases} \qquad \text{(A.2)}$$

$c = 1 - \frac{1}{2}ln2$, $K_{1/3}(z)$ is the McDonald function and $0 < \zeta < 1$ (classically forbidden region for the Bessel equation: turning point $\zeta_0 = (1-1/4n^2)^{1/2} \to 1$ as $n \to \infty$). Unlike WKB formulae, Eq.(A.1) remains valid in direct vicinity of the turning point.

Since

$$K_{1/3}(z) = \begin{cases} e^{-z}(\pi/2z)^{1/2}\left(1 - \frac{5}{72z} + ...\right), & z \to \infty \\ \frac{2^{1/3}\pi}{3^{1/2}\Gamma(2/3)} z^{-1/3}\ [\ 1 + O(z^{2/3})\ ], & z \to 0 \end{cases}$$

it follows from (A.1) that

$$J_n^2(n\zeta) \approx \frac{1}{2\pi n\eta} exp\{-2n\ h(\eta)\}, \quad n \to \infty, \qquad \text{(A.3)}$$

where $\eta = \sqrt{1-\zeta^2}$ and $n\eta^3 \gg 1$ (or $1 - \zeta \gg n^{-2/3}$). At $\zeta = 1$

$$J_n(n) = \frac{2^{1/3}}{3^{2/3}\Gamma(2/3)} n^{-1/3} + O(n^{-5/3}), \qquad \text{(A.4)}$$

see also 9.3.31 in [31]. Further, if $n^{-2/3} \ll 1 - \zeta \ll 1$, then

$$J_n(n\zeta) \approx \left[8\pi^2 n^2 (1-\zeta)\right]^{-1/4} exp\left\{-\frac{2\sqrt{2}}{3}\, n(1-\zeta)^{3/2}\right\} \qquad \text{(A.5)}$$

In our case

$$1-\zeta^2=\frac{sin^2\psi+t_0^2\ cos^2\psi+\gamma^2}{1+\gamma^2}=\psi^2+\gamma^2+... \qquad (A.6)$$

at $\gamma,\psi \ll 1$ (see Eq.(9)). Since $n(1-\zeta^2)^{3/2} > n_0\gamma^3 = \epsilon^{-1} \gg 1$, therefore the use of the asymptotics (A.3) yields Eq.(11) of the paper. Analogously, from (A.5) and (A.6) Eq.(13) immediately follows.

## Appendix B

Here we consider limiting case of the constant electric field, $\omega \to 0$. In this case the energy spectrum $E_n = (n-\nu)\omega$ is practically continuous. Changing summation over $n$ to integration and taking into account that

$$\mathcal{J}_0(\alpha,\beta) = \int_0^\infty e^{-\alpha x}\, a(\beta x)dx = \frac{1}{\sqrt{\alpha(\alpha+2\beta)}}, \qquad a(x) = exp(-x)I_0(x), \quad \text{(B.1)}$$

we obtain from (17) the total ionization rate of $s$-level:

$$W_a(F,\xi) = \kappa^2 C_\kappa^2 \sqrt{\frac{3\epsilon^3}{\pi(1-\xi^2)}}\, exp\left\{-\frac{2}{3\epsilon}\left[1-\frac{1}{10}\left(1-\frac{\xi^2}{3}\gamma^2\right)\right]\right\}, \quad \gamma \ll 1 \quad \text{(B.2)}$$

This equation is valid for $0 \le \xi \lesssim 1-\epsilon$ and coincides with the result of the adiabatic approximation [4,14]. In addition to (B.1), the following identities hold:

$$\mathcal{J}_1 = \frac{\alpha+\beta}{(\alpha^2+2\alpha\beta)^{3/2}}, \qquad \mathcal{J}_2 = \frac{2\alpha^2+4\alpha\beta+3\beta^2}{(\alpha^2+2\alpha\beta)^{5/2}}, \qquad \text{(B.3)}$$

where

$$\mathcal{J}_n \equiv \int_0^\infty e^{-\alpha x}\, a(\beta x)\, x^n dx$$

In the adiabatic region $\alpha \propto \gamma^3$, $\beta \propto \gamma$, so $\alpha \ll \beta$. From (B.1), (B.3) and (18) we have

$$\langle n' \rangle = \frac{\alpha+\beta}{\alpha^2+2\alpha\beta} = \frac{1}{2\alpha}\left(1+\frac{\alpha}{2\beta}+...\right), \quad \frac{\alpha}{\beta} = \frac{2}{3}(1-\xi^2)\gamma^2,$$

$$\Delta n' = \sqrt{\langle n'^2 \rangle - \langle n' \rangle^2} = \frac{\sqrt{\alpha^2+2\alpha\beta+2\beta^2}}{\alpha^2+2\alpha\beta} = \frac{1}{\sqrt{2}\,\alpha}\left(1+\frac{\alpha^2}{8\beta^2}+...\right), \quad \text{(B.4)}$$

and Eqs.(21) follow.

The probability of $n$-photon ionization by linearly polarized radiation is [3,4]

$$w_n = \frac{4}{\pi}A_0\, \epsilon^{3/2}\gamma^2\, w(\sqrt{2\gamma(n-\nu)})exp\left\{-\frac{2}{3}\gamma^3(n-\nu)\right\}, \quad n > \nu, \qquad \text{(B.5)}$$

where $\gamma \ll 1$, $\nu = F^2/4\omega^3$, $A_0$ is defined in Eq.(19) for $\xi = 0$ and $w(x)$ is the well-known transcendental function [31]:

$$w(x) = e^{-x^2} \int_0^x e^{t^2} dt = \begin{cases} x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + ..., & x \to 0 \\ \frac{1}{2}x^{-1} + \frac{1}{4}x^{-3} + ..., & x \to \infty \end{cases} \tag{B.6}$$

Note that

$$w(x) = \frac{\sqrt{\pi}}{2i} erf(ix) \cdot e^{-x^2} = \frac{1}{2}\, x \int_0^1 e^{-x^2 y} \frac{dy}{\sqrt{1-y}}, \tag{B.7}$$

$$\int_0^\infty e^{-\alpha x}\, w(\sqrt{\beta x}\,)\, dx = \frac{1}{2(\alpha+\beta)}\sqrt{\frac{\pi\beta}{\alpha}} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha\beta}}\left(1 - \frac{\alpha}{\beta} + ...\right) \tag{B.8}$$

Integrating the probability distribution $w_n$ (B.5) over $n$ and using (B.8), we arrive at Eq.(B.2) in the case of $\xi = 0$.

Eqs.(18) and (B.5) differ only in the pre-exponential functions, $a(x)$ in (18) and $\tilde{a}(x) = 2\pi^{-1/2} w(\sqrt{2x})$ in (B.5), where $x = \gamma(n - \nu)$. At $x > 1$, $a(x) \approx \tilde{a}(x)$:

$$a(x) = \frac{1}{\sqrt{2\pi x}}\left(1 + \frac{1}{8x} + ...\right), \quad \tilde{a}(x) = \frac{1}{\sqrt{2\pi x}}\left(1 + \frac{1}{4x} + ...\right),$$

see also Fig.8 (the region $0 < x \lesssim 1$ gives negligible contribution to the total ionization rate $W = \sum_{n>\nu} w_n$).

Finally, in the case of circular polarization it follows from (6) that

$$w_{max} = c_5\, W_c, \tag{B.9}$$

where

$$c_5 = \sqrt{\frac{\omega^4 \kappa}{\pi F^3}} = \gamma^2 \sqrt{\frac{\epsilon}{\pi}} = \frac{\gamma^{3/2}}{\sqrt{2\pi K_0}} \ll 1, \tag{B.10}$$

$w_{max} \equiv w_{n_0}$ is the maximal value of the ionization probability and [3,4] $W_c = \kappa^2 C_\kappa^2 \epsilon \cdot exp\left\{-\frac{2}{3\epsilon}\left(1 - \frac{1}{15}\gamma^2\right)\right\}$. This expression can be obtained from (B.2) by the formal change in the pre-exponential factor: $\sqrt{3\epsilon/\pi(1-\xi^2)} \to 1$.

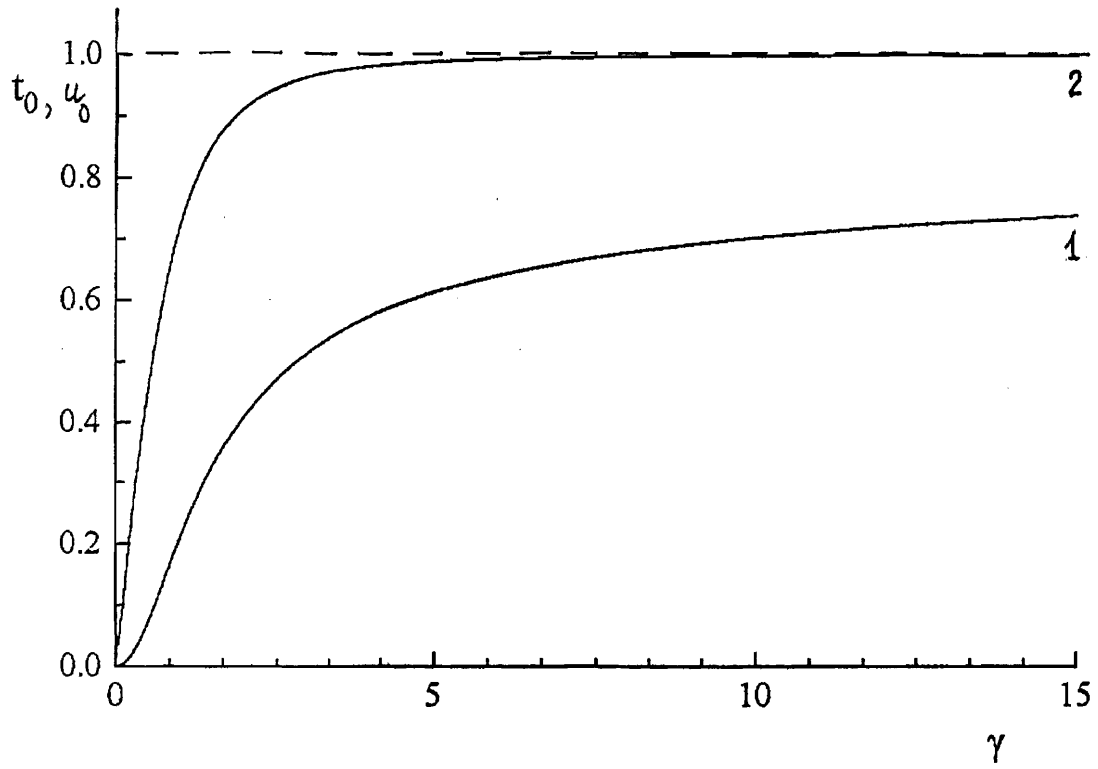


**Fig.1.** The variables $t_0(\gamma)$, curve 1, and $u_0 = \sqrt{(t_0^2+\gamma^2)/(1+\gamma^2)}$, curve 2.

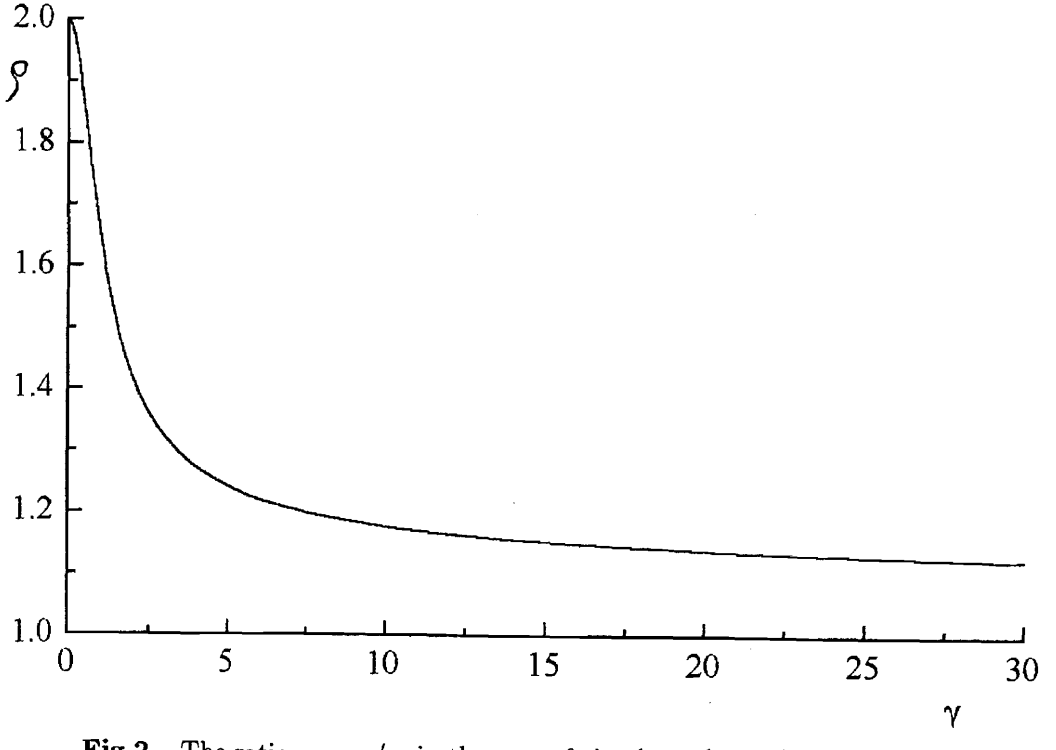


**Fig.2.** The ratio $\rho = n_0/\nu_c$ in the case of circular polarization.

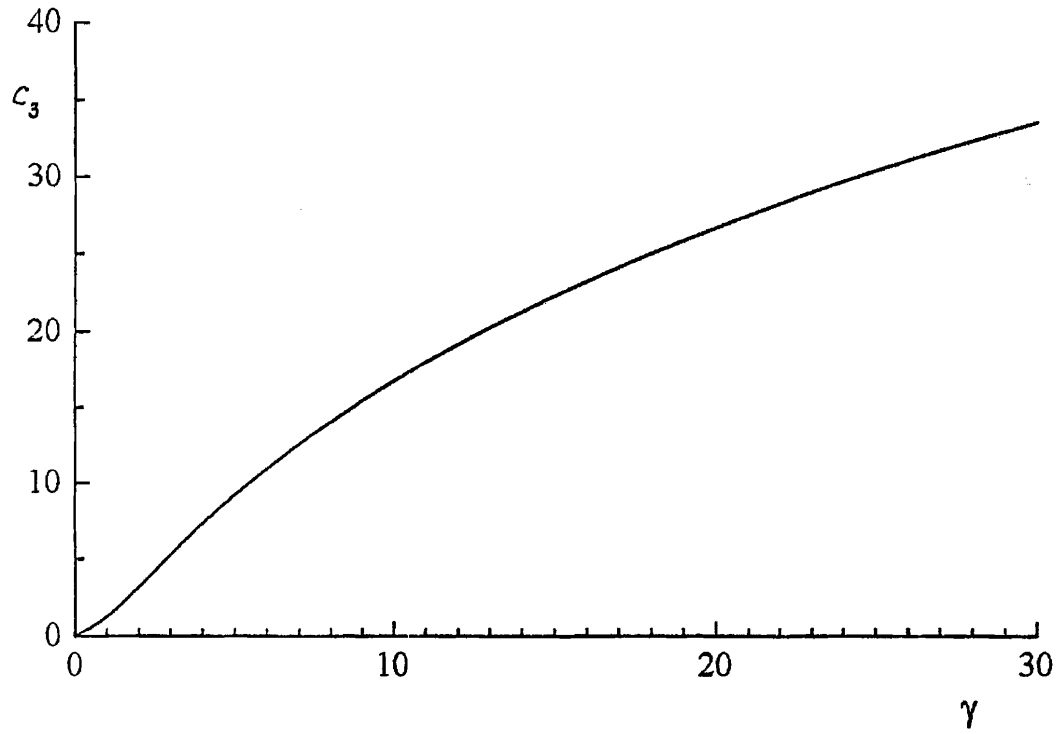


**Fig.3.** The coefficient $c_3(\gamma)$ in Eq.(6) as a function of $\gamma$.

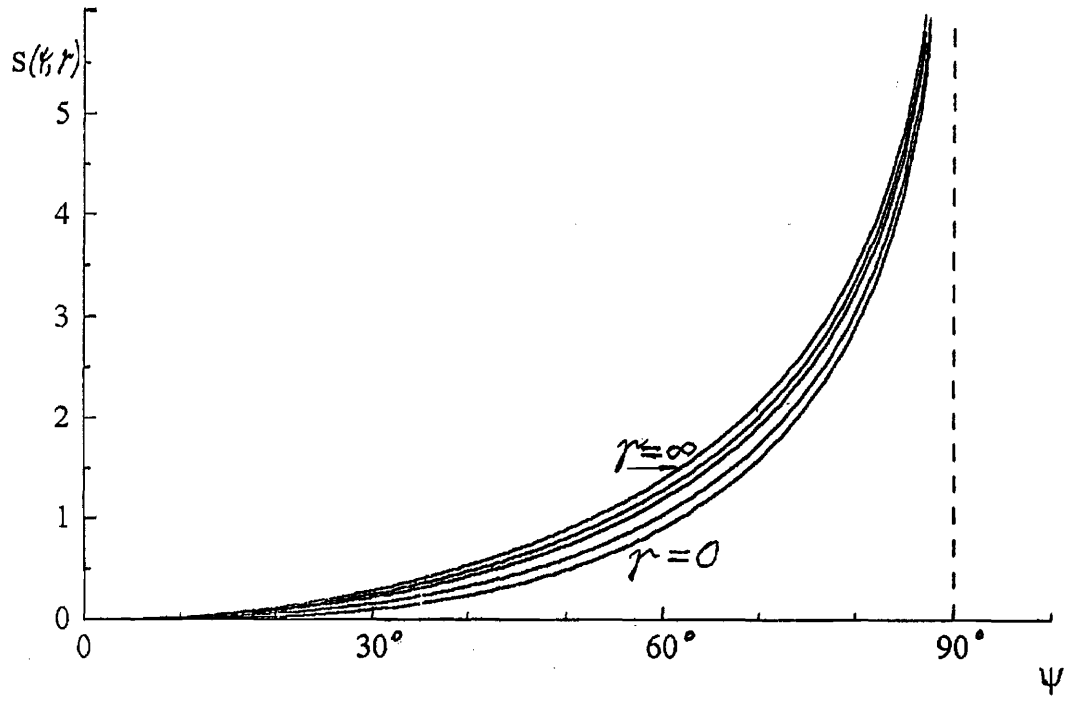


**Fig.4.** The function $s(\psi, \gamma)$ determining the angular distribution of electrons. The curves (from bottom to top) correspond to $\gamma = 0, 0.5, 1.0, 1.5$ and $\infty$.

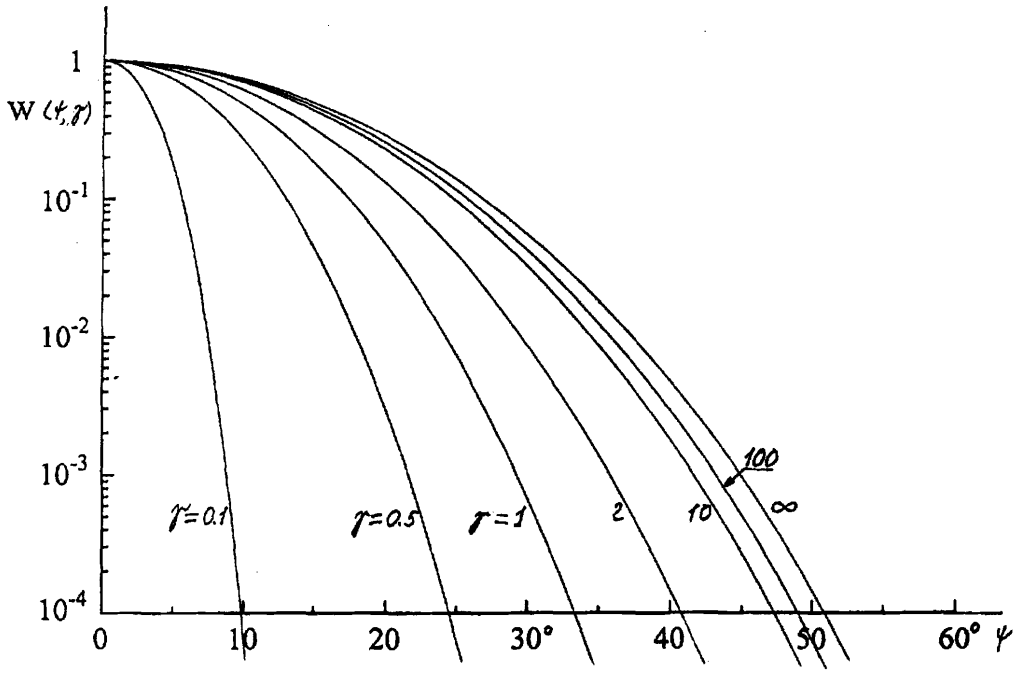


**Fig.5.** Circular polarization. Angular spectra of photoelectrons normalized by the condition $w(0,\gamma) = 1$ ($K_0 = 10$, $\psi$ is the angle between electron momentum $\mathbf{p}$ and polarization plane of laser radiation).

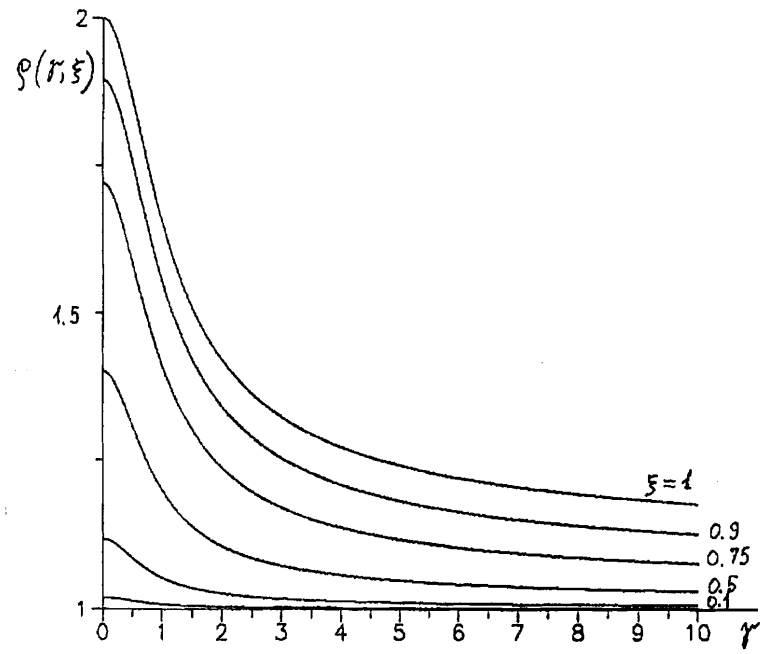


**Fig.6.** The ratio $\rho(\gamma,\xi) = n_0/\nu$ as a function of the Keldysh parameter $\gamma$. Here $n_0$ is the most probable and $\nu$ – the minimal number of absorbed photons, i.e. the ionization threshold. The values of ellipticity $\xi$ are given at the curves, $\xi = 1$ correspond to circular polarization of radiation.

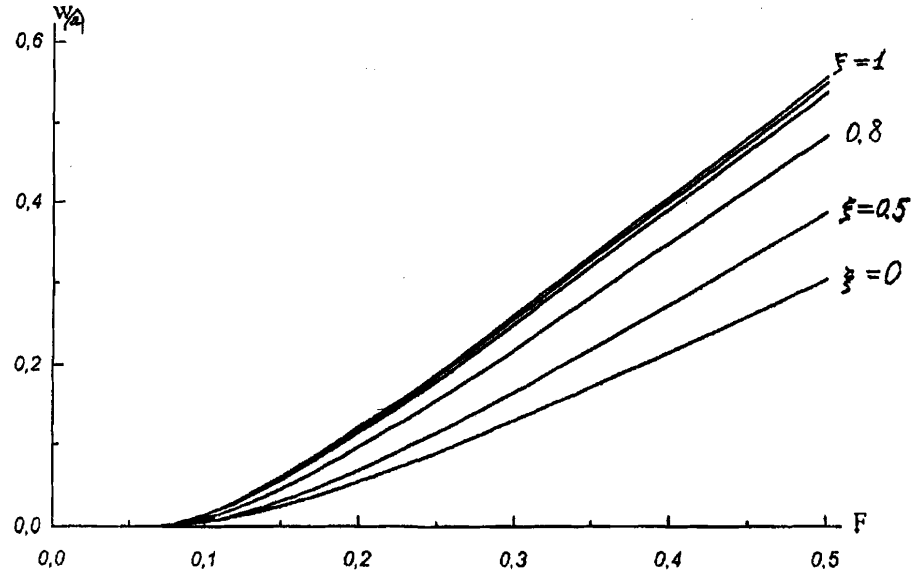


**Fig.7.** The total ionization rate for a hydrogen atom in the ground state by low-frequency laser radiation. The curves (from bottom to top) correspond to the ellipticity $\xi = 0, 0.5, 0.8, 0.9, 0.95$ and 1.0. The values of $F$ and $w_a$ are given in atomic units.

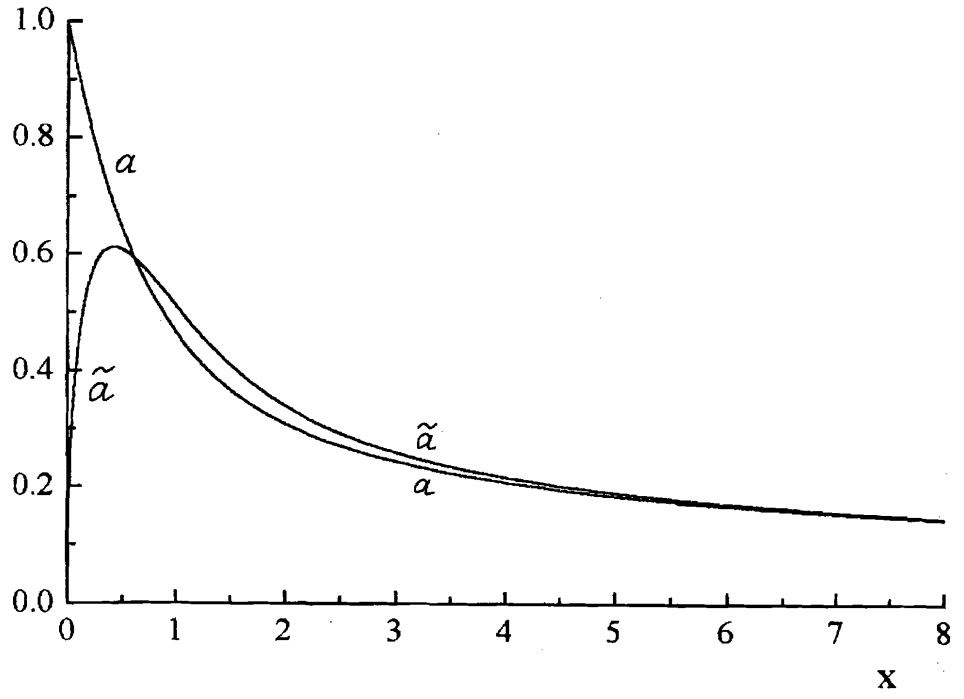


**Fig.8.** The functions $a(x)$ and $\tilde{a}(x)$.

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