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A Generalization of the Temperley-Lieb Algebra

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Abstract

We present a multi-parametric generalization of the Yang-Baxter equation. Using a graphical representation we find a recursive formula for the number of elements of this algebra.

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For various reasons interest in Temperley-Lieb algebras [1] is extremely widespread [2/6]. In addition to their relevance to knot theory and Braids [3], they are related to quantum groups [4], conformal field theory [5], and statistical mechanics [6]. For these reasons the Temperley-Lieb algebra has been under the focus of attention in recent literature. In this paper we present a generalization of this algebra, by constructing a multi-parametric extension of the defining relations. The Temperley-Lieb [1] algebra $TL_n(p)$, is indexed by the number of generators n, and the complex parameter p:

$$\begin{aligned} e_i^2 &= pe_i \\ e_ie_{i\pm1}e_i &= e_i \\ e_ie_j &= e_je_i \qquad |i-j| \geq 2. \end{aligned} \tag{1}$$

One may construct a representation of the generators of Braid group B_i, using the generators of $TL_n(p)$:

$$B_i = q - Ae_i \tag{2}$$

then B_i satisty the Braid group relation on n+1 strands:

$$B_iB_{i\pm1}B_i = B_{i\pm1}B_iB_{i\pm1} \tag{3}$$

The generators B_i , will also satisfy the extra condition (skein relation):

$$A^{-1}B_i - AB_i^{-1} = A^{-1}q - Aq^{-1} \tag{4}$$

The parameters A,q and p are related via the relation

$$p = q^{-1}A + qA^{-1} \tag{5}$$

The elements of $TL_n(p)$ are built as monomials of e_i. Clearly there can only exist a finite number of such elements as a long monomial may be reduced using the defining relations (1).

Let a_k be the numer of independent monomials which can constructed out of exactly k generators. We shall call such a monomial a word, k is the length of the word and a_k is the number of distinct words, Clearly $a_0 = 1$, $a_1 = N$. But the higher values of a_k are harder to determine. If all the a_k were known one could find X_n the cardinality of $TL_n(p)$ by simply adding up all the a_k. We shall derive an

expression for a_k, further along the paper. The cardinality of $TL_n(p)$ in given by the Catalan numbers C_n[2] :

$$C_n = \frac{(2n)!}{n!(n+1)!} \tag{6}$$

further interesting results regarding $TL_n(p)$ may be foumd in reference[2].

A graphical representation of $TL_n(p)$ is given in fig 1. (a). We use this graphical representation for calculating X_n. If we open up the fig 1. (a), we get fig. 2(a), were X_n is the number of ways points may be connected in fig 2(a) without corss over. One can immediately see that X_n satisfy the following recurrsion relation:

$$X_n = \sum_{l=0}^{n-1} X_{n-1-l} X_l \tag{7}$$

Where $X_0 = X_1 = 1$. It is easy to check that the relation (7) generate the Catalan numbers.

We generalise the Temperly-Lieb algebra by introducing an extra set of indices. We now have $e_i^{\alpha\beta}$, as generators of $TL_n^N(M^{\alpha\beta})$:

$$\begin{aligned} &e_i^{\alpha\beta} e_i^{\gamma\theta} = M^{\beta\gamma} e_i^{\alpha\theta} && \alpha, \beta = 1, \cdots, N \\ &e_i^{\alpha\beta} e_{i\pm1}^{\gamma\theta} e_i^{\rho\sigma} = \delta_\gamma^\beta \delta_\rho^\theta e_i^{\alpha\sigma} && \\ &[e_i^{\alpha\beta}, e_j^{\gamma\theta}] = 0 && |i-j| \geq 2. \end{aligned} \tag{8}$$

This algebra has N^2 parameters $M^{\alpha\beta}$. We may construct a representation of the Braid group using $e_i^{\alpha\beta}$:

$$B_i = q - \sum_{\alpha\beta} A_{\alpha\beta} e_i^{\alpha\beta} \tag{9}$$

Where the parameters $A_{\alpha\beta}$ are related to $M^{\alpha\beta}$ and q, via the relation:

$$M^{\alpha\beta} = q A_{\alpha\beta}^{-1} + q^{-1} A_{\alpha\beta} \tag{10}$$

The generators B_i do not satisly a simple skein relation but the inverse of B, has a simple form:

$$B_i^{-1} = q^{-1} - \sum_{\alpha\beta} A_{\alpha\beta}^{-1} e_i^{\alpha\beta} \tag{11}$$

using the relations (9) and (10) we may check that B_i satisfy the braid relations (3).

A graphical representation of this algebra (fig. 1(b)), may be given similar to the simple $TL_n(p)$, by adding the indices α and β to fig. 1(a).

The cardinality of $TL_n^N(M)$ may be calculated using a diagramatic technique. We proceed by defining $X_{n,m}$ to be the number of ways an open graph of n sites to the left and m sites to the right can be connected (fig. 2(b)), with a factor of N for each index. Then $X_{n,n}$ is the cardinality of $TL_n^N(M)$.
The numbers $X_{n,m}$ satisfy the recurrsion relations:

$$\begin{aligned} X_{n,m} &= \overline{X_{n,m}} + \overline{X_{n-2,m}} X_{2,0} + \cdots \\ &\quad + \overline{X_{0,m}} X_{n,0} && m, n = \text{even} \\ \text{or} &\quad + \overline{X_{1,m}} X_{n-1,m} && m, n = \text{odd} \\ &\quad + \cdots + \overline{X_{0,2}} X_{n,m-2} \end{aligned} \tag{12}$$

Where $\overline{X_{n,m}}$ is associated with the graph with its outmost points connected. The following relations are useful for calculating $X_{n,m}$:

$$\begin{aligned} X_{n,m} &= X_{m,n} \\ \overline{X_{n,m}} &= N^{n-m} X_{n-1,m-1} \quad n > m \\ \overline{X_{0,2k}} &= N^{2k-1} X_{0,2k-2} \end{aligned} \tag{13}$$

Using these relations we may find the cardinality of $TL_n^N(M)$, as a polynomial in N. The first few examples are:

$$\begin{aligned} X_{1,1} &= 1 \\ X_{2,2} &= 1 + N^2 \\ X_{3,3} &= 1 + 2N^2 + 2N^4 \\ X_{4,4} &= 1 + 3N^2 + 5N^4 + 6N^6 + N^8 \\ X_{5,5} &= 1 + 4N^2 + 9N^4 + 12N^6 + 10N^8 + 4N^{10} + 2N^{12} \end{aligned}$$

Noting that, if the number of the words, a_k were given, we would have:

$$X_{n,n} = \sum a_k N^{2k} \tag{14}$$

Thus we find a_k by looking at $X_{n,n}$.

Many questions remain unanswered for example, by comparison with $TL_n(p)$, the parameter q, should be fixed and related to the model in question, be it in statistical mechanics, or conformal field theory, whereas the parameters, $A_{\alpha\beta}$ will be temperature dependent. However it is still an open question, as to which model this algebra is related. In fact, so long as we are not in possesion of a matrix representation of $e_i^{\alpha\beta}$ a physical interpretation, will be lacking.

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Figure Captions:

Figure 1.
The diagramatic representation of the generators of $TL_n(p)(a)$ and the generators of $TL_n^N(M)(b)$. Multiplication is represented by placing a diagram on top of the other and connecting corresponding lines.

Figure 2.
The number of elements of the algebra $TL_n(p)$ is given by the number of ways the graph in fig 2(a) can be connected such that lines do not cross each other and not cross the middle line. The same applies to $TL_n^N(M)$, fig. 2(b), provided a track of the indices is kept.

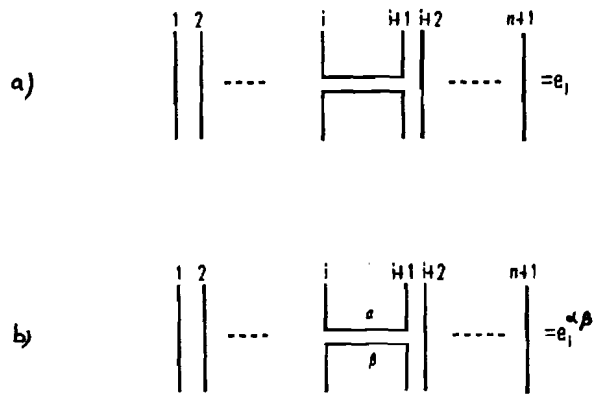

figure 1.

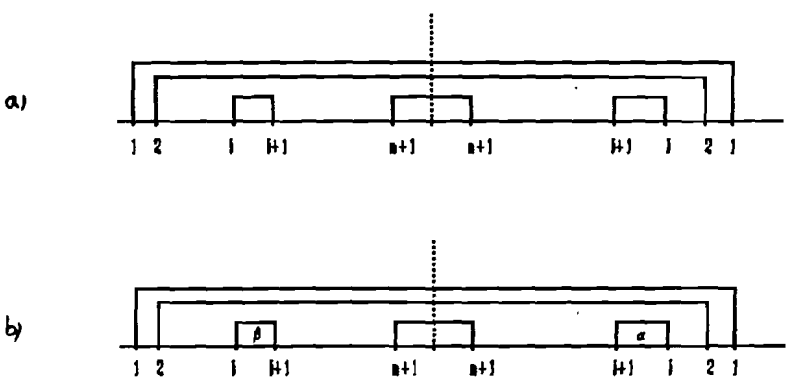

figure 2.