INS-Rep.-937
June 29, 1992



# A version of the Quasiparticle-Phonon Nuclear Model for Doubly-Even Well-Deformed Nuclei

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## Abstract

The basic assumptions concerning the Quasiparticle-Phonon Nuclear Model are formulated and the mathematical apparatus is developed. The Hamiltonian, containing a finite-rank separable isoscalar and isovector multipole, a spin-multipole and a tensor particle-hole as well as particle-particle interactions transforms to a form containing quasiparticle, phonon and quasiparticle-phonon interactions. The general RPA equation is derived and the particular cases are discussed. The very complex interaction does not complicate the description of the fragmentation one-phonon states. It is shown that the three-phonon terms added to the one-and two-phonon terms in the wave function lead to an additional small shift of the two-phonon poles in the secular equation. The influence of the density-dependent separable interaction on the vibrational states is small. A common description of the collective, weakly collective and two-quasiparticle states in doubly-even well-deformed nuclei is obtained.

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## § 1. Introduction

The energies and wave functions of two-qusiparticle and one-phonon states in doubly-even deformed nuclei were calculated in 1960-1975. A sufficiently good enough description was obtained[1] of the available at that time experimental date; predictions were made which were later experimentally confirmed in many cases. It seems to us that new calculations of the vibrational states in deformed nuclei are needed. This is necessitated by the large amount of new experimental data in addition to that involving the first quadrupole and octupole states. Experimental data are available concerning the hexadecapole states as well as the higher-lying collective and weakly collective states. Many experimental data are expected to be obtained at the new generation of accelerators; further, the results of calculations may turn out to be useful.

The vibrational states should be calculated on a new basis, such as the Quasiparticle-Phonon Nuclear Model (QPNM).[2),3)] The QPNM can be used for a microscopic description of the low-spin, small-amplitude vibrational states in spherical nuclei not far from closed shells and in well-deformed nuclei.

Let us consider the specific features of deformed nuclei. In the transition from spherical to axial symmetry, the spherical subshells split into twice-degenerate single-particle states. This splitting of the subshells leads to a decrease in the matrix elements of some operators between the single-particle wave functions of the axial symmetric Woods-Saxon potential, compared with the matrix elements of the same operators between the wave

functions of the pherical symmetric Woods-Saxon potential. Such a decrease in the matrix elements considerably affects the vibrational states of deformed nuclei.

We restrict our investigation to an internal wave function, $\Psi_\nu(K^\pi\sigma)$, with a good quantum number K, parity $\pi$, and $\sigma=\pm1$. We do not consider the Coriolis interaction that mixes states with different K and the same $\pi$. It is possible to take the Coriolis interaction into account in cases when it is necessary to do so.

The specific feature of deformed nuclei is that one-phonon states with the same $K^\pi$ can be formed as the result of different multipole and spin-multipole interactions. The one-phonon states of the electric type or natural parity states with fixed $K^\pi$ can be described by the multipole $\lambda\mu$-KK, K+2 K, K+4 K, etc, and by the spin-multipole $\lambda\lambda\mu$-KKK, K+2 K+2 K, etc, interactions. One-phonon states of the magnetic type or unnatural parity states can be described by the spin-multipole interaction $\lambda'$LK with $\lambda'=L\pm1$ and by the tensor interaction. If in deformed nuclei, as in spherical nuclei, one introduces independent phonons of the electric and magnetic type, the number of states will be doubled. To avoid doubling, the phonon operator has been introduced in [4),5)] which comprises the electric and magnetic parts.

In this paper the mathematical apparatus QPNM for a microscopic description of the doubly-even well-deformed nuclei are presented. The basic assumptions of the QPNM and Hamiltonian are formulated in 2. The general RPA equation and several particular vershions are given in 3. In 4 the wave function of the non-rotational excited states are introduced and basic



equations of the QPNM are derived. The general propertes of the low-spin, small-amplitude vibrational states in doubly-even well deformed nuclei are formulated in 5, and a conclusion is made in 6.

## § 2. Basic assumptions concerning the QPNM.

The initial QPNM Hamiltonian for the nonrotational states of deformed nuclei contains the average field of a neutron and a proton system in the form of the axial-symmetric Woods-Saxon potential, monopole pairing, isoscalar and isovector particle-hole (ph), as well as particle-particle (pp) multipole, spin-multipole and tensor interactions between quasiparticles. The effective interactions between quasiparticles are experessed as a series of multipoles and spin-multipoles. It is essential that the interaction between quasiparticles is represented in a separable form, which was first introduced by Yamaguchi.[6)] A separable interaction of finite rank $n_{max}>1$ is used in cases where the results of calculations are more sensitive to the radial dependence of forces, in compared with a calculation of the structure of complex nuclei within the QPNM. It may be assumed that the finite-rank separable interactions between quasiparticles do not limit the accuracy of calculations.

We introduce a finite-rank separable interaction for deformed nuclei. Consider, for example, the central spin-independent interaction $V(|r_1-r_2|)$ and expend it over multipoles,

$$V(|r_1-r_2|) = \sum_\lambda R^\lambda(r_1,r_2)\frac{4\pi}{2\lambda+1}\sum_{\mu=-\lambda}^{\lambda}(-1)^\mu Y_{\lambda\mu}(\theta_1\psi_1)Y_{\lambda-\mu}(\theta_2\psi_2).$$

We present the radial part, $R^\lambda(r_1,r_2)$, in the form

$$R^\lambda(r_1,r_2) = \sum_{n=1}^{n_{max}} \zeta_n R_n^\lambda(r_1)R_n^\lambda(r_2) \ . \qquad (2.1)$$

Most of the calculations used in the QPNM are performed using a simple separable interaction,

$$R^\lambda(r_1,r_2) = \kappa^\lambda R^\lambda(r_1)R^\lambda(r_2) \ . \qquad (2.2)$$

We now transform the initial QPNM Hamiltonian. For this purpose we perform a canonical Bogolubov transformation,

$$a_{q\sigma} = u_q\alpha_{q\sigma} + \sigma v_q\alpha^+_{q\sigma} \ , \qquad (2.3)$$

in order to replace the particle operators, $a_{q\sigma}$ and $a^+_{q\sigma}$, by quasiparticle operators, $\alpha_{q\sigma}$ and $\alpha^+_{q\sigma}$ . Let us introduce two types of phonon operators. If only interactions of the electric type are taken into account, the phonon creation operator has the following standard form:

$$Q^+_{Ki,\sigma} = 1/2\sum_{q_1q_2}\{\psi^{Ki}_{q_1q_2}A^+(q_1q_2;K\sigma) - \phi^{Ki}_{q_1q_2}A(q_1q_2;K-\sigma)\} \ . \qquad (2.4)$$

If the interactions of the electric and magnetic type are taken into account, we obtain[5),7)]

$$Q^+_{Ki,\sigma} = \frac{1}{2\sqrt{2}}\sum_{q_1q_2}\{\psi^{Ki}_{q_1q_2}(1+i\sigma)[\bar{A}^+(q_1q_2;K\sigma) + \chi(q_1q_2)\bar{A}^+(q_1q_2;K\sigma)]-$$
$$-\phi^{Ki}_{q_1q_2}(1-i\sigma)[\bar{A}(q_1q_2;K-\sigma) + \chi(q_1q_2)\bar{A}(q_1q_2;K-\sigma)]\} \ . \qquad (2.5)$$



The operator (2.5) comprises of the electric and magnetic parts;the coefficients of the electric part are real and those of the magnetic part are imaginary. This form of operator is more convenient than that given earlier.[4)] Here i=1,2,3,...is the root number of the RPA secular

equation and $\psi^{Ki}_{q_1q_2} = \psi^{Ki}_{q_2q_1}$ , $\phi^{Ki}_{q_1q_2} = \phi^{Ki}_{q_2q_1}$ .

The quantum numbers of the single-particle states are denoted by $q\sigma$, where $\sigma=\pm1$; q equals $K^\pi$ and asymptotic quantum numbers $Nn_Z\ \Lambda\ \uparrow$ at $K=\Lambda+1/2$ and $Nn_Z\ \Lambda\ \uparrow$ at $K=\Lambda-1/2$. The operators, $\bar{A}(q_1q_2iK\sigma)$ and $\bar{A}(q_1q_2iK\sigma)$, are given in the Appendix and

$$\chi(q_1q_2)\bar{A}(q_1q_2;K\sigma)=-\chi(q_2q_1)\bar{A}(q_1q_2;K\sigma)=\chi(q_2q_1)\bar{A}(q_2q_1;K\sigma) \ .$$

The RPA one-phonon state is described by the wave function

$$Q^+_{Ki,\sigma}\Psi_0 \ , \qquad (2.6)$$

where $\Psi_0$ is the ground-state wave function of a doubly-even nucleus, determined as being a phonon vacuum. The normalization condition of the wave function (2.6) has the form

$$\frac{1}{2}\sum_{q_1q_2}[(\psi^{Ki}_{q_1q_2})^2 - (\phi^{Ki}_{q_1q_2})^2] = 1 \ . \qquad (2.7)$$

One can easily show that the phonon operators, $Q^+_{Ki,\sigma}$ and $Q_{Ki,\sigma}$, obey the conditions which are usually imposed on RPA phonons.

By using formulae (2.4),(2.5) and others, and after some transformations the QPNM Hamiltonian becomes

$$H_{QPNM} = \sum_{q\sigma} \varepsilon_q \alpha^+_{q\sigma}\alpha_{q\sigma} + H_v + H_{vq} \ , \qquad (2.8)$$

where the first two terms describe quasiparticles and phonons, and $H_{vq}$ describes the quasiparticle-phonon interaction. They have the following form:

$$H_v = H_v^{00} + \sum_\lambda H_v^{\lambda 0} + \sum_K H_v^K \ , \qquad (2.9)$$

$$H_v^K = -\frac{1}{4}\sum_{i_1 i_2} W^K_{i_1 i_2} Q^+_{Ki_1\sigma} Q_{Ki_2\sigma} \ , \qquad (2.10)$$

$$W^K_{i_1 i_2} = W^{KE}_{i_1 i_2} + W^{KM}_{i_1 i_2} + W^{KT}_{i_1 i_2} \ , \qquad (2.11)$$

$$W^{KE}_{i_1 i_2} = \frac{1}{4}\sum_{\lambda\tau}\sum_{n=1}^{n_{max}}\Big[\sum_{\rho=\pm1}(\kappa_0^{\lambda K} + \rho\kappa_1^{\lambda K}) D^{\lambda K i_1}_{n\tau} D^{\lambda K i_2}_{n\rho\tau} + G^{\lambda K}(D^{\lambda K i_1}_{ng\tau} D^{\lambda K i_2}_{ng\tau} + D^{\lambda K i_1}_{nw\tau} D^{\lambda K i_2}_{nw\tau})\Big] - \sum_\lambda W^{\lambda K}_{i_1 i_2} \qquad (2.12)$$

$$W^{KM}_{i_1 i_2} = \frac{1}{4}\sum_{L\tau}\sum_{\lambda'=L,L\pm1}\sum_{n=1}^{n_{max}}\sum_{\rho=\pm1}(\kappa_0^{\lambda' LK} + \rho\kappa_1^{\lambda' LK}) D^{\lambda' LK i_1}_{n\tau} D^{\lambda' LK i_2}_{n\rho\tau} \ , \qquad (2.13)$$

$$W^{KT}_{i_1 i_2} = \frac{1}{2}\sum_{L\tau}\sum_{n=1}^{n_{max}}\sum_{\rho=\pm1}(\kappa^{LK}_{TO} + \rho\kappa^{LK}_{T1}) D^{L-1LK i_1}_{n\tau} D^{L+1LK i_2}_{n\rho\tau} \ , \qquad (2.14)$$

$$H_{vq} = H_{vq}^{00} + \sum_\lambda H_{vq}^{\lambda 0} + \sum_K H_{vq}^K \ , \qquad (2.15)$$

$$H^K_{vq} = -\frac{1}{4\sqrt{2}}\sum_{i_1\tau\sigma}\sum_{n=1}^{n_{max}}\Big(\sum_\lambda H^{\lambda K i_1}_{vq} + \sum_{L,\lambda'=L\pm1} H^{\lambda' LK i_1}_{vq} + \sum_L H^{LKTi_1}_{vq}\Big) \ , \qquad (2.16)$$

$$H^{\lambda K i_1}_{vq} = \sum_{q_1 q_2}^\tau f_n^{\lambda K}(q_1 q_2) V^{(-)}_{q_1 q_2} (\kappa_0^{\lambda K} + \rho\kappa_1^{\lambda K}) D^{\lambda K i_1}_{n\rho^\tau} \qquad (2.17)$$

$$\times\ [(1 - i\sigma)(Q^+_{Ki_1\sigma} + Q_{Ki_1-\sigma}) B(q_1 q_2; K-\sigma) + h.c.] \ ,$$

$$H^{\lambda' LK i_1}_{vq} = \sum_{q_1 q_2}^\tau f_n^{\lambda LK}(q_1 q_2) V^{(+)}_{q_1 q_2} (\kappa_0^{\lambda' LK} + \rho\kappa_1^{\lambda' LK}) D^{\lambda' LK i_1}_{n\rho\tau} .$$

$$\times\ [(\sigma - i)(Q^+_{Ki_1\sigma} - Q_{Ki_1-\sigma}) B(q_1 q_2; K-\sigma) + h.c.] \ , \qquad (2.18)$$

$$H^{LKTi_1}_{vq} = i\sum_{q_1 q_2}^\tau (\kappa^{LK}_{TO} + \rho\kappa^{LK}_{T1})\big[D^{L-1LK i_1}_{n\rho\tau} f_n^{L+1LK}(q_1 q_2) + D^{L+1LK i_2}_{n\rho\tau} f_n^{L-1LK}(q_1 q_2)\big] V^{(+)}_{q_1 q_2} \cdot$$

$$\times\ [(Q^+_{Ki_1\sigma} - Q_{Ki_1-\sigma}) B(q_1 q_2; K-\sigma) + h.c.] \ . \qquad (2.19)$$

Here, $\varepsilon_q$ is the quasiparticle energy with monopole and quadrupole pairing ; $H_v^{00}$, $H_v^{\lambda 0}$, $H_{vq}^{00}$ and $H_{vq}^{\lambda 0}$ are given elsewhere.[8] The functions $D^{\lambda Ki}_{n\tau}$, $D^{\lambda Ki}_{ng\tau}$, $D^{\lambda Ki}_{nw\tau}$ and $D^{\lambda' LKi}_{n\tau}$; the operators $B(q_1 q_2; K\sigma)$, $B(q_1 q_2; K\sigma)$; and the matrix elements of the multipole and spin-multipole operators, $f_n^{\lambda K}(q_1 q_2)$ and $f_n^{L\pm1 LK}(q_1 q_2)$, are given in the Appendix. Further $u^{(\pm)}_{q_1 q_2} = u_{q_1} v_{q_2} \pm u_{q_2} v_{q_1}$, $v^{(\pm)}_{q_1 q_2} = u_{q_1} u_{q_2} \pm v_{q_1} v_{q_2}$. Summation over single-particle states of a neutron and proton system is denoted by $\sum^\tau_{q_1 q_2}$ at $\tau = n$ or $\tau = p$, respectively; $\kappa_0^{\lambda K}$ and $\kappa_1^{\lambda K}$ are the isoscalar and isovector constants of ph and $G^{\lambda\mu} = G_0^{\lambda\mu} + G_1^{\lambda\mu}$ represents the pp multipole interactions; $\kappa_0^{\lambda' LK}$, $\kappa_1^{\lambda' LK}$ and $\kappa^{LK}_{TO}$, $\kappa^{LK}_{T1}$ are constants of the isoscalar and isovector ph spin-multipole and tensor interactions.

Most calculations of the structure of the excited states and $B(E\lambda)$ values have been performed with a phonon operator (2.4) and a simple $n_{max}$=1 multipole interactions with $H^K_{vq}$ in the form

$$\tilde{H}^K_{vq} = \sum_\lambda H^{\lambda K}_{vq} = -\frac{1}{4}\sum_{\tau\sigma i_1}\sum_\lambda\sum_{\rho=\pm1}\sum_{q_1 q_2}^\tau f^{\lambda K}(q_1 q_2) v^{(-)}_{q_1 q_2}$$

$$\times\ (\kappa_0^{\lambda K} + \rho\kappa_1^{\lambda K}) D^{\lambda K i_1}_{\rho\tau} [(Q^+_{Ki_1\sigma} + Q_{Ki_1-\sigma}) B(q_1 q_2; K-\sigma) + h.c.] \ .$$

(2.20)

The Mλ transition probabilities are calculated in a reference article [7] with a phonon operator (2.5), but without a pp spin-multipole interaction.

Calculations using the QPNM are made in four stages. The first stage involves a calculation of the single-particle energies and wave functions of the Woods-Saxon potential. The parameters of the Woods-Saxon, potential are fixed so as to obtain a correct description of the low-lying states in odd-A nuclei, while taking account of the quasiparticle-phonon interaction. Undoubtedly, one uses another form of the average field potential or to calculates the energies and wave functions of the single-particle states within the Hartree-Fock method in order and to use them in calculations involves the QPNM.

The second stage is the canonical Bogolubov transformation and the calculations involving the model of independent quasiparticles. Taking simultaneously into account monopole pairing with the constants $G_\tau$ and quadrupole pairing with the constant $G^{20}$ and under the condition which excludes $0^+$ spurious states, the following equations were derived in [8]

$$1 = \frac{G_\tau}{2} \sum_q^\tau \frac{C_\tau + f^{20}(qq)C_{2\tau}}{C_\tau \varepsilon_q} \quad , \tag{2.21}$$

$$1 = G^{20} \left\{ \sum_q^\tau \frac{f^{20}(qq)C_\tau}{2C_{2\tau}\varepsilon_q} + \sum_{qq'}^\tau \frac{(f^{20}(qq')v_{qq'}^{(+)})^2}{\varepsilon_q} \right\} , \tag{2.22}$$

and

$$N_\tau = \sum_q^\tau \left[1 - \frac{(q)}{\varepsilon_q}\right] \quad . \tag{2.23}$$



By neglecting the nondiagonal matrix elements $f^{20}(qq')$ in eq.(2·22) one arrives at the equations derived earlier in two references. [9),10)] Here,

$$\varepsilon_q = [\Delta_q^2 + \xi^2(q)]^{1/2}, \qquad \xi(q) = E(q) - \lambda_\tau ,$$

$$\Delta_q = C_\tau + f^{20}(qq)C_{2\tau} , \qquad C_\tau = G_\tau \sum_q^\tau u_q v_q , \tag{2.24}$$

and

$$C_{2\tau} = G^{20} \sum_q^\tau f^{20}(qq) u_q v_q ,$$

Where E(q) is the single-particle energy, and $\lambda_\tau$ is chemical potential. The energies of the two-quasiparticle states are calculated while taking the blocking effect into account.

Then, the RPA phonons, (2.4) or (2.5), are introduced and RPA secular equations are solved. In the QPNM, the one-phonon states (2.6) with the operators (2.4) are used as the basis. Therefore, the third stage involves calculations of the one-phonon basis. The phonon basis for calculating of the low-lying states comprises ten ($i_1$=1,2,····,10) phonons of each multipolarity: quadrupole (λμ=20,22), octupole (λμ=30,31,32,33) and hexadecapole (λμ=43,44). The calculated states above 3MeV have been performed with a larger phonon basis with λ>4 and twenty phonon of each multipelarity. The phonon space corresponds to the full space of the two-quasiparticle states in doubly-even deformed nuclei.

The QPNM Hamiltonian is transformed to (2.8). The fourth stage takes into account the quasiparticle-phonon interaction.

The wave functions of the excited states are expressed as a series over the number of phonon operators; in odd nuclei each term is multiplied by a quasiparticle operator. The approximation involves a cut-off of this series. In calculations the Pauli principle is taken into account by using the exact commutation relations between the phonon and quasiparticle operators. To calculate the characteristics of highly excited states, the strength-function method is used. By using a version of the strength-function method one can directly calculate the reduced transition probabilities, spectroscopic factors, transition densities, cross sections and other nuclear characteristics without having to solve the relevant secular equations.

The quasiparticle-phonon interaction is responsible for the fragmentation of quasiparticle and collective motion and, thus, for the complication of the nuclear state structure with increasing excitation energy.

## § 3. RPA equation

We now obtain the RPA equation for energies $\omega_{Ki_1}$ as well as the wave function (2.6) of the one-phonon states. To describe the $K^{\pi}\neq 0^{+}$ state we use the following part of Hamiltonian (2.8), and (2.9):

$$H_{RPA} = \sum_{q\sigma} \varepsilon_q a^{+}_{q\sigma} a_{q\sigma} + H_v^K \quad . \tag{3.1}$$



We determine the average value (3.1) over state (2.6), and using the variational principle we get the following equations:

$$\varepsilon_{q_1q_2} g^{Ki_1}_{q_1q_2} - \omega_{Ki_1} w^{Ki_1}_{q_1q_2} - \sum_{n=1}^{n_{max}} \sum_{\lambda} f_n^{\lambda K}(q_1q_2)\Big[u^{(+)}_{q_1q_2} \sum_{\rho=\pm 1} (\kappa_0^{\lambda K} + \rho\kappa_1^{\lambda K}) D^{\lambda Ki_1}_{n\rho\tau} + v^{(-)}_{q_1q_2} G^{\lambda K} D^{\lambda Ki_1}_{ng\tau}\Big] = 0 \quad . \tag{3.2}$$

$$\varepsilon_{q_1q_2} w^{Ki_1}_{q_1q_2} - \omega_{Ki_1} g^{Ki_1}_{q_1q_2} - \sum_{n=1}^{n_{max}} \Big\{\sum_{\lambda} f_n^{\lambda K}(q_1q_2) v^{(+)}_{q_1q_2} G^{\lambda K} D^{\lambda Ki_1}_{nw\tau} + \sum_{L} \sum_{\rho=\pm} \Big[ \sum_{\lambda'} (\kappa_0^{\lambda' LK} + \rho\kappa_1^{\lambda' LK}) D^{\lambda' LKi_1}_{n\rho\tau} f_n^{\lambda' LK}(q_1q_2) u^{(-)}_{q_1q_2} \chi(q_1q_2) + (\kappa_{T0}^{LK} + \rho\kappa_{T1}^{LK}) u^{(-)}_{q_1q_2} \chi(q_1q_2) \Big( f_n^{L-1LK}(q_1q_2) D^{L+1LKi_1}_{n\rho\tau} + f_n^{L+1LK}(q_1q_2) D^{L-1LKi_1}_{n\rho\tau} \Big) \Big] \Big\} = 0 \quad . \tag{3.3}$$

Here,

$$\varepsilon_{q_1q_2} = \varepsilon_{q_1} + \varepsilon_{q_2}, \quad g^{Ki_1}_{q_1q_2} = \psi^{Ki_1}_{q_1q_2} + \phi^{Ki_1}_{q_1q_2} \text{ and } \quad w^{Ki_1}_{q_1q_2} = \psi^{Ki_1}_{q_1q_2} - \phi^{Ki_1}_{q_1q_2} \quad .$$

From eqs. (3.2) and (3.3) we obtain the functions $g^{Ki}_{q_1q_2}$ and $w^{Ki_1}_{q_1q_2}$ and then substitute them into the formulae for $D^{\lambda Ki_1}_{n\tau}$, $D^{\lambda Ki_1}_{ng\tau}$, $D^{\lambda Ki_1}_{nw\tau}$ and $D^{\lambda' LKi_1}_{n\tau}$. Taking into account that $\tau$=n, $\tau$=p and $\lambda'$=L-1, L, L+1 we derive the secular equality for the energies of the one-phonon states, as an equality to zero of a determinent of rank $12.n_{max}$. The use of a separable interaction of rank $n_{max}$ increase the determinate $n_{max}$ times, in compared with a simple separable interaction, $n_{max}$=1. If we neglect the spin-multipole terms with $\lambda'$=L, then the rank of the determinant is $10\cdot n_{max}$.

Most of our calculations[11),12)] have been performed with $n_{max}=1$ multipole interactions with the matrix elements (A.13) with the radial dependence $R^{\lambda}(r)=\partial V(r)/\partial r$, where V(r) is the central part of the Woods-Saxon potential.

The secular equation for describing the $0^+$ states with a simple ($n_{max}=1$) interaction was obtained in[8)] as an equality to zero of a determinant of rank 10. By including a separable interaction of rank $n_{max}$, the rank of the determinant for the energies $\omega_{20i_1}$ of one-phonon $0^+$ states is $4+6n_{max}$ .

Investigations[11)] of the $0^+$ states have shown the that the role of the pp interactions is essential. Upon increasing $G^{20}$ the low-lying poles of the RPA secular equations are change. As a result, the B(E2) value for an excitation of the first $I^{\pi}K_{i_1}=2^+0_1$ state and the energies of the $0_2^+$, $0_3^+$, $0_4^+$ etc states are decrease at $G^{20}=0.9\ \kappa_0^{20}$, in compared with $G^{20}=0$. Further, the structures of the $0^+$ states change. Inclusion of the pp interactions, on the whole, improves the description of the $0^+$ states.

Calculations of the superdeformed $0^+$ excited states in $^{238}U$ and $^{240}Pu$ are being performed with A. V. Sushkov. The calculated energies of the first excited $0^+$ states are close to the experimental ones, and the EO transition probabilities to fission isomers are in reasonably good agreement with the experimental data. The second and several others $0^+$ states in $^{238}U$ and $^{240}Pu$ are of the isovector type. Quadrupole pairing plays an important role in the description of the superdeformed states.



We now obtain RPA equations for several particular cases. First of all, we derive RPA equations for the multipole interaction with the following radial function:

$$R^{\lambda}(r_1r_2) = \frac{\partial V(r_1)}{\partial r_1}\cdot\frac{\partial V(r_2)}{\partial r_2} + \zeta\frac{1}{r_0^2}V(r_1)V(r_2) \quad .$$

This equation contains the surface- and density-dependent parts, where $\zeta$ is, new free parameter and $r_0=1.2$fm. We denote corresponding matrix elements (A.13) as $f^{\lambda K}(q_1q_2)$ and $f_2^{\lambda K}(q_1q_2)$ . The connection between the separable multipole interaction and the Skyrme force was investigated, [13)]. in which a separable interaction containing a density-depedent part, which is equivalent to the zero-range Skyrme force in framework of RPA was derived. The equivalent separable forces can be written in terms of the transition densities for the vibrational states.

We put the function $W^{KE}_{i_1i_2}$(2.12) in the form

$$W^{KE}_{i_1i_2} = \frac{1}{4}\sum_{\tau}\{\sum_{\rho=\pm1}(\kappa_0^{\lambda K}+\rho\kappa_1^{\lambda K})[D_{\tau}^{\lambda Ki_1}D_{\rho\tau}^{\lambda Ki_2}+ \zeta D_{2\tau}^{\lambda Ki_1}D_{2\rho\tau}^{\lambda Ki_2}] + G^{\lambda K}[D_{g\tau}^{\lambda Ki_1}D_{g\tau}^{\lambda Ki_2}+D_{w\tau}^{\lambda Ki_1}D_{w\tau}^{\lambda Ki_2} + \zeta(D_{2g\tau}^{\lambda Ki_1}D_{2g\tau}^{\lambda Ki_2}+D_{2w\tau}^{\lambda Ki_1}D_{2w\tau}^{\lambda Ki_2})]\} \quad , \qquad (3.4)$$

where $D_{2\tau}^{\lambda Ki}$, $D_{2g\tau}^{\lambda Ki}$ and $D_{2w\tau}^{\lambda Ki}$ have matrix element $f_2^{\lambda K}(q_1q_2)$ instead of $f^{\lambda K}(q_1q_2)$. In this case RPA eqs. (3.2) and (3.3) have the following form:

$$\varepsilon_{q_1q_2}g^{Ki_1}_{q_1q_2} - \omega_{Ki_1}w^{Ki_1} - \sum_{\rho=\pm1}(\kappa_0^{\lambda K}+\rho\kappa_1^{\lambda K})u^{(+)}_{q_1q_2}[f^{\lambda K}(q_1q_2)D_{\rho\tau}^{\lambda Ki_1}+$$

$$+ \zeta f_2^{\lambda K}(q_1q_2)D_{2\rho\tau}^{\lambda Ki_2}] - G^{\lambda K}v_{q_1q_2}^{(-)}[f^{\lambda K}(q_1q_2)D_{g\tau}^{\lambda Ki_1} +$$

$$+ \zeta f_2^{\lambda K}(q_1q_2)D_{2g\tau}^{\lambda Ki_1}] = 0 \qquad (3.5)$$

and

$$\varepsilon_{q_1q_2}w_{q_1q_2}^{Ki_1} - \omega_{Ki_1}g_{q_1q_2}^{Ki_1} - G^{\lambda K}v_{q_1q_2}^{(+)}[f^{\lambda K}(q_1q_2)D_{w\tau}^{\lambda Ki_1} +$$

$$+ \zeta f_2^{\lambda K}(q_1q_2)D_{2w\tau}^{\lambda Ki_1}] = 0. \qquad (3.6)$$

The rank of the secular equation determinant is equal to 12.

The calculation of $K^\pi=2^+$ states in $^{168}$Er using eqs. (3.5) and (3.6) shows that the effect of the density-dependent part of the interaction is small at $\zeta$=0.1-0.2. The quadrupole strength shifts from the first $2_1^+$ state to higher energy at $\zeta>0.4$ in contradiction with experimental data. This means that the role of the density-dependent separable interaction is not important for describing the vibrational states. The transition densities are most sensitive to density-dependent interactions.

The most collective low-lying vibrational states are the quadrupole and octupole ones. Any description of these states should be given in terms of the phonon operator (2.4), while taking the ph and pp multipole interactions into account. The pp interactions improve the description of the energies, the B(E$\lambda$) values as well as the structure of these states. If we consider the E1 transitions to $K^\pi=0^-$ or the $1^-$ states, they should be described in terms of the ph and pp octupole as well as the ph dipole interactions. For calculating the E4 or E5 transitions to the $K^\pi=2^+$ or $K^\pi=0^-,1^-,2^-,3^-$ states the wave functions of these



states should be described in terms of the ph and pp quadrupole and ph hexadecapole interactions or the ph and pp octupole and ph $\lambda$=5 interactions. The RPA equations for such cases have the following form:

$$\varepsilon_{q_1q_2}g_{q_1q_2}^{Ki_1} - \omega_{Ki_1}w_{q_1q_2}^{Ki_1} - f^{\lambda K}(q_1q_2)[u_{q_1q_2}^{(+)}\sum_{\rho=\pm1}(\kappa_0^{\lambda K} + \rho\kappa_1^{\lambda K})D_{\rho\tau}^{\lambda Ki_1} +$$

$$+ v_{q_1q_2}^{(-)}G^{\lambda K}D_{g\tau}^{\lambda Ki_1}] - f^{\lambda\pm2K}(q_1q_2)u_{q_1q_2}^{(+)}\sum_{\rho=\pm1}(\kappa_0^{\lambda\pm2K} + \rho\kappa_1^{\lambda\pm2K})D_{\rho\tau}^{\lambda\pm2Ki_1}$$

$$= 0 \ , \qquad (3.7)$$

$$\varepsilon_{q_1q_2}w_{q_1q_2}^{Ki_1} - \omega_{Ki_1}g_{q_1q_2}^{Ki_1} - G^{\lambda K}f^{\lambda K}(q_1q_2)v_{q_1q_2}^{(+)}D_{w\tau}^{\lambda Ki_1} = 0 \ . \qquad (3.8)$$

Calculations of the E1 transitions from the ground state to the $I^\pi=1^-$ states with K=0 and 1 in doubly-even well-deformed nuclei with dipole and octupole interactions have been performed[14]. The energies, $\omega_{3Ki_1}$, B(E3) values and the largest components of the wave functions are mainly determined by the octupole interaction. The isovector dipole interaction slightly influences the energies, B(E3) values and state structure, but strongly influences the B(E1) values.

For a description of the magnetic M$\lambda$ transitions from the ground to one-phonon states one should use the phonon operator (2.5) and take into account the multipole ph and pp as well as magnetic spin-multipole ph interactions. The RPA equations for such cases has the following form:

$$\varepsilon_{q_1q_2}g_{q_1q_2}^{Ki_1} - \omega_{Ki_1}w_{q_1q_2}^{Ki_1} - f^{\lambda K}(q_1q_2)u_{q_1q_2}^{(+)}\sum_{\rho=\pm}(\kappa_0^{\lambda K} + \rho\kappa_1^{\lambda K})D_{\rho\tau}^{\lambda Ki_1} -$$

$$- f^{\lambda K}(q_1q_2)v^{(-)}_{q_1q_2}G^{\lambda K}D^{\lambda Ki}_{g\tau} = 0 \qquad (3.9)$$

and

$$\varepsilon_{q_1q_2}w^{Ki_1}_{q_1q_2} - \omega_{Ki_1}g^{Ki_1}_{q_1q_2} - f^{\lambda K}(q_1q_2)v^{(+)}_{q_1q_2}G^{\lambda K}D^{\lambda Ki_1}_{w\tau} -$$
$$- \sum_{\rho=\pm}(\kappa_0^{L-1LK} + \rho\kappa_1^{L-1LK})D^{L-1LKi_1}_{\rho\tau}f^{L-1LK}(q_1q_2)u^{(-)}_{q_1q_2}\chi(q_1q_2) = 0 . \qquad (3.10)$$

An investigation of the magnetic M2 and M3 transition probabilities in deformed nuclei has shown[7] that the spin-multipole $\lambda'LK=L+1LK$ and tensor interactions weakly influence the structure of states with excitation energies below 6 MeV and M2 and M3 transition probabilities. Therefore, we do not take them into account in equations (3.9) and (3.10). The energies and structure of the vibrational states below 6 MeV in doubly-even deformed nuclei are mainly determined by multipole interactions. The inclusion of a spin-multipole magnetic interaction in addition to the multipole interactions leads to a shift of part of the M2 and M3 strength from the low-lying states to the region of giant isovector magnetic resonances. The spin parts of the M2 and M3 transition dominantes and the orbital part contribution to the B(M2) and B(M3) values equal (10-40)%.

We do not include in the QPNM Hamiltonian terms comprising the operators $(Q^+_{Ki_1\tau}Q^+_{Ki_2-\sigma} + Q_{Ki_2-\sigma}Q_{Ki_1\sigma})$ and

$B(q_1q_2;K\sigma)B(q_1q_2;K-\sigma)$ . For the roots of the RPA equation the following conditions must be fulfilled:



$$\langle Q_{Ki_1\sigma}|\sum_{q\sigma}\varepsilon_q\alpha^+_{q\sigma}\alpha_{q\sigma} - \sum_{i_2i_3\sigma'}W^K_{i_2i_3}Q^+_{Ki_2\sigma'}Q_{Ki_3\sigma'}|Q^+_{Ki_1\sigma}\rangle = \omega_{Ki_1} \qquad (3.10)$$

and

$$\langle Q_{Ki_1\sigma}|\{\sum_{q\sigma}\varepsilon_q\alpha^+_{q\sigma}\alpha_{q\sigma} - \sum_{i_2i_3}W^K_{i_2i_3}$$
$$\times\frac{1}{2}[Q^+_{Ki_2\sigma}Q^+_{Ki_3-\sigma} + Q_{Ki_3-\sigma}Q_{Ki_2\sigma}]\}|Q^+_{Ki_1\sigma}\rangle = 0 . \qquad (3.11)$$

If we take the Pauli priciple into account for the phonon operators, eq. (3.11) does not equal zero. The terms of Hamiltonian, consisting of operators $(Q^+_{Ki_1\sigma}Q^+_{Ki_2-\sigma} + Q_{Ki_2-\sigma}Q_{Ki_1\sigma})$ are used in the Multi-phonon Method[14] for describing the two-phonon state in deformed nuclei.

The role of the terms of the Hamiltonian containing the operators $B(q_1q_2;K\sigma)\cdot B(q_1q_2;K-\sigma)$ is investigated. Numerical estimations by perturbation theory show that their influence on the vibrational states in well-deformed nuclei is small. In nuclei of the transitional region is important these corrections.

The average number of quasiparticles in the ground states of doubly-even deformed nuclei has been calculated[15]. The ground state correlations increase with increasing collectivity of the first one-phonon states. In nuclei lying at the boundaries of the deformation regions, and especially in the transition nuclei, the number of quasiparticles increase up to 0.5, and, thus, RPA cannot be used. In well-deformed nuclei the average number of quasiparticles in the ground states is small. Inclusion of the pp interaction in addition to that of ph improves the applicability RPA.

It is possible to conclude that the RPA when it takes into account ph and pp interactions can be used to calculate the states in well-deformed nuclei in the regions $150<A<186$ ($90<N<112$, $60<Z<86$) and $A>232$. The one-phonon states can be used to forming the phonon basis of the QPNM.

## § 4. QPNM Equations

Our aim is to describe within the QPNM the low-lying, low-spin non-rotational states in well-deformed doubly-even nuclei.

We take into account the ph and pp simple multipole interactions. Usually, our wave function comprises of one- and two-phonon components. We investigated the contribution of the two-phonon components to the wave functions of the low-lying states. To investigate these contributions, since we take into account the three-phonon terms, the wave function of the excited state has following form:

$$\Psi_\nu(K_0^{\pi_0}\sigma_0) = \{\sum_{i_0} R^\nu_{i_0} Q^+_{g_0\sigma_0} + \sum_{\substack{g_1g_2\\ \sigma_1\sigma_2}} \frac{(1+\delta_{g_1g_2})^{1/2}}{2[1+\delta_{K_00}(1-\delta_{\mu_10})]^{1/2}} \times \delta_{\sigma_1\mu_1+\sigma_2\mu_2,\sigma_0K_0} P^\nu_{g_1g_2} Q^+_{g_1\sigma_1} Q^+_{g_2\sigma_2} + \sum_{\substack{g_1g_2g_3\\ \sigma_1\sigma_2\sigma_3}} b_{g_1g_2g_3} \delta_{\sigma_1\mu_1+\sigma_2\mu_2+\sigma_3\mu_3,\sigma_0K_0} F^\nu_{g_1g_2g_3} Q^+_{g_1\sigma_1} Q^+_{g_2\sigma_2} Q^+_{g_3\sigma_3}\}\Psi_0 . \quad (4.1)$$



Here $g=\lambda\mu i$, $\mu_0=K$ and $b_{g_1g_2g_3}$ is a numerical factor; $\nu=1,2,3,\cdots$ is the number of $K_0^{\pi_0}$ state. To take the Pauli principle into account regarding the two- and three-phonon terms of the wave function (4.1), we introduce the function

$$k_0(g_1,g_1^1|g_1,g_2) = (1+\delta_{g_1g_2})^{-1} \sum_{\sigma_1\sigma_2} \delta_{\sigma_1\mu_1+\sigma_2\mu_2,\sigma_0K_0} \times \langle Q_{g_2\sigma_2}[[Q_{g_1\sigma_1}, Q^+_{g_1\sigma_1}], Q^+_{g_2\sigma_2}]\rangle , \quad (4.2)$$

$$\kappa^{K_0}(g_1,g_2) \equiv \kappa^{K_0}(g_2,g_1|g_1g_2) . \quad (4.3)$$

It's explicit form is given in[3]. The normatization condition of the wave function (4.1) in the diagonal of the $k^{K_0}$ approximation has the form

$$\sum_{i_0}(R^{\nu^2}_{i_0})^2 + \sum_{g_1>g_2}(P^\nu_{g_1g_2})^2[1+k^{K_0}(g_1g_2)] + \sum_{g_1>g_2>g_3}(F^\nu_{g_1g_2g_3})^2 \{1+\tfrac{1}{2}[k^{K_0\pm\mu_1}(g_2,g_3)+k^{K_0\pm\mu_2}(g_1,g_3)+k^{K_0\pm\mu_3}(g_1,g_2)]\} = 1 . \quad (4.4)$$

We now find the average value of

$$H^{\lambda_0K_0}_{QPNM} = \sum_{q\sigma_0}\varepsilon_q\alpha^+_{q\sigma}\alpha_{q\sigma_0} - \sum_{i_1i_2\sigma} W^{\lambda_0K_0}_{i_1i_2} Q^+_{\lambda_0K_0i_1\sigma}Q_{\lambda_0K_0i_2\sigma} + H^{\lambda_0K_0}_{vq}$$

over the state (4.1):

$$(\Psi^*_\nu(K_0^{\pi_0}\sigma_0)H^{\lambda_0K_0}_{QPNM}\Psi_\nu(K_0^{\pi_0}\sigma_0)) = \sum_{i_0}\omega_{g_0}(R^\nu_{i_0})^2 + \sum_{g_1<g_2}(P^\nu_{g_1g_2})^2[\omega_{g_1}+\omega_{g_2}+\Delta\omega(g_1g_2)][1+k^{K_0}(g_1g_2)]$$

$$+ \sum_{g_1<g_2<g_3} (F^{\nu}_{g_1g_2g_3})^2[(\omega_{g_1}+\Delta\omega_{g_1})(1+k^{K_0\pm\mu_1}(g_2,g_3)) +$$

$$(\omega_2+\Delta\omega_2)(1+k^{K_0\pm\mu_2}(g_1,g_3)) + (\omega_3+\Delta\omega_3)(1+k^{K_0\pm\mu_3}(g_1,g_2))] -$$

$$- \sum_{i_0 g_1 g_2} \frac{(1+\delta_{g_1g_2})^{1/2}}{[1+\delta_{K_0,0}(1-\delta_{\mu_1,0})]^{1/2}} R^{\nu}_{i_0} P^{\nu}_{g_1g_2} U^{g}_{g_1g_2}(1+k^{K_0}(g_1,g_2))$$

$$- \sum_{g_1g_2} \sum_{g_1'g_2'g_3'} \frac{(1+\delta_{g_1g_2})^{1/2}}{[1+\delta_{K_0,0}(1-\delta_{\mu_1,0})]^{1/2}} b_{g_1'g_2'g_3'} P^{\nu}_{g_1g_2} F^{\nu}_{g_1'g_2'g_3'} U^{g_1g_2}_{g_1'g_2'g_3'} , \tag{4.5}$$

where

$$\Delta\omega(g_1g_2) = -\sum_{i_1}[k^{K_0}(g_2,g_1'|g_1,g_2)W^{\lambda_1\mu_1}_{i_1i_1'} + k^{K_0}(g_2',g_1|g_1g_2)W^{\lambda_2\mu_2}_{i_1'i_1}] , \tag{4.6}$$

Here, $g_1' = \lambda_1\mu_1 i'$, $g_2' = \lambda_2\mu_2 i'$ and $W^{\lambda\mu}_{ii'}$ is determined by using (2.12) and $H^{\lambda_0K_0}_{vg}$ by using (2.20);

$$\Delta\omega_{g_1} = -\sum_{i'} W^{\lambda_1\mu_1}_{i_1i_1'}[k^{K_0\pm\mu_3}(g_2,g_1'|g_1g_2) + k^{K_0\pm\mu_2}(g_3,g_1';\, g_1,g_2)], \tag{4.7}$$

$$U^{g_0}_{g_1g_2}(1+k^{K_0}(g_1,g_2)) = -\frac{1}{2}\sum_{\sigma_1\sigma_2}\delta_{\sigma_1\mu_1+\sigma_2\mu_2,\sigma_0K_0}$$

$$\times [\langle Q_{g_0\sigma_0}\tilde{H}^{K_0}_{vq}Q^{+}_{g_1\sigma_1}Q^{+}_{g_2\sigma_2}\rangle + \text{h.c.}] , \tag{4.8}$$

$$U^{g_1g_2}_{g_1'g_2'g_3'} = \delta_{g_1g_1'}U^{g_2}_{g_2'g_3'}(1+k^{K_2}(g_2',g_3')) +$$

$$+ \delta_{g_1g_2'}U^{g_2}_{g_1'g_3'}(1+k^{K_2}(g_1',g_3')) + \cdots . \tag{4.9}$$

Using the variational principle in the form

$$\delta\{(\Psi^{*}_{\nu}(K^{\pi}_{0}\sigma_0)H^{\lambda_0K_0}_{QPNM}\Psi_{\nu}(K^{\pi}_{0}\sigma_0)) - E_{\nu}[(\Psi^{*}_{\nu}(K^{\pi}_{0}\sigma_0)\Psi_{\nu}(K^{\pi}_{0}\sigma_0) - 1]] = 0 ,$$



(4.10)

we obtain a set of three equations for the energies, $E_{\nu}$, and the wave function (4.1). The first one is

$$(\omega_{g_0} - E_{\nu})R^{\nu}_{i_0} - \sum_{g_1>g_2}\frac{(1+\delta_{g_1g_2})^{-1/2}}{[1+\delta_{K_0,0}(1-\delta_{\mu_1,0})]^{1/2}} P^{\nu}_{g_1g_2}$$

$$\times U^{g_0}_{g_1g_2}(1+k^{K_0}(g_1,g_2)) = 0 . \tag{4.11}$$

If we find $F^{\nu}_{g_1g_2g_3}$ from third equation and $R^{\nu}_{i_0}$ from (4.11) and substitute it into second one we obtain the following equation:

$$\sum_{g_1>g_2} P^{\nu}_{g_1g_2}\{[\omega_{g_1}+\omega_{g_2}+\Delta\omega(g_1g_2) - E_{\nu}](1+k^{K_0}(g_1,g_2)\delta_{g_1,g_1^{\circ}}\delta_{g_2,g_2^{\circ}} -$$

$$- \sum_{i_0}\frac{(1+\delta_{g_1g_2})^{-1/2}}{[1+\delta_{K_0,0}(1-\delta_{\mu_1,0})]^{1/2}}\frac{(1+\delta_{g_1^{\circ}g_2^{\circ}})^{-1/2}}{[1+\delta_{K_0,0}(1-\delta_{\mu_1^{\circ},0})]^{1/2}}$$

$$\frac{U^{g_0}_{g_1^{\circ}g_2^{\circ}}U^{g_0}_{g_1g_2}(1+k^{K_0}(g_1^{\circ},g_2^{\circ}))(1+k^{K_0}(g_1,g_2))}{\omega_{g_0}-E_{\nu}} -$$

$$- \sum_{g_1'g_2'g_3'}\frac{(1+\delta_{g_1g_2})^{-1/2}}{[1+\delta_{K_0,0}(1-\delta_{\mu_1,0})]^{1/2}}\frac{(1+\delta_{g_1^{\circ}g_2^{\circ}})^{-1/2}}{[1+\delta_{K_0,0}(1-\delta_{\mu_1^{\circ},0})]^{1/2}} b^2_{g_1'g_2'g_3'}$$

$$\times \frac{U^{g_1^{\circ}g_2^{\circ}}_{g_1'g_2'g_3'}\; U^{g_1g_2}_{g_1'g_2'g_3'}}{J(g_1',g_2',g_3') - E\nu[1+k^{K_0\pm\mu_1'}(g_2'g_3')+k^{K_0\pm\mu_2'}(g_1'g_3')+k^{K_0+\mu_3'}(g_1'g_3')]}\} = 0, \tag{4.12}$$

$$J(g_1',g_2',g_3') = (\omega_{g_1'}+\Delta\omega_{g_1'})(1+k^{K_0\pm\mu_1'}(g_2'g_3')) + (\omega_{g_2'}+\Delta\omega_{g_2'})(1+k^{K_0\pm\mu_2'}(g_1'g_3'))$$

$$+(\omega_{g_1'}+\Delta\omega_{g_3'})(1+k^{K_0\pm\mu'_3}(g_1'g_2'))\ .$$

For description of the fragnentation of the two-phonon state must be a diagonalised set of eq.(4.12) in the space of the two-phonon states. In this case we cannot neglect the non-diagonal terms in eq.(4.12), due to the appearance of extraneous solutions (6). We investigate the influence of the three-phonon terms in the wave function (4.1) concerning the contribution of the two-phonon configuration to the low-lying states. We can therefore neglect the non-diagonal terms in eq.(4.12), and take the terms with $g_1=g_1^0$ and $g_2=g_2^0$ into account. In this approximation the influence of the three-phonon states is reduced to a shift of the two-phonon poles, which we denote as $\Delta(g_1g_2)\ (1+k^{K_0}(g_1g_2))$. This shift is positive and differs from zero if the Pauli principle is not taken into account in the three-phonon terms of the wave function (4.1).

By taking into account eqs. (4.11) and (4.12) in the diagonal approximation we obtain the following equation:

$$[\omega_{g_1}+\omega_{g_2}+\Delta\omega(g_1g_2)-\Delta(g_1g_2)-E_\nu]P^\nu_{g_1g_2}-$$

$$-\sum_{i_0}\frac{(1+\delta_{g_1g_2})^{-1/2}}{[1+\delta_{K_0,0}(1-\delta_{\mu_i0})]^{1/2}}U^{g_0}_{g_1g_2}R^\nu_{i_0}=0\ . \qquad (4.13)$$

Hence, we get the secular equation

$$\det||(\omega_{g_0}-E_\nu)\delta_{i_0,i'_0}-\sum_{g_1>g_2}\frac{1+k^{K_0}(g_1,g_2)}{1+\delta_{g_1g_2}}$$



$$\times\frac{U^{g_0}_{g_1g_2}U^{g'_0}_{g_1g_2}}{\omega_{g_1}+\omega_{g_2}+\Delta\omega(g_1,g_2)-\Delta(g_1,g_2)-E_\nu}||=0\ . \qquad (4.14)$$

This secular equation differs from the one used in [3),11),12)] due to an additional shift $\Delta(g_1g_2)$. Now we can neglect the three-phonon terms in (4.1) and (4.4) and from (4.4), (4.11) and (4.13) find $R^\nu_{i_0}$ and $P^\nu_{g_1g_2}$ for each value of $E_\nu$ .

The rank of the determinant (4.14) equals the number of one-phonon terms in the wave function (4.1). Inclusion of the Pauli principle in the two-phonon terms (4.1) generates in (4.14) a factor of $(1+k^{K_0}(g_1g_2))$ and a shift of $\Delta\omega(g_1g_2)$ of the two-phonon pole. The three-phonon terms in (4.1) lead to an additional shift of $\Delta(g_1g_2)$.

The form of equations (4.11) and (4.13) and the rank of the determinant (4.14) are independent of what ph and pp multipole and spin-multipole interactions are taken into account, and are independent of the rank, $n_{max}$, of the separable interactions. Equations (4.11), (4.13) and (4.14) coincide in form with the equations in previous papers[1),3)] in which only the ph multipole interactions were taken into account. All of the complications caused by the form of the interactions were concentrated on the RPA equations. This means that calculations using the QPNM can be made with any complex interactions preseted in a separable form.

## § 5. General Properties of Low-lying Vibrational States in Deformed Nuclei

The energies and structure of low-lying non-rotational states of doubly-even deformed nuclei are mainly determined by single-particle energies and wave functions of the Woods-Saxon potential, monopole pairing and isoscalar ph multipole interactions. Inclusion of the pp multipole interactions improves the description, especially $0^+$ states. By including the isovector ph spin-multipole magnetic interactions, the energies, $B(E\lambda)$ values and largest two-quasiparticle components of the wave functions of the one-phonon states change slightly.

Regarding the general properties of the vibrational states of doubly-even deformed nuclei with an excitation energy of up to 6 MeV, we can make the following conclusions: 1) For states with energies up to (2.5-3.0) MeV the anharmonicity of nuclear vibrations is small. This indicates that the contribution of a one-phonon component to the normalization of the wave function exceeds 90%. The states with fixed $K^\pi$ values are mainly of the one-phonon type corresponding to the first, second, etc., solutions of the RPA secular equation. A small anharmonicity involves the essential feature of the vibrational states in deformed nuclei, compared with spherical open-shell nuclei. 2) At excitation energies higher than 2.5 MeV, in the energy region of two-phonon poles, there is fragmentation of one-phonon states.

The small anharmonicity of the low-lying vibrational states of deformed nuclei is due to two factors: first, the numerical values of the function $U^{g_0}_{g_1 g_2}$ are small; second, due to a shift of the two-phonon poles their energies become larger than 2.5 MeV.

The function $U^{g_0}_{g_1 g_2}$ is a non-coherent sum of many terms containing the matrix elements $f^{\lambda K}(q_1 q_2)$, forward $\psi^{Ki}_{q_1 q_2}$ and backward $\phi^{Ki}_{q_1 q_2}$ RPA amplitudes. In deformed nuclei, terms with different signs suppress each other. As a result, the numerical values of the function $U^{g_0}_{g_1 g_2}$ range from 0.01 to 0.20 MeV, and only in some cases, $U^{g_0}_{g_1 g_2}$, takes valus larger than 0.2 MeV. In spherical nuclei the largest terms in $U^{\lambda i}_{\lambda_1 i_1, \lambda_2 i_2}$ have the same sign for the first roots of secular equations the numerical values of $U^{\lambda i}_{\lambda_1 i_1 \lambda_2 i_2}$ are one or two orders of magnitude larger in the open-shell spherical nuclei compared with the deformed ones.

To investigate the influence of a density-dependent interaction on function $U^{g_0}_{g_1 g_2}$ we estimate the $U^{g_0}_{g_1 g_2}$ values using the RPA wave functions calculated with interaction $W^{KE}_{i_1 i_2}$ in the form presented in (3.4). We take into account the surface- and density- dependent separable interaction. According our calculation the numerical value, $U^{441}_{221,221}$ in $^{168}$Er increases by 10% at $\zeta$=0.1, compared with $\zeta$=0. This means that the density-dependent separable interaction only weakly influences the low-lying vibrational states in well-deformed nuclei.

The terms of Hamiltonians containing operators

$$(Q^+_{Ki_1 0} Q^+_{Ki_2 -0} + Q_{Ki_2 -0} Q_{Ki_1 0})$$

do not change the poles of a secular eq.(4.40), but do affect the function $U^{g_0}_{g_1 g_2}$. According to our estimation this influence is small.

The calculated eneries of the low-lying states described by wave functions (4.1) are slightly smaller compared with RPA calculations. The wave function has a dominating one-phonon component and several small two-phonon components. The contribution of a one-phonon component to a wave function normalization condition is more than 80% for states with energy up to 2 MeV. Such structured excited states are exemplified by five $K^{\pi}=2^{+}$, four $K^{\pi}=0^{+}$ and four $K^{\pi}=3^{-}$ states in $^{168}$Er.

Let's consider the contribution of the two-phonon components to the wave function of low-lying states. According our calculation, these contributions to the normalization condition is less than 10% for most states with energies less than 2 MeV. If the contribution of the two-phonon components to the wave function normalization exceeds 50%, this state is determined as a two-phonon state. Based on QPNM calculations of the energy centroids of two-phonon states, it has been concluded in[17),18)] that collective two-phonon states cannot excist in deformed nuclei. According to the Multiphonon Method[14)] sdg IBM calculations[19)] and the Selfconsistent-Collective-Coordinate Method[20)] the first $K^{\pi}_{\nu}=4^{+}_{1}$ state in $^{168}$Er should be a collective two-phonon state. A new QPNM calculation of the first $K^{\pi}_{\nu}=2^{+}_{1}$, $3^{+}_{1}$ and $4^{+}_{1}$ states in $^{168}$Er has been performed [21)]. It has been shown that the contribution of a double gamma vibrational component to the normalization of the wave function of the $4^{+}_{1}$ state can be achived up to 20% by fitting constants $\kappa_0^{22}$ and $G^{22}$.

A new experimental investigation[22)] has established a large double gamma vibrational component in the first $K^{\pi}_{\nu}=4^{+}_{1}$ state in



$^{168}$Er. Accordingly a multi-Coulomb-excitation experiment on $^{164}$Dy [23)] concluded that a two-phonon gamma vibrational state with large collectivity does not exist in the energy region between 1.5 and 2.0 MeV in $^{164}$Dy. We can state that the available experimental data do not contradict the conclusion concerning the absence of collective two-phonon states in deformed nuclei. Additional experiments are needed to establish the contribution of two-phonon configurations to the wave functions of low-lying states in deformed nuclei.

The collectivity of the first quadrupole and octupole states and its absence in higher-lying states up to giant resonances undelie phenomenological models including IBM. In ref.[24)], for the first time it was experimentally shown that the most collective is not the first, but higher lying states $K^{\pi}_{\nu}=3^{-}_{4}$ in $^{168}$Er the largest part of the E2 strength is concentrated in $^{172}$Yb, not in the first and second quadrupole states, but, within the 2-3 MeV energy interval. A non-standart distribution of the E3 strength in several deformed nuclei has been predicted in [12),25)].

According to RPA calculation there exist states within the energy range 3÷4 MeV which may be strongly excited by γ-transitions from a ground states with B(E2)>0.5 s.p.u. and B(E3)>0.8 s.p.u.. It has been experimentally observed[26)] in $^{150}$Nd the octupole collective states with B(E3)>0.6 s.p.u. with energies up to 3.4 MeV.

At excitation energies of 2.5-4.0 MeV there are many two-phonon poles of the secular eq. (4.14). At these energies there

exist states with large one- or two-phonon components in their wave functions. The one-phonon strength is fragmented over two or three states. In some cases mixing of two one-phonon states is observed. Fragmentation of the one-phonon strength proceeds in a different way for various states; it is stronger for collective states with large B(Eλ) values for transitions to the ground state. It is possible to experimentally observed the electric and magnetic collective states within this energy region.

The discovery of the orbital M1 transitions in deformed nuclei[27] creates interest in searching for collective $1^+$ states. As a result, many collective $1^+$ states have been observed in deformed nuclei. According to QPNM calculations these $1^+$ states in $^{164}$Dy are weakly fragmented. The concentrations of the M2 and M3 strengths in narrow energy intervals below 4 MeV in $^{164}$Dy and $^{168}$Er have been predicted in[7].

Experimental investigation concerning the Eλ and Mλ strength distributions within the energy range (2÷6) MeV is very useful for understanding nuclear structure.

Collective vibrational states are not limited by quadrupole and octupole states. It has been shown[28] that in some cases multipole interactions with λ=5-9 lead to a mixing of two-quasiproton and two-quasineutron configurations in those states with large K. Perhaps the necessity of including high-multipolarity interactions is related to the inclusion[29] of high-multipolarity deformations with λ=5, 6 and 7, which have been found to be important in the regions of barium and radium.



Quasiparticle-phonon interactions lead to a rather strong fragmentation of the one-phonon states with energies larger than 4 MeV. This makes experimental detection of the Eλ and Mλ strength concentration to the discrete levels difficult at excitation energies of (4÷6) MeV.

## § 6. Conclusion

The investigation concerning the framwork of the QPNM for the vibrational states in doubly-even well-deformed nuclei has shown that they can be treated as small-amplitude collective states in regions $150<A<186$ and $A>230$. In the QPNM common descriptions of the collective, weakly collective and two-quasiparicle states has been obtained. All non-rotational states up to fixed energy in well-deformed nuclei can be calculated using the QPNM.

The basic ideas found in the investigation of the vibrational states in the doubly-even well deformed nuclei with an excitation energy up to 6 MeV were as follows:

1) One-phonon states with energies below 2.5 MeV are slighty fragmented due to quasiparticle-phonon interactions. The wave function of those states have a dominant one-phonon component. Many states at excitation energies in 2.5-3.5 MeV range also have a large one-phonon component of their wave function.
2) The many collective states up to 4 MeV have B(Eλ) and B(Mλ) values and can be observed experimentally.

3) The quasiparticle-phonon interactions lead to a fragmentation of the one-phonon states with energies greater than 4 MeV. This makes their experimental observation difficult.

The non-standart behavior of the several most collective states, the collective vibrational states in the (1÷4)MeV energy range and the necessity for a description of the γ-transitions between excited states shows that should not divided in the well deformed nuclei the non-rotational states on the collective and non-collective ones. Extraction of a collective subspace led to a limitation of all non-rotational states. The common description of the collective, weakly collective and two-quasiparticle states allowed us to calculate the γ-transitions probabilities between all nuclear states. In many cases the Coriolis interaction should be taken into account in a description of the γ-transitions between rotational bands.

The QPNM can serve as a basis for calculating the energies and wave functions of the excited states in well-deformed nuclei.

It is reasonable to expect that an experimental study of the excited states of well-deformed nuclei within the 2÷4 MeV energy range will be carried out at a new generation of accelearators and detectors with high energy resolution.

In conclusion I would like to thank A. V. Sushkov and N. Yu. Shirikova for their joint investigations, some results have been presented in this paper. This work was party undertaken during my stay at INS of the University of Tokyo. I am very thankful to professor B. Imanishi and physicists of the Theory Department of INS for his hospitality. I also acknowledge the financial support of the Reserarch fellowship of Ministry of Education, Science and Culture.

## Appendix

The operators, matrix elements and functions entering into the QPNM Hamiltonian are

$$A^+(q_1q_2;K\sigma) =$$

$$\tilde{A}^+(q_1q_2;K\sigma) = \sum_{\sigma'}{}' \delta_{\sigma'(K_1-K_2),\sigma K}\sigma' \alpha^+_{q_1\sigma'}\alpha^+_{q_2-\sigma'}, \quad \text{if } |K_1-K_2| = K$$
$$\bar{A}^+(q_1q_2;K\sigma) = \sum_{\sigma'}{}' \delta_{\sigma'(K_1+K_2),\sigma K}\alpha^+_{q_2\sigma'}\alpha^+_{q_1\sigma'}, \quad \text{if } K_1+K_2 = K; \tag{A.1}$$

$$A^+(q_1q_2;K\sigma) =$$

$$\tilde{a}^+(q_1q_2;K\sigma) = \sum_{\sigma'}{}' \delta_{\sigma'(K_1-K_2),\sigma K}\alpha^+_{q_1\sigma'}\alpha^+_{q_2-\sigma'}, \quad \text{if } |K_1-K_2| = K$$
$$\bar{a}^+(q_1q_2;K\sigma) = \sum_{\sigma'}{}' \delta_{\sigma'(K_1+K_2),\sigma K}\sigma' \alpha^+_{q_2\sigma'}\alpha^+_{q_1\sigma'}, \quad \text{if } K_1+K_2 = K; \tag{A.2}$$

$$\tilde{a}^+(q_1q_2;K\sigma) = \sigma\chi(q_1q_2)\tilde{A}^+(q_1q_2;K\sigma),$$
$$\bar{a}^+(q_1q_2;K\sigma) = \sigma\bar{A}^+(q_1q_2;K\sigma). \tag{A.3}$$

$$\chi(q_1q_2) = 1, \quad \chi(q_1q_2) = -1, \quad \chi^2(q_1q_2) = 1,$$

$$\chi(q_1q_2)a^+(q_1q_2;K\sigma) = a^+(q_1q_2;K\sigma),$$

$$\chi(q_2q_1)a^+(q_1q_2;K\sigma) = -a^+(q_1q_2K\sigma) = a^+(q_2q_1;K\sigma);$$

$$\delta_{\sigma_1K_1+\sigma_2K_2,\sigma K} = \begin{cases} 1, & \text{if } \sigma_1K_1 + \sigma_2K_2 = \sigma K, \\ 0, & \text{if } \sigma_1K_1 + \sigma_2K_2 \neq \sigma K, \end{cases}$$

with all $K>0$, $\sigma = \pm 1$.



After simple transformation we express the operator $\tilde{A}(q_1q_2;K\sigma)$, $\bar{A}(q_1q_2;K\sigma)$ and $a(q_1q_2;K\sigma)$ through the phonon operators (2.5)

$$\tilde{A}^+(q_1q_2;K\sigma) = \frac{1-i\sigma}{\sqrt{2}} \sum_{i_0} [\psi^{Ki_0}_{q_1q_2} Q^+_{Ki_0\sigma} + \phi^{Ki_0}_{q_1q_2} Q^+_{Ki_0-\sigma}],$$

$$\bar{A}^+(q_1q_2;K\sigma) = \frac{1-i\sigma}{\sqrt{2}} \chi(q_1q_2) \sum_{i_0} [\psi^{Ki_0}_{q_1q_2} Q^+_{Ki_0\sigma} + \phi^{Ki_0}_{q_1q_2} Q^+_{Ki_0-\sigma}], \tag{A.4}$$

$$a^+(q_1q_2;K\sigma) = \frac{\sigma-i}{\sqrt{2}} \chi(q_1q_2) \sum_{i_0} [\psi^{Ki_0}_{q_1q_2} Q^+_{Ki_0\sigma} + \phi^{Ki_0}_{q_1q_2} Q^+_{Ki_0-\sigma}].$$

$$B(q_1q_2;K\sigma) = \begin{cases} \sum_{\sigma'}{}' \delta_{\sigma'(K_1-K_2),\sigma K}\alpha^+_{q_1\sigma'}\alpha_{q_2\sigma'}, & \text{if } |K_1-K_2| = K; \\ \sum_{\sigma'}{}' \delta_{\sigma'(K_1+K_2),\sigma K}\sigma'\alpha^+_{q_1\sigma'}\alpha_{q_2-\sigma'}, & \text{if } K_1+K_2 = K. \end{cases} \tag{A.5}$$

$$B(q_1q_2;K\sigma) = \begin{cases} \sum_{\sigma'}{}' \delta_{\sigma'(K_1-K_2),\sigma K}\sigma'\alpha^+_{q_1\sigma'}\alpha_{q_2\sigma'}, & \text{if } |K_1-K_2| = K, \\ \sum_{\sigma'}{}' \delta_{\sigma'(K_1+K_2),\sigma K}\alpha^+_{q_1\sigma'}\alpha_{q_2-\sigma'}, & \text{if } K_1+K_2 = K. \end{cases} \tag{A.6}$$

$$D^{\lambda Ki}_{n\tau} = \sum^{\tau}_{q_1q_2} f^{\lambda K}_n(q_1q_2) u^{(+)}_{q_1q_2} g^{Ki}_{q_1q_2}, \tag{A.7}$$

$$D^{\lambda Ki}_{ng\tau} = \sum^{\tau}_{q_1q_2} f^{\lambda K}_n(q_1q_2) v^{(-)}_{q_1q_2} g^{Ki}_{q_1q_2}, \tag{A.8}$$

$$D^{\lambda Ki}_{nw\tau} = \sum^{\tau}_{q_1q_2} f^{\lambda K}_n(q_1q_2) v^{(+)}_{q_1q_2} w^{Ki}_{q_1q_2}, \tag{A.9}$$

$$D^{\lambda'LKi}_{n\tau} = \sum^{\tau}_{q_1q_2} f^{\lambda'LK}_n(q_1q_2) u^{(-)}_{q_1q_2} w^{Ki}_{q_1q_2}\chi(q_1q_2), \tag{A.10}$$

where

$$g^{Ki}_{q_1q_2} = \psi^{Ki}_{q_1q_2} + \phi^{Ki}_{q_1q_2}, \quad w^{Ki}_{q_1q_2} = \psi^{Ki}_{q_1q_2} - \phi^{Ki}_{q_1q_2}.$$

The matrix elements of the multipole and spin-multipole operators are expressed through

$$f_n^{\lambda K}(q_1q_2) = \langle q_1|R_n^{\lambda}(\tau)Y_{\lambda K}(\theta\phi)|q_2\rangle = \tilde{f}^{\lambda K}(q_1q_2) + \chi(q_1q_2)\bar{f}^{\lambda K}(q_1q_2)$$

$$f^{\lambda K}(1_1q_2) = \begin{cases} \tilde{f}^{\lambda K}(q_1q_2), & \text{if } |K_1 - K_2| = K; \\ \bar{f}^{\lambda K}(q_1q_2)\chi(q_1q_2). & \text{if } K_1 + K_2 = K: \end{cases} \quad (A.11)$$

$$f^{L-1LK}(q_1q_2) = \langle q_1|R_n^{\lambda}(r)\{\sigma Y_{L-1}(\theta\phi)\}_{LK}|q_2\rangle \ . \quad (A.12)$$

$$f^{\lambda K}(q_1q_2) = \langle q_1|\frac{\partial V(r)}{\partial r} Y_{\lambda K}(\theta\phi)|q_2\rangle \ . \quad (A.13)$$



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