Fermilab Library
RC
NAM
0 1160 0013635 2
INP-1585-PL
JUL 0 1992

Report INP No 1585/PL

MICROSCOPIC GROUP THEORETICAL FORMULATION OF NUCLEAR COLLECTIVE DYNAMICS

Marcin Cerkaski * and Igor N. Mikhailov **

ABSTRACT

Assuming an $O(N-1)$ symmetry for the interaction term in the N-body Hamiltonian we find a closed subsystem of equations describing the collective motion in a classical way. When studying, in the group geometric way, the mutual corespondency of $O(N-1)$ invariant approach with the $Sp(6,R)$ collective model we find that the nucleons move along trajectories determined by an effective one-body time-dependent harmonic potential being a function of the collective variables. The relation between the equations for the collective motion and the system of equations found elsewhere for the second order moments of the Wigner distribution function is discussed. A class of stationary solutions to the collective equations of motion leads to the cranking model with the selfconsistency relations depending on the $O(N-1)$ scalar part of the potential.

* permanent address: Henryk Niewodniczański Institute of Nuclear Physics, Department of Theoretical Physics, Ul. Radzikowskiego 152, 31-342, Kraków, Poland

** permanent address: Joint Institute for Nuclear Research, Laboratory of Theoretical Physics, Head Post Office, P.O.B.79, Moscow, USSR.

1. Introduction

The investigation of nuclear collective motion, which is currently a very important field of research, has been approached along several different lines.

The first approach springs from the time-dependent Hartree-Fock theory (TDHF) and its simplified versions in which a selfconsistent field is approximated by the time dependent (cranked) mean field. In particular the cranking model [1,2] has been successfully applied to describe the yrast states of nuclei.

Important mathematical developments have made it possible to arrive in a fully microscopic way at the unified Bohr and Mottelson Hamiltonian (Ref.[3],V.II). An elegant theory of generalized hyperspherical wave functions (GHSF) [4] has been formulated using the original set of many-body coordinates [5-7].

Other microscopic approaches based on the concept of dynamical groups G^{dyn} have been very intensively developed [9-13]. At the top of the considered group-dynamical models there is the $Sp(6,R)$ model introduced by Rosensteel and Rowe [8-10]. In the work by Vanagas *et al* [11,12] the formulation of the nuclear dynamics is approached within the context of an exact or approximate symmetry of the many-body Hamiltonian. In this work the $O(N-1)$ invariant model appears as a model for restricted dynamics. The geometrical approach to this model has been developed in the papers by Kramer [13].

During the last two decades important progress has been achieved using the concept of nuclear fluid, whose dynamics could be described in terms of the Wigner distribution function [14] or of quantities related to it [15-19]. This approach is closely related to the TDHF. Due to these investigations the nuclear hydrodynamic model has been transformed into the Fermi-liquid droplet model (FLDM) by taking into account the distortions of the nuclear Fermi surface accompanying the collective flow.

All the approaches have their merits and their drawbacks. The group– theoretical formulation is very transparent mathematically, which is very important in formulating the quantum theory of nuclei free from *ad hoc* assumptions typical of the TDHF approach. The latter approach however leaves much room for using physical intuition and is, in fact, quite effective in digesting the wealth of recent experimental data.

By now it has become clear that the direct application of either of the approaches demands great computing power to obtain physical predictions. The FLDM meets with difficulties when the restrictions arising from the antisymmetry of the wave–functions become important. The algebraic approach demands further development, especially in studying the collective motion which is related to the evolution of the higher moments of matter distribution.

The aim of this paper is to study the relations between the fully microscopic theory [9–13,20–23] built on the group $Sp(6,R)\times O(N-1)$ and theories based on the concept of the time–dependent mean field.

In Section II the Hamiltonian and the equations of motion are presented in terms of Zickendraht–Dzyublik–Filippov coordinates. In essence, this section has an introductory character summarizing the formalism needed for the subsequent parts. However, the analysis of the phase–space corresponding to this parameterization and of the Poisson bracket structure of observables is given here in a more complete manner than elsewhere. This allows us to write down a complete system classical of dynamical equations.

In Section III the classical equations of motion for the $O(N-1)$ invariant model are examined. Considering the geometric formulation of the dynamics we show that the evolution of the system is governed by the one–body time–dependent harmonic field. The correspondence is established between the collective motion of a nucleus and that of a classical liquid drop.

In Section IV the group–theoretical structure of the collective space is studied and the relation between the collective $Sp(6,R)$ and intrinsic $SO(N-1)$ invariants of the motion are discussed.

In Section V the $Sp(6,R)$–dynamics is compared with the dynamics described by the Fermi–liquid droplet model (FLDM).

In Section V a class of stationary solutions of the classical equations of the collective motion is examined. The single–particle motion corresponding to such solutions is compared with the motion of particles in the cranked harmonic oscillator model (CHOM). The selfconsistency conditions of the nuclear CHOM are discussed.

In Section VII the material of the paper is reviewed.

Some additional formalism necessary for the understanding of the paper is given in the three *Appendices*. The most important original results are presented under the title of *Propositions*.

II. The Hamiltonian and the equations of motion in the overcomplete system of coordinates.

The leading role of the quadrupole (ellipsoidal) deformation in determining the nature of nuclear structure is well established [3]. One expects that the main axes of the inertia tensor in heavy nuclei do not change much due to quantum effects. For this reason, we start with the introduction of a coordinate system in which the components of the nuclear inertia tensor play the role of collective coordinates, and we proceed to study the nuclear motion within the classical mechanics. This is achieved by considering the Zickendraht–Dzyublik–Filippov system of coordinates [4–7,9] which we use to introduce the overcomplete parameterization of the elements of the $6N$–6 dimensional phase space including the relative Jacobi coordinates $\vec{x}_k$ and the momenta $\vec{p}_k$ of N nucleons (k=1,2,...,N–1). We have:

$$x_{\alpha k}=\sum_X \mathcal{D}_{X\alpha}(\theta)\lambda_X\mathcal{R}_{Xk}(\varphi_1,\varphi_2,\ldots,\varphi_{N-9}). \tag{2.1}$$

The coordinates on the r.h.s. satisfy the conditions:

$$\sum_{\alpha=1}^{3}\mathcal{D}_{X\alpha}\mathcal{D}_{Y\alpha}=\delta_{XY},\qquad \sum_{k=1}^{N-1}\mathcal{R}_{Xk}\mathcal{R}_{Yk}=\delta_{XY}, \tag{2.2a, b}$$

Using the projection matrices:

$$f^{coll}_{\alpha k,\beta l} = \delta_{\alpha\beta} \sum_X \mathcal{R}_{Xk}\mathcal{R}_{Xl}, \tag{2.3a}$$

$$f^{intr}_{\alpha k,\beta l} = \delta_{\alpha\beta} f'_{kl}, \qquad f'_{kl} = \delta_{kl} - \sum_X \mathcal{R}_{Xk}\mathcal{R}_{Xl}, \tag{2.3b, c}$$

we decompose the momentum space of the relative motion into two orthogonal subspaces: $\mathcal{P} = \mathcal{P}^{coll} \oplus \mathcal{P}^{intr}$

$$p_{\alpha k} = p^{coll}_{\alpha k} + p^{intr}_{\alpha k}, \tag{2.4a}$$

It is convenient to parameterize the terms in eq.(2.4a) as follows [9]:

$$p^{coll}_{\alpha k} = \sum_X \mathcal{D}_{X\alpha}(\theta)\left\{\mathcal{R}_{Xk}p_X + \sum_{Y(\neq X)} \mathcal{R}_{Yk}\frac{(\tilde{L}_{XY}\lambda_Y + \lambda_X\tilde{J}_{XY})}{\lambda_X^2 - \lambda_Y^2}\right\}, \tag{2.4b}$$

For the second term in eq.(2.4a) we propose the form:

$$p^{intr}_{\alpha k} = \sum \mathcal{D}_{X\alpha}(\theta)\lambda_X^{-1}\pi_{Xk}. \tag{2.4c}$$

The overcomplete system of the momenta $\{\pi_{Xk}\}$ in eq.(2.4c) obeys the following linear relations:

$$\sum_{k=1}^{N} \mathcal{R}_{Xk}\pi_{Yk} = 0. \tag{2.5}$$

which we immediately obtain noticing that $\sum_n \mathcal{R}_{Xn}f'_{nk}$=0 for X=A, B, C, k=1,..., $N-1$. Equations (2.1, 2.4a-c) allow the inverse transformation:

$$\sum_\beta Q_{\alpha\beta}\mathcal{D}_{X\beta} = \lambda_X^2\mathcal{D}_{X\alpha}, \qquad \text{for } X = A, B, C, \tag{2.6a}$$

$$\mathcal{R}_{Xk} = \frac{1}{\lambda_X}\sum_\alpha \mathcal{D}_{X\alpha}x_{\alpha k}, \qquad \text{for } X = A, B, C, \tag{2.6b}$$

$$p_X = \sum_{\alpha,k} \mathcal{D}_{X\alpha}\mathcal{R}_{Xk}p_{\alpha k}, \qquad \text{for } X = A, B, C, \tag{2.7a}$$

$$\tilde{L}_{AB} \equiv L_C = \sum_{X,Y,\alpha,k} \epsilon_{CXY}\lambda_X\mathcal{R}_{Xk}\mathcal{D}_{Y\alpha}p_{\alpha k} = \sum_{X,Y,k} \epsilon_{CXY}x_{Xk}p_{Yk}, \qquad \text{cycl } A, B, C, \tag{2.7b}$$

$$\tilde{J}_{AB} \equiv J_C = \sum_{X,Y,\alpha,k} \epsilon_{CXY}\lambda_X\mathcal{R}_{Yk}\mathcal{D}_{X\alpha}p_{\alpha k} = \sum_{X,Y,k} \epsilon_{CXY}\frac{\lambda_X}{\lambda_Y}x_{Yk}p_{Xk}, \quad \text{cycl } A, B, C, \tag{2.7c}$$

$$\pi_{Xm} = \lambda_X \sum_{\alpha,n} \mathcal{D}_{X\alpha}f'_{mn}p_{\alpha n}, \qquad \text{for } X = A, B, C. \tag{2.7d}$$

In the equations (2.6a) we have used the quadrupole tensor $Q_{\alpha\beta}$= $\sum_{k=1}^{N-1} x_{\alpha k}x_{\beta k}$. The matrix $\mathcal{D}_{A\alpha}$ which depends on the three Euler angles $\theta_1, \theta_2, \theta_3$ describes the $SO^L(3)$ transformation from the laboratory (l-frame) to the body frame (b-frame) of reference, the latter being defined as the frame where the matrix of the inertia tensor is diagonal: Q_b=diag$(\lambda_A^2, \lambda_B^2, \lambda_C^2)$. The quantities $\lambda_A, \lambda_B, \lambda_C$ characterize both the size and the quadrupole deformation of the system. Here we assume that $\lambda_A > \lambda_B > \lambda_C$, then the matrix $\mathcal{D}(\theta)$ is determined up to the transformations of the group D_2 of rotations through the angle π around the principal axes of the inertia tensor.

The quantity $\vec{L}$ is the angular momentum. We call the quantity $\vec{J}$ the vortex spin vector. This quantity appears in the Riemann equations of the classical gravitating liquid droplet [24,25]. In the microscopic theory it is introduced in several papers [4,26,27]. Our definition of the vortex spin coincides with that of ref.[27], however, let us note that some authors [26] use this name for the vector $\vec{\mathcal{L}}$=$-\vec{J}$.

Throughout the paper we use the characters α, β, γ for the indices of the $SO^L(3)$ tensors in the l-frame, and X, Y, Z for their indices in the b-frame. When convenient, we use pseudovectors instead of antisymmetric tensors, and the summation convention for the repeated indices, such as $F_Z = \frac{1}{2}\epsilon_{ZXY}\tilde{F}_{XY}$.

The conditions (2.2a, b) are valid both in the classical and in the quantum mechanics. The quantum analogue of (2.5) is obvious:

$$\sum_{k=1}^{N-1} [\mathcal{R}_{Xk}\pi_{Yk} + \pi_{Yk}\mathcal{R}_{Xk}] = 0, \qquad \text{for } X, Y = A, B, C. \tag{2.8}$$

The momenta π_{Yk} which have been defined in eq.(2.7d) and also J_A, J_B, J_C are the differential operators with respect to the factor space $M \equiv O(N-4)\backslash O(N-1)$. This factor space can be parameterized by some angles $\varphi_1, \varphi_2, \ldots, \varphi_{3N-9}$ [4]. One can

also understand the quantities $\vec{\mathcal{R}}_k \equiv (\mathcal{R}_{Ak}, \mathcal{R}_{Bk}, \mathcal{R}_{Ck})$, $\vec{\pi}_k = (\pi_{Ak}, \pi_{Bk}, \pi_{Ck})$, and $\vec{J}$ as functions of $6N-18$ independent coordinates and momenta defined so that the relations $(2.2a-b, 2.5)$ are satisfied identically.

In the following we use extensively the bracket notation $\{a, b\}$. The latter can be understood either in the classical sense as the Poisson bracket, or quantum mechanically as the commutator times the factor $(i\hbar)^{-1}$.

Proposition (2.1). The canonical Poisson bracket relations

$$\{x_{\alpha k}, x_{\beta l}\} = 0, \qquad \{p_{\alpha k}, p_{\beta l}\} = 0,$$

$$\{x_{\alpha k}, p_{\beta l}\} = \delta_{\alpha\beta}\delta_{kl},$$

hold if and only if the following relations are satisfied:

$$\{\pi_{Xk}, \pi_{Yl}\} = -f'_{kl}\tilde{J}_{XY} + (\mathcal{R}_{Xl}\pi_{Yk} - \mathcal{R}_{Yk}\pi_{Xl}), \tag{2.9a}$$

$$\{\pi_{Xk}, \mathcal{R}_{Yl}\} = -\delta_{XY}f'_{kl}, \qquad \{J_X, \pi_{Yk}\} = -\epsilon_{XYZ}\pi_{Zk}, \tag{2.9a, c}$$

and [26]

$$\{J_X, J_Y\} = -\epsilon_{XYZ}J_Z, \qquad \{J_X, \mathcal{R}_{Yk}\} = -\epsilon_{XYZ}\mathcal{R}_{Zk}, \tag{2.9d, e}$$

$$\{L_X, L_Y\} = -\epsilon_{XYZ}L_Z, \qquad \{L_X, \mathcal{D}_{Y\alpha}\} = -\epsilon_{XYZ}\mathcal{D}_{Z\alpha}, \tag{2.9f, g}$$

$$\{\lambda_X, p_Y\} = \delta_{XY}, \tag{2.9h}$$

with all the other omitted brackets $\{\ldots\}$ equal to zero.

In arriving to the nontrivial part of the Poisson bracket relations, namely $(2.9a-c)$, it is useful to mention the equivalent forms of the quantities introduced before

$$\pi_{Xk} = (\mathcal{R}\cdot j\cdot f)_{Xk}, \qquad J_{XY} = (\mathcal{R}\cdot j\cdot {}^t\mathcal{R})_{XY}, \tag{2.7.c', d'}$$

where

$$j_{kl} = \sum_\alpha (x_{\alpha kn}p_{\alpha l} - x_{\alpha k}p_{\alpha l}), \quad k, l = 1, \ldots, N-1, \tag{2.10a}$$

are the generators of the $SO(N-1)$ group. The parameterization of the coordinates and momenta introduced before allows us to write

$$j_{kl} = ({}^t\mathcal{R}\cdot\pi - {}^t\pi\cdot\mathcal{R} + {}^t\mathcal{R}\cdot\tilde{J}\cdot\mathcal{R})_{kl}. \tag{2.10b}$$

Let us also write the well known relations

$$\{j_{mn}, t_{k_1,\ldots,k_p}\} = \sum_i \left(\delta_{mk_i}t_{k_1,\ldots,n,\ldots,k_p} - \delta_{nk_i}t_{k_1,\ldots,m,\ldots k_p}\right), \tag{2.11}$$

which are valid for an the $SO(N-1)$ tensors $t_{k_1\ldots k_p}$. The relations $(2.9b)$ follow immediately from eqs.$(2.7d', 2.11)$. Next, we have

$$\sum_{n,m}\{\mathcal{R}_{An}, \pi_{Bl}\}j_{nm}f'_{mk} = \delta_{AB}(f'\cdot j\cdot f')_{lk} = 0.$$

The first equality is the consequence of the just proved relation $(2.9b)$, while the second one is obtained from the first using eq.$(2.10b)$ and the conditions $(f'\cdot\mathcal{R})_{nX} = (\mathcal{R}\cdot f')_{Xn} = 0$ following from eq.$(2.2b)$. Since, we arrive at

$$\{\pi_{Ak}, \pi_{Bl}\} = \sum_{mn}\mathcal{R}_{An}\left(\{j_{nm}, \pi_{Bl}\}f'_{mk} + j_{mn}\{f'_{nk}, \pi_{Bl}\}\right).$$

Applying eq.(2.11) to the first term under the sum and noticing that $(\vec{\pi}\cdot f')_n = \vec{\pi}_n$, while using eqs.$(2.3c, 2.9b)$ for the second term we recover the expression $(2.9a)$. The equations $(2.9c)$ we find applying eqs.$(2.7c', 2.9a, b)$. The canonical relations follow immediately $(\Leftarrow)$ from the system $(2.9a-h)$.

The expressions on the l.h.s. of the conditions $(2.2a, b)$ and (2.5) are elementary invariants for the Poisson structure $(2.9a-g)$. This means that our constraints are consistent with it.

In the new coordinates the Hamiltonian has the form:

$$H = T^{vib} + T^{rot} + T^{intr} + U, \tag{2.12}$$

where

$$T^{vib} = \frac{1}{2m}\sum_X p_X^2, \qquad T^{intr} = \sum_X \sum_k \frac{\pi_{Xk}^2}{2m\lambda_X^2}, \tag{2.13a, b}$$

$$T^{rot} = \frac{1}{2m}\sum_{A>B}\frac{1}{(\lambda_A^2-\lambda_B^2)^2}\left\{(\lambda_A^2+\lambda_B^2)(\tilde{L}_{AB}^2+\tilde{J}_{AB}^2)+4\lambda_A\lambda_B\tilde{L}_{AB}\tilde{J}_{AB}\right\}, \tag{2.14}$$

The potential U contains the part U^0 which is a scalar with respect to the $O(N-1)$ group and the tensor part U^{tens}:

$$U = U^0(\lambda_A,\lambda_B,\lambda_C) + U^{tens}(\lambda_A\lambda_B\lambda_C;\mathcal{R}(\varphi)), \tag{2.15}$$

In principle both terms here may be calculated once the particle–particle interaction is known. The $O(N-1)$ scalar potential has been calculated for the central nucleon–nucleon interaction in ref.[28]. The role of the tensor two–body interaction is examined in ref.[29].

Let us introduce the following definitions

$$\Omega_X \stackrel{\text{def}}{=} \frac{\partial H}{\partial L_X}, \qquad \Lambda_X \stackrel{\text{def}}{=} \frac{\partial H}{\partial J_X}, \qquad \omega_X^2 \stackrel{\text{def}}{=} -\frac{1}{m\lambda_X}\frac{\partial H}{\partial \lambda_X}, \tag{2.16a, b, c}$$

$$Y_{XY} \stackrel{\text{def}}{=} \sum_{k=1}^{N-1}\pi_{Xk}\pi_{Yk}, \qquad \mathcal{L}_{XY} \stackrel{\text{def}}{=} (m\lambda_X^2)^{-1}\delta_{XY}, \tag{2.16d, e}$$

where $\vec{\Omega}$ and $\vec{\Lambda}$ are the pseudovectors of the angular rotation velocity and the vortex rotation velocity [4,24] respectively, while Y will be called the Fermi–energy tensor. We have

$$\begin{bmatrix}\Omega_C\\ \Lambda_C\end{bmatrix} = \frac{1}{m(\lambda_A^2-\lambda_B^2)^2}\begin{bmatrix}\lambda_A^2+\lambda_B^2 & 2\lambda_A\lambda_B\\ 2\lambda_A\lambda_B & \lambda_A^2+\lambda_B^2\end{bmatrix}\begin{bmatrix}L_C\\ J_C\end{bmatrix}, \qquad \text{cycl } A,B,C. \tag{2.17}$$

$$\omega_A^2 = \frac{Y_{AA}}{m^2\lambda_A^4} + \frac{1}{2m^2\lambda_A}\sum_{X=B,C}\left\{\frac{(\tilde{L}_{AX}+\tilde{J}_{AX})^2}{(\lambda_A-\lambda_X)^3}+\frac{(\tilde{L}_{AX}-\tilde{J}_{AX})^2}{(\lambda_A-\lambda_X)^3}\right\}, \qquad \text{cycl } A,B,C, \tag{2.18a}$$

or when we express the momenta $\vec{L},\vec{J}$ by the velocities $\vec{\Omega},\vec{\Lambda}$ we get

$$\omega_A^2 = \frac{Y_{AA}}{m^2\lambda_A^4}+\Omega_B^2+\Omega_C^2+\Lambda_B^2+\Lambda_C^2-2(\lambda_C/\lambda_A)\Omega_B\Lambda_B-2(\lambda_B/\lambda_A)\Omega_C\Lambda_C, \qquad \text{cycl } A,B,C. \tag{2.18b}$$

Proposition (2.2). The Poisson bracket relations lead to the following system of equations ($X=A,B,C$):

$$\frac{d}{dt}\lambda_X - \frac{1}{m}p_X = 0, \tag{2.19a}$$

$$\frac{d}{dt}p_X - m\omega_X^2\lambda_X + \partial U^0/\partial\lambda_X = -\partial U^{tens}/\partial\lambda_X, \tag{2.19b}$$

$$\frac{d}{dt}L_X + \sum_{Y,Z}\epsilon_{XYZ}\Omega_Y L_Z = 0, \tag{2,19c}$$

$$\frac{d}{dt}J_X + \sum_{Y,Z}\epsilon_{XYZ}(\Lambda_Y J_Z + \mathcal{L}_{YY}Y_{YZ}) = -\sum_{Y,Z,k}\epsilon_{XYZ}\frac{DU^{tens}}{D\mathcal{R}_{Yk}}\mathcal{R}_{Zk}, \tag{2.19d}$$

$$\frac{d}{dt}\begin{bmatrix}\vec{\mathcal{R}}_k\\ \vec{\pi}_k\end{bmatrix} - \begin{bmatrix}\tilde{\Lambda} & \mathcal{L}\\ -Y\mathcal{L} & \tilde{\Lambda}-\tilde{J}\mathcal{L}\end{bmatrix}\begin{bmatrix}\vec{\mathcal{R}}_k\\ \vec{\pi}_k\end{bmatrix} = -\begin{bmatrix}0\\ DU^{tens}/D\vec{\mathcal{R}}_k\end{bmatrix}, \tag{2.19e, f}$$

$$\frac{d}{dt}\mathcal{D}_{X\alpha} + \sum_{Y,Z}\epsilon_{XYZ}\Omega_Y\mathcal{D}_{Z\alpha} = 0, \tag{2.19g}$$

The r.h.s. of eqs.(2.19d, e, f) contains the $SO(N$–1) tensorial part of the interaction. The derivative appearing there is defined as:

$$DF/D\mathcal{R}_{Xk} \equiv \{\pi_{Xk},F\} = \sum_{m=1}^{N-1} f'_{km}(\partial F/\partial\mathcal{R}_{Xm}). \tag{2.20}$$

Equations (2.19g) describing the orientation of the main axes of the nuclear inertia tensor in the space are separated from the remaining equations (2.19$a-f$). Thus, noticing that the total angular momentum $L^2 = L_A^2+L_B^2+L_C^2$ is conserved and because the matrices $\mathcal{R},\pi$ introduce only $6N$–21 independent variables (see eqs.(2.2b,2.5)), we conclude that the system (2.19a–f) contains $6N$–10 independent equations of the motion ($L\neq 0$). The system reduces by two more equations in the case $L=0$.

III. The Sp(6,R) classical collective dynamics

Drastic reduction of the system of dynamic equations takes place when the $O(N$–1) tensorial part of the interaction vanishes (U^{tens}=0). The rest of the paper is devoted to this particular case. Let us introduce 3–dimensional square matrices $\mathcal{U},\mathcal{G},\mathcal{F},\lambda$ and $\mathcal{E}$

$$\mathcal{U} = \dot{\lambda} + \lambda\cdot\tilde{\Lambda} - \tilde{\Omega}\cdot\lambda \tag{3.1}$$

$$\mathcal{G}_{XY} = -(\dot{\lambda}\cdot\tilde{\Lambda} - \tilde{\Omega}\cdot\dot{\lambda})_{XY} + (m^{-2}\lambda^{-1}\cdot Y\cdot\lambda^{-2} - \lambda\cdot\tilde{\Lambda}^2 - \tilde{\Omega}^2\cdot\lambda + 2\tilde{\Omega}\cdot\lambda\cdot\tilde{\Lambda})_{XY}, \quad (3.2)$$

$$\mathcal{F} = -\mathrm{diag}(\partial U^0/\partial\lambda_A, \partial U^0/\partial\lambda_B, \partial U^0/\partial\lambda_C), \qquad \lambda = \mathrm{diag}(\lambda_A, \lambda_B, \lambda_C). \quad (3.3a, b)$$

$$\mathcal{E} = \lambda\cdot(2\tilde{\Omega} - \lambda\cdot\tilde{\Lambda}\cdot\lambda^{-1})\cdot\lambda^{-1}; \quad (3.3c)$$

Proposition (3.1). When $U^{tens} = 0$, the time derivative of the matrices $\mathcal{U}$ and Y close the system of the differential equations

$$\frac{d}{dt}\mathcal{U} = \mathcal{G} + m^{-1}\mathcal{F}. \quad (3.4)$$

$$\frac{d}{dt}Y = \mathcal{E}\cdot Y + Y\cdot{}^t\mathcal{E}. \quad (3.5)$$

The closure of the above subsystem of the fifteen equations for the variables λ_X, Ω_X, Λ_X, Y_{XY}, $X, Y = A, B, C$ follows immediately from the Poisson bracket relations. Indeed, all the terms of the $O(N-1)$ invariant Hamiltonian are built on eighteen variables $\lambda_X, p_X, L_X, J_X, Y_{XY}$, and the relations

$$\{L_X, Y_{YZ}\} = \{\lambda_X, Y_{YZ}\} = \{p_X, Y_{YZ}\} = 0, \quad (3.6a, b, c)$$

$$\{J_X, Y_{YZ}\} = -(\epsilon_{XYS}Y_{SZ} + \epsilon_{XZS}Y_{YS}), \quad (3.7)$$

$$\{Y_{XY}, Y_{ZS}\} = -\tilde{J}_{XZ}Y_{YS} - \tilde{J}_{XS}Y_{ZY} - \tilde{J}_{YZ}Y_{XS} - \tilde{J}_{YS}Y_{XZ}. \quad (3.8)$$

together with eqs.(2.9$a-d$) close their Poisson algebra.

Proof of the Proposition (3.1). Using eqs.(2.19$a-b$) and noticing that

$$\mathcal{G}_{XX} = \lambda_X\omega_X^2, \quad (3.9)$$

we arrive at eqs.(3.4) for the diagonal elements of the matrix $\mathcal{U}$. We get the equations (3.4) for the off–diagonal terms considering the time derivative of the relations

$$(\lambda\cdot\tilde{\Lambda} - \tilde{\Omega}\cdot\lambda)_{XY} = m^{-1}(\lambda_X^2 - \lambda_Y^2)^{-1}(\lambda_X\tilde{J}_{XY} + \tilde{L}_{XY}\lambda_Y), \quad (3.10)$$

which follow from eq.(2.17). We have

$$m^{-1}(\lambda_X^2-\lambda_Y^2)^{-2}[-2(\lambda_X\dot{\lambda}_X-\lambda_Y\dot{\lambda}_Y)(\lambda_X\tilde{J}_{XY}+\tilde{L}_{XY}\lambda_Y)+(\lambda_X^2-\lambda_Y^2)(\dot{\lambda}_X\tilde{J}_{XY}+\tilde{L}_{XY}\dot{\lambda}_Y)]$$
$$= -(\dot{\lambda}\cdot\tilde{\Lambda} - \tilde{\Omega}\cdot\dot{\lambda})_{XY}. \quad (3.11)$$

$$m^{-1}(\lambda_X^2-\lambda_Y^2)^{-1}(\lambda_X\dot{\tilde{J}}_{XY}+\dot{\tilde{L}}_{XY}\lambda_Y) = (m^{-2}\lambda^{-1}Y\lambda^{-2}-\lambda\tilde{\Lambda}^2-\tilde{\Omega}^2\lambda+2\tilde{\Omega}\lambda\tilde{\Lambda})_{XY}, \quad (3.12)$$

since, summing r.h.s. for both terms we obtain the required result. In establishing the relation (3.12) it is convenient to present eqs.(2.19c,d) in the form

$$\frac{d}{dt}\tilde{L} = \tilde{\Omega}\cdot\tilde{L} - \tilde{L}\cdot\tilde{\Omega}, \qquad \frac{d}{dt}\tilde{J} = \tilde{\Lambda}\cdot\tilde{J} - \tilde{J}\cdot\tilde{\Lambda} + Y\cdot\mathcal{L} - \mathcal{L}\cdot Y. \quad (3.13)$$

This calculation shows that the nine equations (3.4) are equivalent to the twelve first order differential equations (2.19a–d). Using the Leibnitz formula for the time derivative of the definition (2.16d) and applying eqs.(2.19f, 2.5) one arrives at equations

$$\frac{d}{dt}Y_{XY} - \sum_{Z,Z'}[(\tilde{\Lambda}_{XZ}\delta_{ZZ'} - \tilde{J}_{XZ}\mathcal{L}_{ZZ'})Y_{Z'Y} + (\tilde{\Lambda}_{YZ}\delta_{ZZ'} - \tilde{J}_{YZ}\mathcal{L}_{ZZ'})Y_{XZ'}] = 0. \quad (3.14)$$

Equations (3.5) follow immediately from the above, given the identity

$$\mathcal{E} = \tilde{\Lambda} - \tilde{J}\cdot\mathcal{L}. \quad (3.15)$$

The compact and elegant form of the equations of motion (3.4,3.5) follows from their geometrical character. We show it in the following way. Let ρ be the $6 \times N-1$ dimensional matrix of the Jacobi coordinates and momenta

$$\rho = \begin{Bmatrix} \vec{x}_1 \ldots \vec{x}_{N-1} \\ \vec{p}_1 \ldots \vec{p}_{N-1} \end{Bmatrix}, \quad (3.16)$$

and let us define the collective tensor:

$$\mathbf{C}(\rho) = \rho\cdot{}^t\rho\cdot\mathbf{K}_{(6)} = \begin{Bmatrix} (L+S)/2 & -Q \\ P & (L-S)/2 \end{Bmatrix}, \quad \text{where } \mathbf{K}_{(6)} = \begin{Bmatrix} 0 & -1 \\ 1 & 0 \end{Bmatrix}, \quad (3.17)$$

The elements of $\mathbf{C}(\rho)$ are the Poisson generators of the $Sp(6,R)$ group:

$$S_{\alpha\beta} = \sum_{k=1}^{N-1}(x_{\alpha k}p_{\beta k} + p_{\alpha k}x_{\beta k}), \qquad L_{\alpha\beta} = \sum_{k=1}^{N-1}(x_{\alpha k}p_{\beta k} - p_{\alpha k}x_{\beta k}), \tag{3.18a, b}$$

$$P_{\alpha\beta} = \sum_{k=1}^{N-1} p_{\alpha k}p_{\beta k}, \qquad Q_{\alpha\beta} = \sum_{k=1}^{N-1} x_{\alpha k}x_{\beta k}. \tag{3.18c, d}$$

Let the six–dimensional block–diagonal matrix $\bar{\mathcal{D}}(\theta(t))=\mathcal{D}(\theta(t)) \oplus \mathcal{D}(\theta(t))$ determine the transformation of ρ to the b–frame $\rho \to \rho_b=\bar{\mathcal{D}}(\theta(t))\rho$. Using relations (3.10) and eqs.(2.1, 2.4$a-c$) we find

$$\rho_b = \begin{Bmatrix} \lambda & 0 \\ m\mathcal{U} & \lambda^{-1} \end{Bmatrix} \begin{Bmatrix} \vec{\mathcal{R}}_1 ... \vec{\mathcal{R}}_{N-1} \\ \vec{\pi}_1 ... \vec{\pi}_{N-1} \end{Bmatrix}. \tag{3.19}$$

Applying the above and eq.(2.2b,2.5) we arrive at the formulas

$$\mathbf{C} = {}^t\bar{\mathcal{D}}(\theta)\cdot\mathbf{C}_b\cdot\bar{\mathcal{D}}(\theta), \qquad \mathbf{C}_b = \begin{Bmatrix} m\lambda\cdot{}^tU & -\lambda^2 \\ \lambda^{-1}\cdot Y\cdot\lambda^{-1} + m^2U\cdot{}^tU & -mU\cdot\lambda \end{Bmatrix}. \tag{3.20}$$

The mutual correspondence between the $O(N-1)$ invariant dynamics and the $Sp(6,R)$ model comes from the last mapping and from the analysis of the evolution of the system in the phase space. Within the $O(N-1)$ model the evolution is determined by the action of the $Sp(6,R)$ group and we have [13]:

$$\rho(t) = \mathcal{G}(t)\cdot\rho(0), \qquad \mathbf{C}(t) = \mathrm{Ad}_{\mathcal{G}(t)}(\mathbf{C}(0)), \qquad \mathrm{Ad}_{\mathcal{G}}(\mathbf{X}) \stackrel{\mathrm{def}}{=} \mathcal{G}\cdot\mathbf{X}\cdot\mathcal{G}^{-1}, \tag{3.21a, b, c}$$

where $\mathcal{G}(t)$ is the 6–dimensional symplectic matrix:

$$Sp(6,R) = \left\{\mathcal{G} \in GL(6,R);\ \mathbf{K}_{(6)}\mathcal{G}\mathbf{K}_{(6)}{}^{-1} = {}^t\mathcal{G}^{-1}\right\}. \tag{3.21d}$$

It is convenient to factorize the matrix $\mathcal{G}(t)$ as follows:

$$\mathcal{G}(t) = {}^t\bar{\mathcal{D}}(\theta(t))\cdot\mathcal{G}_b(t)\cdot\bar{\mathcal{D}}(\theta(0)), \tag{3.22}$$

where the matrix $\mathcal{G}_b(t)$ describes the evolution of the system in the b–frame:

$$\rho_b(t) = \mathcal{G}_b(t)\cdot\rho_b(0), \tag{3.23}$$

Proposition (3.2). The Lee derivative of the matrix $\mathcal{G}_b(t)$

$$\dot{\mathcal{G}}_b(t)\cdot\mathcal{G}_b^{-1}(t) = \mathbf{H}_b(q(t)^{coll}), \tag{3.24}$$

has the form:

$$\mathbf{H}_b(q(t)^{coll}) = \begin{Bmatrix} \tilde{\Omega} & \frac{1}{m}\mathbf{1} \\ \mathcal{F}\cdot\lambda^{-1} & \tilde{\Omega} \end{Bmatrix}, \tag{3.25}$$

where $\tilde{\Omega}$ denotes the angular velocity tensor $\dot{\mathcal{D}}(\theta)\cdot{}^t\mathcal{D}(\theta)$.

Proof. Taking the time derivative of the equations (3.19) and substituting equations (2.19e, f, 3.15) we get

$$\frac{d}{dt}\rho_b(t) = \mathbf{H}_b(q^{coll}(t))\rho_b, \tag{3.26}$$

$$\mathbf{H}_b = \left\{\frac{d}{dt}\begin{pmatrix} \lambda & 0 \\ m\mathcal{U} & \lambda^{-1} \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ m\mathcal{U} & \lambda^{-1} \end{pmatrix}\begin{pmatrix} \tilde{\Lambda} & \mathcal{L} \\ -Y\cdot\mathcal{L} & \mathcal{E} \end{pmatrix}\right\}\begin{pmatrix} \lambda^{-1} & 0 \\ -m\lambda\cdot\mathcal{U}\cdot\lambda^{-1} & \lambda \end{pmatrix}, \tag{3.27}$$

Next, calculating the matrix $\mathbf{H}_b$ from the last formula and comparing the result with eq.(3.25) we come to conditions

$$\tilde{\Omega} = \dot{\lambda} + \lambda\cdot\tilde{\Lambda}\cdot\lambda^{-1} - \mathcal{U}\cdot\lambda^{-1}, \qquad m^{-1}\mathbf{1} = \lambda\cdot\mathcal{L}\cdot\lambda, \tag{3.28a, b}$$

$$\mathcal{F}\cdot\lambda^{-1} = m\left(\frac{d}{dt}\mathcal{U} - \tilde{\Omega}\cdot\mathcal{U} + \mathcal{U}\cdot\tilde{\Lambda} - m^{-1}\lambda^{-1}\cdot Y\cdot\mathcal{L}\right)\lambda^{-1}, \tag{3.28c}$$

$$\tilde{\Omega} = -\dot{\lambda}\cdot\lambda + \mathcal{U}\cdot\lambda^{-1} + 2\tilde{\Omega} - \lambda\cdot\tilde{\Lambda}\cdot\lambda^{-1}, \tag{3.28d}$$

The equations (3.28a, b, d) follow immediately from the definitions (3.1–3.3b,2.16e) while eq.(3.28c) is obtained applying eq.(3.4), which finishes the proof.

Let us also examine the relations (3.21a, b, d). Formula (3.21a) results from (3.26). The matrix $\mathbf{H}_b$ belongs to the $sp(6,R)$ algebra:

$$sp(6,R) = \left\{\mathbf{X} \in gl(6,r):\ \mathbf{X}_b\cdot\mathbf{K}_{(6)} + \mathbf{K}_{(6)}\cdot{}^t\mathbf{X}_b = 0\right\}.$$

This enshure the simplecticity of the matrix $\mathcal{G}_b$, hence also $\mathcal{G} \in Sp(6,R)$ and finally the matrix $\mathbf{C} \in sp(6,R)$ evolves in time as follows: $\mathbf{C} = \mathcal{G}(t)\cdot\mathbf{C}(0)\cdot(\mathbf{K}_{(6)}^{-1}\cdot{}^t\mathcal{G}(t)\cdot\mathbf{K}_{(6)}) = \mathcal{G}(t)\cdot\mathbf{C}(0)\cdot\mathcal{G}^{-1}(t)$.

Proposition(3.2) says: *the $O(N-1)$ invariant classical system is dynamically equivalent to the system of independent particles moving in the local time dependent harmonic field. The evolution of this field is described by the closed subsystem of equations of motion for the collective variables $\{q^{coll}\}$.*

Indeed, let us consider the field–frame (f-frame) $\rho_f = \bar{\mathcal{D}}^f(\alpha(t))\rho$ and let the particles move in the specific time dependent (translational invariant) single–particle field:

$$U^{eff} = m \sum_{Y,k} \tilde{\omega}_Y^2(t)(x_{Yk}^f)^2, \tag{3.29}$$

Noticing that the transformation–generating function has the form:

$$S(x, p_f, t) = \sum_{\alpha, A, n} x_{\alpha n} \mathcal{D}_{A\alpha}^f(\alpha(t)) p_{An}^f, \tag{3.30}$$

we obtain the time dependent Routhian function $R(\rho_f; t)$

$$R(\rho_f; t) \equiv H(\rho(\rho_f)) - \partial S(\rho(\rho_f), t)/\partial t =$$

$$\sum_Y \sum_k [m^{-1}(p_{Yk}^f)^2 + (\tilde{\omega}_Y^2(t) x_{Yk}^f)^2]/2 - \vec{\Omega}_f(t) \cdot \vec{L}_f, \tag{3.31a}$$

or in more compact form

$$R(\rho_f; t) = \frac{1}{2} \mathrm{tr}\, [\mathbf{H}_f(t) \cdot \mathbf{C}(\rho_f)]. \tag{3.31b}$$

where $\tilde{\Omega}_f = \dot{\mathcal{D}}^f \cdot {}^t\mathcal{D}^f$. If $\theta_i(t) = \alpha_i(t)$ and $m\tilde{\omega}_X^2(t) = -(\mathcal{F} \cdot \lambda^{-1})_{XX}(t)$ then $\tilde{\Omega}_f(t) = \tilde{\Omega}(t)$, the matrix $\mathbf{H}_f = \mathbf{H}_b$ and the canonical equations of motion for ρ_f generated by Routhian $R(\rho_f; t)$ coincide with equations (3.26) for ρ_b.

In this approach the angle functions $\alpha_i(t)$ are assumed to be known and independent of the canonical variables ρ; thus the variables ρ_f are canonical too. Then, denoting $\mathbf{C}_f \equiv \mathbf{C}(\rho_f)$ we find that generators $\{Q_f, S_f, P_f, L_f\}$ and $\{Q, S, P, L\}$ obey the same Poisson bracket relations. On the other hand, the system of time–independent coordinates ρ_b does not obey the canonical relations (see the equations in *Proposition*(2.1)). The cranking term $-\vec{\Omega} \cdot \vec{L}$ should be added to the Hamiltonian function to compensate in equations of motion for the differences between the structure of the ρ_b and ρ_f Poisson

relations. Applying the Leibnitz formula for the time derivative of the definition (3.21) and using eqs.(3.23,3.24) one gets

$$\frac{d}{dt}\mathbf{C}_b = \mathbf{H}_b \mathbf{C}_b - \mathbf{C}_b \mathbf{H}_b. \tag{3.32}$$

The physically most important configurations are those (i) for which the matrices $\mathbf{H}_b$ and $\mathbf{C}_b$ commute and (ii) which are stable in regard to small variations of the collective variables. Such configurations distinguish the physical domain in which the stable effective mean field exists. We will return to the discussion of point (i) in Section VI.

Formally, eqs.(3.4,3.5) admit the trivial solution $Y_{XY}=0$. In this case the above equations and eqs.(2.19g) become equivalent with the Riemann equations of motion to the Dirichlet Liquid Droplet (DLD). The dynamics of DLD was studied for more than a century by great mathematicians and astronomers (see ref.[24,25]). In application to the nuclear dynamics [26,27,30–34] this model has been formulated in a fully microscopic way using the semisimple group GCM(3). The Lie algebra of this group is spanned by fifteen elements $\{Q, S, L\}$ and constitutes the dynamical algebra of the DLD. This allows us to render the quantum–mechanical form to the DLD model and to study its place in the microscopic nuclear theory. One obtains an another model based on the group $G^{dyn} = Sp(4, R)$ reducing the particle motion to a single plane [35,36]. The two–dimensional dynamics leads to a great simplification in the structure of the collective space. In *Appendix A* we show that the four variables $Y_{AA}, Y_{AB}, Y_{BB}, J_C$ can be expanded in terms of the generators of the $u(2)$ algebra.

IV. The collective space

We begin this section with the discussion of the reduced collective space (see ref.[13] by Kramer) on which the $Sp(6, R)$ group acts as a transitive group of canonical transformations. On the grounds of Section III we define the reduced collective space as the manifold of the orbits $\mathcal{O}_{\mathbf{X}}$:

$$\mathcal{O}_{\mathbf{X}} = \{\mathbf{C} \in sp(6, R) : \ \mathbf{C} = \mathrm{Ad}_{\mathcal{G}}(\mathbf{X}), \ \mathcal{G} \in Sp(6, R)\}. \tag{4.1}$$

A useful classification of these orbits is given by the identification of an orbit $\mathcal{O}_{\mathbf{X}}$ with the coset space of the $Sp(6,R)$ group modulo the stability group $G_{\mathbf{X}}$:

$$G_{\mathbf{X}} = \{\mathcal{G} \in Sp(6,R),\ \mathrm{Ad}_{\mathcal{G}}(\mathbf{X}) = \mathbf{X}\}$$

Hence:

$$\dim \mathcal{O}_{\mathbf{X}} = \dim(Sp(6,R)/G_{\mathbf{X}}) = \dim Sp(6,R) - \dim G_{\mathbf{X}}$$

Within the orbit $\mathcal{O}_X \equiv \mathcal{O}_{\vec{f}}$ we may choose the representative point β so that the matrix $\mathbf{C}(\beta)$ with elements given in $\hbar$ units is of the form

$$\bar{\mathbf{C}}_{\vec{f}} = \mathbf{C}(\beta) = \begin{Bmatrix} \mathbf{0} & -\mathbf{f} \\ \mathbf{f} & \mathbf{0} \end{Bmatrix}, \qquad \mathbf{f} = \mathrm{diag}(f_1, f_2, f_3), \quad f_1 \le f_2 \le f_3 \le 0. \tag{4.2}$$

Leaving aside the classes of the orbits with $f_3 = 0$, which are physically interesting only for $N \le 6$, we distinguish the following four classes of $\mathcal{O}_{\vec{f}}$, $\vec{f}=(f_1, f_2, f_3)$:

$$\dim \mathcal{O}_{\vec{f}} = \begin{cases} 12 & \text{if } f_1 = f_2 = f_3, \text{ (A)}; \\ 16 & \text{if } f_1 > f_2 = f_3, \text{ (B)}; \\ 16 & \text{if } f_1 = f_2 > f_3, \text{ (C)}; \\ 18. & \text{if } f_1 > f_2 > f_3, \text{ (D)}. \end{cases}, \tag{4.3}$$

The stability groups $G_{\mathbf{X}} \subset Sp(6,R)$ corresponding to these classes are isomorphic with the groups $U(3)$, $U(1) \times U(2)$, $U(2) \times U(1)$ and $U(1) \times U(1) \times U(1)$, respectively [13].

Further reduction of the collective space follows due to $SO^L(3)$ rotational symmetry. We may exclude the three Euler angles from the collective space by introducing the space denoted as $\mathcal{O}_{\vec{f}}/SO^L(3)$ of the elements conjugated by the $SO^L(3)$ group (see eqs.(3.19)). Then, the factor space $\mathcal{O}_{\vec{f}}/SO^L(3)$ should be decomposed into the subspaces $\mathcal{O}_{\vec{f};L}$ enumerated by the value of the angular momentum L. We get

$$\dim \mathcal{O}_{\vec{f};L} = \begin{cases} 8,\ (6) & \text{if } f_1 = f_2 = f_3,\ L > 0, (L = 0), \text{ (A case)}; \\ 12, (10) & \text{if } f_1 > f_2 = f_3,\ L > 0, (L = 0), \text{ (B case)}; \\ 12, (10) & \text{if } f_1 = f_2 > f_3,\ L > 0, (L = 0), \text{ (C case)}; \\ 14, (12) & \text{if } f_1 > f_2 > f_3,\ L > 0, (L = 0), \text{ (D case)}. \end{cases} \tag{4.4}$$

The physical meaning of the above classification is contained in the following *Propositions.*

Proposition (4.1). On the manifolds $\mathcal{O}_{\vec{f};L}$ the eigenvalues y of the Fermi-energy tensor are found in the intervals $f_1 \ge y_1 \ge f_2$, $y_1 \ge y_2 \ge y_3$, $f_2 \ge y_3 \ge f_3$.

Proposition (4.2). For the degenerate orbits (A), (B) and (C) the tensors Y and J take the following general forms

$$Y_b = f^2 \mathbf{1}, \qquad \vec{J}_b = \mathbf{O}, \qquad \text{for case } A, \tag{4.5}$$

$$Y_b = {}^t\mathcal{D}_Y \cdot Y_{in} \cdot \mathcal{D}_Y, \qquad \vec{J}_b = {}^t\mathcal{D}_Y \vec{j}, \qquad \text{for case } B, \tag{4.6a}$$

$$\mathcal{D}_Y \in \mathrm{SO}(3), \quad Y_{in} = \mathrm{diag}(y_1, y_2, y_3), \quad j_1 = j_2 = 0,\ j_3 = j, \quad |\, j \,| \le f_1 - f_2$$

$$y_{1(2)} = [f_1^2 + f_2^2 - j^2]/2 \pm \tfrac{1}{2}\sqrt{[(f_1+f_2)^2 - j^2][(f_1-f_2)^2 - j^2]}, \tag{4.6b, c}$$

$$y_3 = f_2^2, \tag{4.6d}$$

while in case (C) Y_b and y_i are found as in case (B) after exchanging f_1 with f_3 and assuming $|\, j \,| \le f_1 - f_3$.

For the class (D) the eigenvalues $y_i = y_i(\vec{f}, \vec{j})$ may be found by solving a sixth order equation which could be obtained using formulas presented below. We skip the detailed discussion of this case.

Proposition (4.3). The invariants of the symplectic $Sp(6,R)$ action in the orbits space $\mathcal{O}_{\vec{f}}$

$$C_{2p} = (-1)^p \mathrm{tr}\ \mathbf{C}(\rho)^{2p}, \qquad p = 2, 4, 6, \tag{4.7}$$

are expressed by the collective variables as follows

$$C_2 = \mathrm{tr}\ (2Y - \vec{J}^2), \tag{4.8a}$$

$$C_4 = \mathrm{tr}\ \{(2Y - \vec{J}^2)^2 - 2Y^2\}, \tag{4.8b}$$

$$C_6 = \text{ tr } \{(Y - \tilde{J}^2)^3 + 3Y\tilde{J}^4 - 3Y\tilde{J}Y\tilde{J} \ - 3Y^2\tilde{J}^2 + Y^3\}, \tag{4.8c}$$

while for the two other invariants

$$C_4' = \frac{1}{8}C_2^2 - \frac{1}{4}C_4 \qquad C_6' = \frac{1}{48}C_2^3 - \frac{1}{8}C_2C_4 + \frac{1}{6}C_6, \tag{4.9a, b}$$

we have

$$C_4' = \sum_{X,Y}(Y_{XX}Y_{YY} - Y_{XY}^2) + \sum_{X>Y} J_X Y_{XY} J_Y, \tag{4.8b'}$$

$$C_6' = \det\ Y \tag{4.8c'}$$

Proof of Propositions (4.1–4.3). Let us consider the transformation $\rho \to \rho' = \mathcal{G}^{-1}\cdot\rho$ and let $\mathcal{G} \in GCM(3) \subset Sp(6,R)$ be of the form $\mathcal{G} = {}^t\bar{\mathcal{D}}(\theta)\cdot\bar{\lambda}\cdot\bar{X}$ where

$$\bar{\mathcal{D}}(\theta) = \begin{Bmatrix} \mathcal{D}(\theta) & 0 \\ 0 & \mathcal{D}(\theta) \end{Bmatrix}, \quad \bar{\lambda} = \begin{Bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{Bmatrix}, \quad \bar{X} = \begin{Bmatrix} 1 & 0 \\ X & 1 \end{Bmatrix}, \tag{4.10}$$

and where the symmetric matrix X chosen is $X = -m({}^t\mathcal{U}\cdot\lambda + \lambda\cdot\mathcal{U})/2$. Noticing that $\tilde{J} = -m({}^t\mathcal{U}\cdot\lambda - \lambda\cdot\mathcal{U})$, we find

$$\mathbf{C} \to \mathbf{C}' = \mathrm{Ad}_{\mathcal{G}^{-1}}(\mathbf{C}) = \begin{Bmatrix} \frac{1}{2}\tilde{J} & -1 \\ -\frac{1}{4}\tilde{J}^2 + Y & \frac{1}{2}\tilde{J} \end{Bmatrix} \tag{4.11}$$

The relations (4.8a–c) are arrived at applying formulas (4.7) for the matrix $\mathbf{C}'$. Then eqs.(4.8b', c') follow after long but elementary calculations. Next, using formula (4.2) we determine the values of the invariants

$$C_{2p} = 2\sum_{i=1}^{3} f_i^{2p}, \qquad C_4' = \sum_{i>j} f_i^2 f_j^2 \qquad C_6' = f_1^2 f_2^2 f_3^2. \tag{4.12}$$

and using eqs.(4.8a–c, 4.12) we get the following equations for the eigenvalues y_i:

$$(y_2 - y_1)j_2^2 + (y_3 - y_1)j_3^2 = F(y_1), \qquad \text{cycl } 1,2,3 \tag{4.13a, b, c}$$

with $F(y) = \Pi_i(y - f_i^2)/y$. If we choose the order $y_1 \geq y_2 \geq y_3$, the l.h.s. of eq.(4.13a) (eq.(4.13c)) becomes negative (positive). Noticing that $y_1 \geq y_3$ and that $F(y)$ is nonnegative for $f_2 \geq y \geq f_3$ and for $y \geq f_1$ we obtain *Proposition* (4.1). Applying it to case B in the *Proposition* (4.2) we conclude that $y_3 = f_2^2$. Thus, using eqs.(4.8a, 4.9b, 4.12, 4.13c) we have

$$y_1 + y_2 + \sum_i j_i^2 = f_1^2 + f_2^2, \quad y_1y_2 = f_1^2 f_2^2, \quad (y_1 - f_2^2)j_1^2 + (y_2 - f_2^2)j_2^2 = 0,$$

which entail the result in eqs.(4.6a – d). The proof for case (C) is quite similar and equations (4.5) (case (A)) immediately follow from the formulas in case (B).

Another class of invariants of motion constitute the generators $j_{kl}(\rho)$ (see eq.(2.10)) of the $SO(N\text{–}1)$ group. Let us consider the following functions (Casimirs of the $SO(N\text{–}1)$ group)

$$D_{2p} = (-1)^p \mathrm{tr}\ j(\rho)^{2p}, \qquad p = 1,2,3, \tag{4.14}$$

In the classical N–body problem these functions and the functions C_{2p} are dependent and coincide. We verify it substituting into formulas (4.14) the expressions (2.10). Then applying eqs.(2.2b,2.5) we find for D_{2p}, p=1,2,3 the same relations which have been established already in eqs.(4.8a–c) for the invariants C_{2p}. Latter result expresses in the classical mechanics an important fact well known from quantum investigations. Considering the space of the quantum N–body states $\mathcal{S}^N$ and a chain of the subgroups $Sp(6N-6,R) \subset Sp(6,R) \times SO(N-1)$ one shows [37] that: (i) the even and the odd parity states form the two $Sp(6N-6,R)$ multiplets $< \frac{q}{2}(\frac{1}{2})^{3N-4} >$, q=1,3 respectively and (ii) the groups $Sp(6,R)$, $SO(N\text{–}1)$ are a complementary pair in respect to the above reduction, which means that the reduction for both representations is multiplicity free. One has

$$\mathcal{S}^N = \sum_{f_1' \geq f_2' \geq f_3' \geq 0} \mathcal{S}(f_1' f_2' f_3')^N, \tag{4.15a}$$

with

$$\mathcal{S}(f_1' f_2' f_3')^N = < f_1 f_2 f_3 >^{sp(6,R)} \otimes (f_1' f_2' f_3' 0^{(N-7)/2})^{so(N-1)},$$
$$f_i = f_1' + \frac{N-1}{2}\hbar. \tag{4.15b}$$

The eigenvalues of the Casimir invariants of the $so(N-1)$ algebras are well known [38,39]. The domain of validity of the classical approximation given by eqs.(4.12) is examined in Appendix B.

To apply the above results to the nuclear system, the selection of the states allowed by the Pauli principle should be made. We say that the $Sp(6,R)$ representation $< f_1, f_2, f_3 >$ is allowed when the set of the antisymmetric wave functions with the $O(N-1)$ orbital structure $(f_1' f_2' f_3' 0 \ldots 0)$ is not empty. Due to the group reduction $O(N-1) \supset S_N^{orb}$, where S_N^{orb} is the symmetric group of the spatial particle coordinates permutations, the latter problem may be solved in a purely group-theoretical way [40]. We also say that the $O(N-1)$ invariant classical model is prequantized when the initial conditions are chosen so that the part of the phase space associated with the forbidden values of the invariants C_{2p}' is truncated.

V. Relation between the collective dynamics in O(N–1) invariant model and in the FLDM

Equation (3.4–3.5) may be deduced from the TDHF theory formulated in terms of the Wigner distribution function [14] which is the Fourier transform of the one-body density matrix. The relations obtained by integrating the equation for the Wigner distribution function $f(\mathbf{r}, \mathbf{p}, \mathbf{t})$ over momenta $(\mathbf{p})$ with the weights $1, p_i, p_i p_j$, etc., constitute an infinite set of equations for various local fields: for density, collective velocity, pressure, etc. [19]. This set includes the continuity equation.

The analogy of these equations with equations of the classical hydrodynamics opens the way to the use of well-developed mathematical methods. One way to do it is realized in ref.19, where the method of "virial theorems" by Chandrasekhar and Lebovitz [24] is

generalized to the description of the collective motion in the droplet of a Fermi liquid. Following this way, one establishes the generalized virial theorems, i.e. the relations between the integral characteristics appearing after taking the moments from the equation for the Wigner function over the full phase space of a nucleon. Then one introduces approximations leading to physical models parameterizing the collective velocity field $\mathbf{u}(\mathbf{x},t)= m^{-1} \int dp^3 \mathbf{p} f(\mathbf{x},\mathbf{p},t)/\rho(\mathbf{x},t)$ $(\rho(\mathbf{x},t)= \int dp^3 f(\mathbf{x},\mathbf{p},t))$ and other local fields.

When the collective velocity is approximated by a linear function of coordinates, the following moments of the distribution function are involved in the equations of motion: $< x_X x_Y >, < x_X p_Y >, < p_X p_Y >$, where $< \phi(\mathbf{x},\mathbf{p},t) >= \int\int dp^3 dx^3 \phi(\mathbf{x},\mathbf{p},t) f(\mathbf{x},\mathbf{p},t)$. These moments constitute a set of collective variables identical to the one discussed in Section III. In particular, the Fermi energy tensor Y and the vorticity vector $\vec{J}$ appear in FLDM, and equations (3.5) follow exactly.

In the FLDM the same expression for the kinetic energy is obtained as in the formulas in Section II. No explicit statement of the invariance properties of the nucleon–nucleon interaction is made. The potential and its gradients entering into the dynamical equations for the collective variables are related to the nucleon-nucleon interaction following the selfconsistency rules of the HF theory. Their dependence on the collective variables is obtained using these rules and the continuity equation to adjust variations of the integrals entering equations of the motion with collective currents. In applications the selfconsistent potential is often substituted by empirically defined one describing the properties of nuclear matter [42,43], e.g. by using the surface tension plus Coulomb potential. Disregarding the difference in the definition of the potential functions $U^0(\lambda_A, \lambda_B, \lambda_C)$, one may say that eq.(3.4) and with it the collective dynamics considered before is satisfied exactly in the FLDM as well.

The parallels between the description of the $O(N-1)$-invariant system given in the previous sections and in the hydrodynamical picture rest on the properties of the single-particle orbits: as follows from eq.(3.16b), the particle collective velocity $\mathbf{p}^{coll}/m$ is a linear

function in its coordinates with the same coefficients for all particles. Thus, the velocity field corresponding to the $O(N-1)$ invariant system is a linear function in the coordinates. It is precisely this property of the collective flow which underlies the Dirichlet-Riemann theory embodied in FLDM.

As is known from studies by many authors (see ref.[19,42] and the references quoted there), the Fermi-energy tensor Y plays a decisive role in the description of the nuclear giant quadrupole resonance: it determines the part of the elasticity coming from the quadrupole deformation of the Fermi-surface. In quantum systems the deformation of the geometrical surface is accompanied by the distortion of the Fermi-surface. The stresses due to the deformation of the Fermi-surface turn out to be much more important for the description of the energy of the giant resonances than those which are originated by the potential energy.

The low–energy excited states are quite sensitive to the potential. The rules of FLDM are quite flexible and much simpler than those obtained by the direct application of algebraic methods. A number of potentials have already been tested in different physical problems in the framework of FLDM [19,41-44]. However, the field of applications of this method is limited by the fact, that an information on the properties relevant to the noncollective motion is missing here. This drawback may be overcome using equations described in the previous sections with simple but realistic potential functions.

VI. Model of S-bands

Among the solutions to the differential equations (3.4–5) of special importance are the stationary ones describing the states of equilibrium of atomic nuclei. Within the DLD model one distinguishes the two families of the stationary solutions which are called in the literature S and P–Riemann ellipsoids. The problem of classification of the stationary solutions of $Sp(6,R)$ model is more complicated, and it will be discussed in a forthcoming paper. Below we restrict our attention to a family of stationary solutions which generalize

the concept of the S–Riemann ellipsoids. We define them stating the following set of conditions:

$$L_A = L_B = J_A = J_B = 0, \tag{6.1a–d}$$

$$Y_{AB} = Y_{BC} = Y_{CA} = 0, \tag{6.2a–c}$$

$$p_A = p_B = p_C = 0, \tag{6.3a–c}$$

$$\Lambda_C = \frac{J_C}{m}\frac{Y_{AA}/\lambda_A^2 - Y_{BB}/\lambda_B^2}{Y_{AA} - Y_{BB}}, \tag{6.4}$$

$$\omega_X^2 = (m\lambda_X)^{-1}(\partial U^0/\partial\lambda_X), \qquad (X = A, B, C), \tag{6.5a, b, c}$$

where the frequencies ω_X (see eq.(2.20)) can be rewritten in the following form:

$$\omega_A^2 = \frac{Y_{AA}}{m^2\lambda_A^4} + \Omega_C^2 + \Lambda_C^2 - 2(\lambda_B/\lambda_A)\Omega_C\Lambda_C, \tag{6.6a}$$

$$\omega_B^2 = \frac{Y_{BB}}{m^2\lambda_B^4} + \Omega_C^2 + \Lambda_C^2 - 2(\lambda_A/\lambda_B)\Omega_C\Lambda_C, \tag{6.6b}$$

$$\omega_C^2 = \frac{Y_{CC}}{m^2\lambda_C^4}. \tag{6.6c}$$

It is straightforward to verify that the above set of conditions describes stationary solutions to the system of motion equations (3.4 –5). However, the relations (6.1a – 6.5c) are insufficient to determine the elements of the matrix $\mathbf{C}(\rho)$. On the grounds of Section IV the extremes of the Hamiltonian function should be searched in the orbit space $\mathcal{O}_{\vec{f}:L}$. Since, using the ***Proposition*** (4.3) and eqs.(4.12) we find the following expressions for the diagonal elements of Fermi-energy tensor

$$Y_{AA(BB)} = \frac{1}{2}(h_A^2 + h_B^2 - J_C^2) \pm \frac{1}{2}\sqrt{[(h_A + h_B)^2 - J_C^2][(h_A - h_B)^2 - J_C^2]}, \tag{6.7a, b}$$

$$Y_{CC} = h_C^2. \tag{6.7c}$$

where the vector $\vec{h} = (h_A h_B h_C)$ can be chosen in the three nonequivalent ways:

$$\vec{h}_L = (f_1, f_2, f_3), \qquad \vec{h}_M = (f_1, f_3, f_2), \qquad \vec{h}_H = (f_3, f_1, f_2). \tag{6.8}$$

Now, for given values of L^2, $\vec{h}$ the equations (6.1a–6.7c) provide a complete set of algebraic equations. The fact that Y_{XX} are positively defined restricts the range of the variable J_C (see eqs.(6.7a,b)) to the interval $|J_C \leq h_A - h_B$. The transformation $\vec{L} \to -\vec{L}$, $\vec{J} \to -\vec{J}$ admitted by the Hamiltonian symmetry allows us to use only the nonpositive values of J_C from the above interval. In *Appendix C* the equilibrium conditions (6.4, 6.5a,b,c) are presented in a form more convenient for calculations. For given values $\vec{h}, J_C$ the equations *eq.*(C.4a,b,c) supplemented by expressions (6.7$a-c$) allow us to calculate the principal axes $\lambda_A, \lambda_B, \lambda_C$, while $L = |L_C|$ is calculated from eq.(C.1).

In the space $\mathcal{O}_{\vec{f}}$ the stationary solutions form one parameter families. The three vectors $\vec{h}$ appearing in eq.(6.8) divide the solutions into the three subfamilies. We will denote them as $S(\vec{h})$-bands.

Due to equations (3.25,6.5$a-c$) the matrix $\mathbf{H}_b$ depends on the four parameters $\omega = (\omega_A \omega_B \omega_C \Omega_C)$, $(\Omega_A = \Omega_B = 0)$ in exactly the same way as the matrix $\mathbf{H}_f$ in the standard CHOM. Furthermore, since for the equilibrium configurations the matrixes $\mathbf{C}_b$ and $\mathbf{H}_b$ commute (see eq.(3.32)), thus under a symplectic transformation $\mathcal{G}(\omega)$ known from Valatin work [45], we might simultaneously represent both matrices at the canonical form $\bar{\mathbf{H}}(-\vec{\omega}')=\mathrm{Ad}_{\mathcal{G}(\omega)}(\mathbf{H}_b(\omega))$, $\bar{\mathbf{C}}(\vec{h})=\mathrm{Ad}_{\mathcal{G}(\omega)}(\mathbf{C}_b)$, i.e. in the same form as we used in formula (4.2). Owing to the conditions (6.7$a-c$) the vector $\vec{h}$ remains unchanged throughout the band. Let us note that the same is true for the CHOM bands. Hence for the equilibrium band we can write $\mathbf{C}_b = \mathbf{C}_b(\vec{h}, \omega) = \mathbf{C}_f$. The eigenvalues of $\mathbf{H}_b(\omega)$ are already known [47,48]

$$\omega'^2_{A(B)} = (\omega_A^2 + \omega_B^2)/2 + \Omega_C^2 \mp \frac{1}{2}\sqrt{(\omega_A^2 - \omega_B^2)^2 + 8 \cdot \Omega_C^2(\omega_A^2 + \omega_B^2)}, \quad \omega'_C = \omega_C. \tag{6.9}$$

Noticing the canonically invariant character of expression (3.31b) for the Routhian function we immediately find $R = \vec{h} \cdot \vec{\omega}'$. Inserting here $\vec{\omega}'$ from eq.(6.9) and comparing the derivatives $\partial/\partial\omega_X^2$ of R obtained from the last expression with the derivatives of R found from expression (3.31a) we restate the standard formulas of CHOM [47,48]

$$\begin{aligned} \lambda_X^2 &= \frac{1}{2m}\left\{\left(\frac{h_A}{\omega'_A} + \frac{h_B}{\omega'_B}\right) \pm \frac{\omega_A^2 - \omega_B^2 \pm 4\Omega_C^2}{\omega'^2_A - \omega'^2_B}\left(\frac{h_A}{\omega'_A} - \frac{h_B}{\omega'_B}\right)\right\}, \quad X = A, B; \\ \lambda_C^2 &= \frac{h_C}{m\omega_C}, \end{aligned} \tag{6.10}$$

Using the formulas (3.1, 3.17, 3.20) and (6.1a – 6.4, 6.6$a-c$) we arrive at the following equations for the remaining elements of matrix $\mathbf{C}_b$:

$$X_{AC} = X_{CA} = X_{CB} = X_{BC} = 0, \qquad \text{for } X = S, L, P, \tag{6.11}$$

$$S_{AA} = S_{BB} = S_{CC} = 0, \tag{6.12}$$

$$P_{AA} = m^2[(\omega_A^2 - \Omega_C^2)\lambda_A^2 + \Omega_C^2\lambda_B^2], \qquad P_{BB} = m^2[(\omega_B^2 - \Omega_C^2)\lambda_B^2 + \Omega_C^2\lambda_A^2], \tag{6.13a,b}$$

$$P_{CC} = m^2\omega_C^2\lambda_C^2, \tag{6.13c}$$

$$S_{AB} = m(\lambda_A^2 - \lambda_B^2)\Omega_C, \qquad P_{AB} = 0, \tag{6.14a,b}$$

$$L_{AB} = m(\lambda_A^2 + \lambda_B^2)\Omega_C - 4m(\omega_A^2 - \omega_B^2)^{-1}[\lambda_A^2(\omega_A^2 - \Omega_C^2) - \lambda_B^2(\omega_B^2 - \Omega_C^2)]\Omega_C. \tag{6.14c}$$

Within the frequently used semiclassical approach for CHOM [46-51] under the Valatin transformation $\mathcal{G}(\omega)$ one writes $\hat{R} = \sum \omega'_X \hat{h}_X$, where $\hat{h}_X$ are commuting elements of the $sp(6, R)$ algebra. The quantum solutions are found as the eigenstates of the Routhian operator $\hat{R}$

$$\hat{R}(\omega)|\vec{f}, \vec{h}; \omega, t> = \sum_X \omega'_X h_X |\vec{f}, \vec{h}; \omega, t> \tag{6.15}$$

In this formula h_X denote the eigenvalues of the operators $\hat{h}_X$, $\vec{f}$ is the lowest weight vector of the IUR of the $sp(6, R)$ algebra, while the other quantum numbers distinguishing the state are omitted. From the above we conclude that the stationary solutions constituting the three bands $S(\vec{h}_X)$,$X = L, M, H$ are represented as the coherent vector states. In particular, the solutions consist of the $S(\vec{h}_L)$, (and forming the yrast line within $sp(6, R)$

multiplet), are associated with lowest weight vector state $|\vec{f},\vec{f};\omega,t>$. Moreover, since the formulas (6.10-6.14c) remain valid in the semiclassical level as expectation values of $sp(6,R)$ observables, the bridge between the $S(\vec{h})$ bands and between the bands of the CHOM ensures the consistence of the semiclassical rules stated at the end of Section IV with the semiclassical CHOM approach.

The algebraic model of the collective bands presented above extends the range of the application of the CHOM model. The new element is contained in the formulas $(6.5a-c)$. They express the selfconsistency conditions generated by collective potential U^0, which connect the distribution of matter with distribution of the mean field.

Let us consider, as an example, a very simple form of the isotropic volume-dependent potential:

$$U(\lambda_A,\lambda_B,\lambda_C)^0 = 3/p\cdot(\lambda_A\lambda_B\lambda_C/\mathring{\lambda}^3)^{p/3}\frac{\hbar^2}{m\mathring{\lambda}^2}. \tag{6.16}$$

In eq.(6.16) the parameter p fixes the compressibility of a nucleus, while the length parameter $\mathring{\lambda}$ fixes the value of the geometrical nuclear radius $R_0=(R_AR_BR_C)^{1/3}$ in the ground state, i.e. for $J_C=L_C=0$. Assuming nuclear surface having an ellipsoidal form $\sum x_X^2/R_X^2=1$ and a homogeneous distribution of the nuclear matter we obtain $\lambda_0^2=NR_0^2/5$. Here the ground state value of $\lambda_0=(\lambda_A\lambda_B\lambda_C)^{1/3}$ is given by eqs.(C.10a,b) and we get

$$\mathring{\lambda}^2 = NR_0^2F_0^{-4/(p+2)}/5, \qquad F_0=(f_1f_2f_3)^{1/3}. \tag{6.17a,b}$$

Equations $(6.5a-c)$ take the form

$$\omega_A^2\lambda_A^2-\omega_B^2\lambda_B^2=0, \qquad \omega_A^2\lambda_A^2+\omega_B^2\lambda_B^2-2\omega_C^2\lambda_C^2=0, \tag{6.18a,b}$$

$$2T=pU^0, \tag{6.18c}$$

and from relations $(6.17a, 6.18c, C.9a,b)$ we find for the total energy E

$$E=\frac{(p+2)}{p}(F/F_0)^{2p/(p+2)}a, \qquad a=\frac{15}{2NR_0^2m}F_0^2. \tag{6.19a,b}$$

The function F is defined in eq.(C.8) and depends only on the invariants of motion $\vec{f}$ and J_C ($L_C=L_C(J_C)$). The parameter a determines the noncollective part of the kinetic energy T^{intr}.

It is interesting to consider the following function:

$$\bar{F}_0(\mathcal{Z},\mathcal{N}) = \min_{\vec{f}} F_0(\vec{f}), \tag{6.20}$$

where the minimization is done by taking into account the Pauli principle for fixed proton and neutron particle numbers $\mathcal{Z},\mathcal{N}$. The weight vector $\vec{f}_0$ corresponding to F_0 determines the ground $Sp(6,R)$ multiplet of collective states. The estimate of the smooth part of $\bar{F}_0$ may be found using equations given in [3] (Vol:2,Ch.6,eq.(6. 510)). One has: $\bar{F}_0^{sm}(\mathcal{Z},\mathcal{N})\approx[(3\mathcal{Z})^{4/3}+(3\mathcal{N})^{4/3}]/12\approx(3N/2)^{4/3}/6$. Then $\bar{a}$ establishes the Fermi-energy and one finds for the smooth part of this quantity

$$\bar{a}^{sm}\approx 17.7\cdot N\cdot \mathrm{MeV}, \tag{6.21}$$

Notice that the Fermi-energy tensor is important for the description of the nuclear shape. In particular, the shape in the ground state is determined by this tensor. This could be seen even without calculation by considering eqs.$(6.5a-c)$ and $(6.6a-c)$. Assuming that the potential energy $U^0(\lambda_A,\lambda_B,\lambda_C)$ is a symmetric function of its arguments one concludes that eq.$(6.5a-c)$ could be fulfilled at $\lambda_A=\lambda_B=\lambda_C$ only if $Y_{AA}=Y_{BB}=Y_{CC}$. For the volume dependent potential $\lambda_X\sim f_X$. Thus, only those nuclei are spherical in their ground states which have the ground-state vector $\vec{f}$ with three equal f_X indices. The Pauli principle admits equal values of f_X for the lowest configurations only in doubly magic nuclei. In other words, the $O(N-1)$ invariant model explains the spherical symmetry of the ground states of nuclei with closed harmonic oscillator shells and the loss of the spherical symmetry in nuclei with open shells.

The influence of the Fermi-energy tensor on the nuclear shape is particularly important in the case of a volume dependent potential. Such potential considered within the DLD

model leads to a quadrupole instability of the nuclear shape for $L > 0$. Notice, however, that in nuclei the surface and Coulomb forces play an important role in establishing the shape.

The equilibrium conditions (6.18a, b) have been determined earlier for two particular submodels discussed in detail by Troudet and Arvieu [47,49]. The first results from assuming that the product of oscillator frequencies $\omega_A\omega_B\omega_C$ is constant through all the band. We leave out the proof that substituting $p = 2$ we will reproduce this model. Considering the limit $p \to \infty$ (or $p \to -\infty$) we arrive at the second model, for which the geometrical radius R_0 is common for all values of L. The equilibrium relations to this model have been found in ref.[49]. A simplified version of the selfconsistent CHOM bands based on the approximation suggested by Bohr and Mottelson (Ref.[3],Vol.2,Ch.4.§3) has been investigated in [50] (see also *Appendix C*).

From equations in *Appendix C* it is easy to notice that the functional dependence of $L, z = (\lambda_A^2 - \lambda_B^2)/(\lambda_A^2 + \lambda_B^2)$ and $\lambda_A\lambda_B/\lambda_C^2$ on the vortex spin J is given by universal functions. This means that the results of this Section reduce the investigation of the model of S–bands for all the potentials belonging to the class (6.16), i.e. for all values of p, to a problem that has already been worked out [47,49].

VII. Summary

The content of previous sections sheds light on the interplay of nuclear collective and individual–particle phenomena.

The nuclear dynamics is approached in this paper using the mathematically transparent parameterization of the complete set of the N–body coordinates in which the components of the inertia tensor are taken as the six collective coordinates. A rather formal way of introducing the complementary coordinates and the complexity of N–body eigenfunctions musk the individual-particle motion and create a gap between this formulation of the theory and alternative microscopic approaches. The token remains true in the case when

the assumptions of the $O(N-1)$ invariance of the Hamiltonian is introduced. However, in the classical limit of exact solutions of the $O(N$–$1)$–invariant model the nature of the collective and single–particle motions as well as their mutual dependence are amazingly transparent. It is precisely this limit what is examined in a detailed way here. Our findings may be summarized as follows:

1) The nuclear collective motion is described by the system of equations including in addition to the inertia tensor and the angular momentum vector two additional elements: Fermi energy tensor Y and vorticity (circulation) vector $\vec{J}$. These two quantities depend on the coordinates and on the momenta conjugated to the coordinates complementary to the inertia tensor components.

2) The motion of individual particles is determined by the time dependent harmonic field. The parameters of the field are unambiguously given by the collective variables and by general invariants of the collective motion. In the other words, in the classical limit to the $O(N$–$1)$ invariant model, the motion of the nucleons is governed exactly by the time dependent selfconsistent field.

3) The variables describing the collective motion may be expressed in terms of the generators of the $Sp(6, R)$ group, which allows the geometrical analysis of the model. The evolution of the system is described by the $Sp(6, R)$–group action in the $sp(6, R)$ vector space spanned by the collective variables algebra. Due to the participation of Y and $\vec{J}$ in the enveloping field of this algebra, this space is larger than the collective space of the most existing models considering the quadrupole deformation of nuclei. This is true, in particular, in respect to the unified model where neither of these elements participates in the collective dynamics.

4) Considering the $Sp(6, R)$ orbits in the phase space one finds explicit relations for invariants of the motion in terms of Y and $\vec{J}$. Some important geometrical constraints on these quantities are obtained in the general case. A detailed analysis is given for the classes of orbits with dimensions lower than the maximal one.

5) Equations of the collective motion allow the stationary solutions. Hereabout a family of such solutions called as S–bands is examined. The S–bands establish a link between the theory of S–Riemann stationary solutions and the selfconsistent CHOM. Properties of S–bands manifest an important role which play the circulation of the collective velocity $(\vec{J})$ and the anisotropy in the distribution of the nucleons in the momentum space (Y) in determining the shapes and inertial properties of rotating nuclei.

6) It turns out that in the case of a one parametric family of the volume dependent potentials including two limits considered in the literature the solutions for S-bands admit a scaling which reduces the problem to the one investigated before.

7) Parallels are established between the description of the collective motion in the $O(N-1)$ invariant model and in the FLDM. In the latter the same equations of the collective motion as derived in this paper appear as approximate solutions of the TDHF theory without any constraints on the local part of the selfconsistent field. The FLDM is successfully applied to the description of the bulk properties of the distribution of strengths of the multiple transitions. This may serve as a test of applicability of the model described in this paper.

We hope that the material presented before will be useful for a number of future investigations, for example, for the classification of all stationary solutions and for the description of nuclear collective rotational and vibrational bands. The need of the generalization of existing models for such an analysis is clearly manifested by the recent discovery of the $SU(3)$ multiplets in the superdeformed rotating nuclei [52].

Acknowledgements.

One of the authors (M.C.) acknowledges the hospitality of the Laboratory of Theoretical Physics at JINR where the bulk of the work on this paper was done. Both express their gratitude to CSNSM (Orsay) for providing excellent conditions to accomplish this work.

This work is partially supported by the KBN grant no 2.0204.91.01.

Appendix A

The algebra of variables Y_{AA}, Y_{BB}, Y_{AB} and J_{AB} becomes transparent after mapping onto the four-dimensional algebra $u(2)$ $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ with the structural constants given by the equations

$$f^k_{ij} = 2\epsilon_{ijk} \qquad f^j_{0i} = f^j_{i0} = f^0_{ij} = 0 \qquad \text{for } i,j,k = 1,2,3,$$

The realization of the quantities $\tilde{J}_{XY}, Y_{XY} (X, Y \in \{A, B\})$ reads

$$Y = Y_{AA} + Y_{BB} = \frac{1}{2}(\sigma_0^2 + \sigma_1^2 - \sigma_2^2 + \sigma_3^2), \tag{A.1a}$$

$$\Delta Y = Y_{AA} - Y_{BB} = \sigma_1 \times (\sigma_0^2 - \sigma_2^2)^{1/2}, \tag{A.1b}$$

$$Y_{AB} = \sigma_3 \times (\sigma_0^2 - \sigma_2^2)^{1/2}, \tag{A.1c}$$

$$\tilde{J}_{AB} = \sigma_2. \tag{A.1d}$$

In fact, using the relation

$$\{F, G\} = (\partial^i F)(\partial^j G) f^k_{ij} \sigma_k,$$

we recover the Poisson brackets

$$\{\tilde{J}_{AB}, Y\} = 0, \quad \{\tilde{J}_{AB}, \Delta Y\} = -4Y_{AB}, \quad \{\tilde{J}_{AB}, Y\} = \Delta Y, \tag{A.2a – c}$$

$$\{Y, \Delta Y\} = 8\tilde{J}_{AB} Y_{AB}, \quad \{Y, Y_{AB}\} = 2\tilde{J}_{AB}\Delta Y, \quad \{Y_{AB}, \Delta Y\} = 2\tilde{J}_{AB} Y, \tag{A.2d–f}$$

identical with those in eqs.(3.3,3.4).

The relations $(A.1a-d)$ may be used to quantise the Hamiltonian of the two dimensional version of the $O(N-1)$-invariant model. It is achieved by mapping $\{q\}$ onto the set of the self-adjoint operators of the algebra $u(2)$. A remark should be added concerning the generalization of the theory to the three-dimensional case. In [22] the formulas are developed for the matrix elements of the operators $\tilde{J}$ defined as operators in the many body subspaces $\mathcal{S}^N(f_1 f_2 f_3)$ (see eq.(4.13b)). It is found that the dimension of the matrix

representation of the $\tilde{J}$ obtained in this way coincides with that of the irreducible representation of the $u(3)$ algebra. However, no explicit relations are known for $\tilde{J}, Y$ operators in terms of the generators of the $u(3)$ algebra.

Appendix B

Some comments are necessary to establish the domain in which the classical expressions (4.12) for the eigenvalues of the Casimir invariants C_{2p} and semiclassical relations $(6.7a-c)$ are valid. We are going to show that the approximation (4.12) works much better than one might think, looking at the exact formulas for these invariants, and that eqs.$(6.7a-c)$ can be applied even if the number of nucleons is not very large. To discuss this problem, we apply the quantum expression for the Hamiltonian [9]

$$\hat{H}^{vib} = -\frac{\hbar^2}{2m}\sum_A \left\{ \frac{\partial^2}{\partial\lambda_A^2} + \frac{N-4}{\lambda_A}\frac{\partial}{\partial\lambda_A} + 2\sum_{B(\neq A)} \frac{\lambda_A}{\lambda_A^2-\lambda_B^2}\frac{\partial}{\partial\lambda_A} \right\}, \qquad (B.1)$$

The transformation

$$\hat{F}' = \hat{f}^{-1}\hat{F}\hat{f}, \qquad (B.2)$$

$$\hat{f} = (\lambda_A\lambda_B\lambda_C)^{-(N-4)/2}[(\lambda_A^2-\lambda_B^2)(\lambda_B^2-\lambda_C^2)(\lambda_A^2-\lambda_C^2)]^{-1/2}, \qquad (B.3)$$

affects only the term $\hat{H}^{vib}$ which transforms into

$$(\hat{H}^{vib})' = -\frac{\hbar^2}{2m}\sum_A \frac{\partial^2}{\partial\lambda_A^2} + \hat{U}_1^{quan} + \hat{U}_2^{quan}, \qquad (B.4)$$

$$\hat{U}_1^{quan} = \frac{\hbar^2}{2m}\frac{(N-5)^2-1}{4}\sum_A \frac{1}{\lambda_A^2}, \qquad (B.5)$$

$$\hat{U}_2^{quan} = -\frac{\hbar^2}{2m}\sum_A \left\{ \frac{\lambda_A^2+\lambda_B^2}{(\lambda_A^2-\lambda_B^2)^2} + \frac{\lambda_A^2+\lambda_C^2}{(\lambda_A^2-\lambda_B^2)^2} \right\}$$
$$+\frac{\hbar^2}{2m}\sum_A \lambda_A^2 \left\{ \frac{1}{(\lambda_A^2-\lambda_B^2)} + \frac{1}{(\lambda_A^2-\lambda_C^2)} \right\}^2, \qquad (B.6)$$

The term $\hat{U}_1^{quan}$ proportional to N^2 is much more important than $\hat{U}_2^{quan}$. Neglecting $\hat{U}_2^{quan}$ and adding $\hat{U}_1^{quan}$ to the intrinsic kinetic energy one obtains (B.7) with Y' being given by $(B.8)$

$$(\hat{T}^{intr})' = \frac{1}{2m}\sum_X Y'_{XX}/\lambda_X^2, \qquad (B.7)$$

$$Y'_{XY} = Y_{XY} + \frac{1}{4}[(N-5)^2-1]\hbar^2\delta_{XY}. \qquad (B.8)$$

The Y' tensor has a slightly different meaning as compared with the definition given in the principal part of this paper. For IUR $(f_1'f_2'f_3'0\ldots0)$ of the algebra $so(N-1)$ we have $C_2 = 2\sum_{i=1}^3 f_i'(f_i'+(N-5)\hbar)+4(f_1'-f_3')\hbar$. Hence, using eq.$(4.8a)$ and noticing from the formula $4.16b)$ that $f_i' = f_i-(N-1)\hbar/2$ we arrive at

$$\sum_A (Y'_{XX}+J_X^2) = \frac{1}{2}C_2 + \frac{3}{4}((N-5)^2-1)\hbar^2 \approx f_1^2+f_2^2+f_3^2. \qquad (B.9)$$

For the same reason the eigenvalues of the quantities C_4' and C_6' should be renormalized as well. In quantum mechanics the change of the order of j-operators in eqs.(4.14) changes the invariant operators and their eigenvalues. One may use this ambiguity and choose undefined terms such that the relations (4.12) become valid also for C_4', C_6'.

Appendix C

It is useful to rewrite the system of equations (6.4), $(6.5a-c)$ in a somewhat different way. Equation (6.4) may be written in the form

$$L_C = -J_C\frac{1}{\sqrt{1-z^2}}\left[1-\frac{2z^2}{1-z^2}(1-zY/\Delta Y)\right], \qquad (C.1)$$

where

$$z = \frac{\lambda_A^2-\lambda_B^2}{\lambda_A^2+\lambda_B^2}, \qquad Y = Y_{AA}+Y_{BB}, \qquad \Delta Y = Y_{AA}-Y_{BB}, \qquad (C.2a,b,c)$$

while eqs.(6.4$a-c$) read

$$(m\lambda_A\lambda_B)^{-1}(1-z^2)^{-5/2}\sum_{n=0}^{5}A_n z^n = (\lambda_A\frac{d}{d\lambda}-\lambda_B\frac{d}{d\lambda})U^0, \qquad (C.3a)$$

$$T_{AA}+T_{BB}-2T_{CC} = \frac{1}{2}(\lambda_A\frac{d}{d\lambda_A}+\lambda_B\frac{d}{d\lambda_B}-2\lambda_C\frac{d}{d\lambda_C})U^0, \qquad (C.3b)$$

$$T_{AA}+T_{BB}+T_{CC} = \frac{1}{2}\sum_X \lambda_X\frac{d}{d\lambda_X}U^0. \qquad (C.3c)$$

In eqs.(C.3a) the coefficients A_n are found to be:

$$A_0=\Delta Y, \qquad A_1=\frac{5}{2}J_C^2-Y, \qquad A_2=-2\Delta Y-6B_1,$$
$$A_3=-2A_1+4B_2, \qquad A_4=\Delta Y+2B_1, \qquad A_5=A_1-2B_2, \qquad (C.4)$$
$$B_1=J_C^2Y/\Delta Y, \qquad B_2=J_C^2[1+(Y/\Delta Y)^2].$$

The quantities T_{XX} in eqs.(C.3b,c) are the components of the kinetic energy tensor corresponding to the stationary solution

$$T_{AA}+T_{BB} = (m\lambda_A\lambda_B)^{-1}(1-z^2)^{-5/2}\sum_{n=0}^{5}E_n z^n, \qquad (C.5)$$

$$T_{CC} = Y_{CC}/(2m\lambda_C^2). \qquad (C.6)$$

where

$$E_0=J_C^2/4+Y/2, \qquad E_1=-\Delta Y/2, \qquad E_2=-2E_0,$$
$$E_3=-B_1+\Delta Y, \qquad E_4=B_2+E_0, \qquad E_5=-B_1-\Delta Y/2. \qquad (C.7)$$

The left-hand side of eq.($C.3c$) is the total kinetic energy of a nucleus.

Let us examine the above system choosing the potential U^0 in the form given in eq.(6.16). Let z_{eq}, λ_{eq} denote equilibrium values of $z, (\lambda_A\lambda_B\lambda_C)^{1/3}$ respectively and

$$F = \left[f_C(1-z_{eq}^2)^{-5/2}\sum_n E_n z_{eq}^n\right]^{1/3}. \qquad (C.8)$$

From eq.(C3.b) we get $T_{AA}+T_{BB}=2T_{CC}$, thus for the total kinetic energy T we find

$$T = \frac{3}{2}[2T_{CC}(T_{AA}+T_{BB})^2]^{1/3} = \frac{3}{2}(F/\lambda_{eq})^2/m,$$

and using eq.(C.3c) we get $\lambda_{eq} = \lambda\!\!\!^{-}(F/\hbar)^{2/(p+2)}$. Let λ_0, F_0, z_0 denote values of λ_{eq}, F and z in the ground state i.e. at the point J_C=L_C=0. From eq.($C.3a, C4, 6.7$) we get $z_0=(h_A^2-h_B^2)/(h_A^2+h_B^2)$ and by simple calculation we arrive at the following relations

$$T = a(F/F_0)^{2p/(p+2)}, \qquad a=\frac{3}{2m}(F_0/\lambda_0)^2, \qquad (C.9a,b)$$

$$F_0 = (f_Af_Bf_C)^{1/3}, \qquad \lambda_0 = \lambda\!\!\!^{-}(F_0/\hbar)^{2/(p+2)}. \qquad (C.10a,b)$$

Let us assume also that $\mid z_{eq} \mid \ll 1$. Keeping the first two terms in the polynomial of z on l.h.s. of eq.($C.3a$) and using eqs.(6.7a,b, 6.8) we find

$$z_{eq} \approx \frac{2S}{(1+S^2)}\sqrt{1-[J/(h_A-h_B)]^2}, \quad S=\frac{(h_A-h_B)}{(h_A+h_B)}, \quad \vec{h}=\vec{h}_G, \; G=L,M,H. \quad (C.11)$$

Hence, the parameter S must be small too. For these assumptions the range of angular momentum within the nuclear band is limited: $O \le L_C \le (h_A-h_B)$. Indeed, from eq.(C.1) we get $L_C \approx -J_C$, so that we may assume J_C to be negative $-(h_A-h_B) \le J_C \le 0$. Considering the above equations, while preserving the lowest terms in the power of L_C, we obtain

$$F \approx [h_C(h_Ah_B+L_C^2/4)]^{1/3}. \qquad (C.12)$$

Substituting this factor into the energy formula (6.19a) and choosing the parameter $p=2$ we arrive at the result which coincides with an approximation which has been discussed in [50].

REFERENCES

[1] R.Bengtsson and S.Frauendorf, Nucl. Phys. **A 314**, 27(1979); **A 327**, 189(1979).

[2] P.Ring, R.Beck and H.J.Mang, Z.Phys. **231**, 10(1970).

[3] A.Bohr, B.R.Mottelson, *Nuclear Structure,* vol.I, W.A.Benjamin INC, New York, Amsterdam, 1969, vol.II, 1974.

[4] G.F.Filippov, V.I.Ovcharenko, Yu.F.Smirnov, *Microscopic theory of collective excitations in atomic nuclei*, Naukova Dumka, Kiev 1981.

[5] A.Ya.Dzyublik, preprint ITF−71 − 122*R*, Inst.Theor.Phys., Kiev,USSR.

[6] W.Zickendraht, J.Math.Phys. **10**, 30(1969); **12**, 1663(1971).

[7] A.Ya.Dzyublik, V.I.Ovcharenko, A.I.Steshenko, G.F.Filippov, Yad.Fiz **15**, 869(1972) [Sov.J.Nucl.Phys. **15**, 487(1972)]

[8] G.Rosensteel and D.J.Rowe, Ann.Phys. **123**, 36(1978).

[9] G.Rosensteel and D.J.Rowe, Ann.Phys. **126**, 198(1980).

[10] G.Rosensteel and D.J.Rowe, Ann.Phys. **126**, 343(1980).

[11] V.V.Vanagas and R.K.Kalinauskas, Yad.Fiz. **18**, 768(1973), [Sov.J.Phys. **18**, 395 (1974)].

[12] V.V.Vanagas, *The Microscopic Nuclear Theory within the Framework of the Restricted Dynamics*, Lecture Notes. Toronto, University Toronto,Dep. of Physics, 1977.

[13] P.Kramer, Ann.Phys. **141**, 254(1982); **141**, 269(1982).

[14] E.Wigner, Phys.Rev. **40**, 749(1932).

[15] G.F.Bertsh, in *Nuclear Physics with heavy ions and mesons*, V1, p.175-262, Editors: R.Balian, M.Rho, G.Ripka; N.H.197

[16] J.R.Nix, A.J.Sierk, Phys.Rev. **C21**,396(1980).

[17] D.M.Brink, M.Di Toro, Nucl.Phys. **A372**, 151(1981).

[18] V.M.Kolomietz, H.K.Tang, Physica Scripta (1981).

[19] E.B.Balbutsev, I.N.Mikhailov, Journal of Phys. **G14**,545(1988).

[20] D.J.Rowe, Rep.Prog.Phys. **48**, 1419 (1985).

[21] M.Moshinsky, J.Math.Phys. **25**, 1555(1984).

[22] O.Castanos, A.Frank, E.Chacon and M.Moshinky, J.Math.Phys. **23**, 2537(1982).

[23] E.Chacon, M.Moshinsky and V.Vanagas, J.Math.Phys. **22**, 605(1981).

[24] S.Chandrasekhar, *Ellipsoidal figures of equilibrium*, New Haven and London, Yale University, 1969.

[25] H.Lamb, *Hydrodynamics*, England, Cambridge University Press, 1932

[26] P.Gulshani and D.J.Rowe, Can.J.Phys. **54**, 970(1976).

[27] O.L.Weaver, R.Y.Cusson and L.C.Biedenharn, Ann.Phys **102**, 493(1976).

[28] O.A.Katkiavichius, V.V.Vanagas, Liet.Fiz.Rink., v.XXI, No 2, 3(1976).

[29] V.V.Vanagas, Liet.Fiz.Rink., v.XXII, No 2, 3(1982).

[30] R.Y.Cusson, Nucl.Phys. **A 114**, 289(1968).

[31] O.L.Weaver, R.Y.Cusson and L.C.Biedenharn, Ann.Phys **77**, 250(1973).

[32] G.Rosensteel and D.J.Rowe, Ann.Phys. **96**, 1(1976).

[33] B.Buck, L.C.Biedenharn and R.Y.Cusson, Nucl.Phys. **A3**,205(1979).

[34] G.Rosensteel Ann.Phys **186**, 230(1988)

[35] O.Castanos,E.Chacon and M.Moshinsky,J.Math.Phys. **25**,2995(1984).

[36] E.Chacon, P.O.Hess, M.Moshinsky, J.Math.Phys. **28**, 2223(1987).

[37] M.Moshinsky and P.Quesne, J.Math.Phys. **12**, 1772(1971).

[38] A.M.Perelomov and V.S.Popov, Izv.Acad.Sci. **2**, (Math. USSR), 1313(1968).

[39] C.O.Nwachuku, J.Math.Phys. **20**, 1260(1979).

[40] V.V.Vanagas, *Algebraic Methods in Nuclear Theory*, Mintis, Vilnius 1971.

[41] E.B.Balbutsev, I.N.Mikhailov, Z.Vaishvila, Nucl.Phys **A 457**, 222(1986).

[42] C.E.Rosenkilde J.Math.Phys. **8**, 98(1967)

[43] E.B.Balbutsev, I.E.Piperova, Sov.Nucl.Phys. **50**, 961(1989).

[44] E.B.Balbutsev, S.U.Bastrukov, I.N.Mikhailov, B.P.Sinichkin, Sov.Nucl.Phys. **50**, 1264 (1989).

[45] J.G.Valatin, Proc.Roy.Soc. **238**, 132(1956).

[46] G.Ripka, J.P.Blazoit N.Kassis, Invited lecture Internat. Extended Seminar, Trieste 1973, ed.IAEA, Vienna 1975, vol. I, p.445.

[47] T.Troudet and R.Arvieu, Ann.Phys. **134**, 1(1981).

[48] T.Troudet and R.Arvieu, Z.Phys. **A 291** (1979).

[49] T.Troudet and R.Arvieu, Phys.Lett. **B 82**, 308(1979).

[50] M.Cerkaski and Z.Szymanski A.Phys.Pol. **B10**, 163(1978).

[51] V.G.Zelevinski, Sov.J.Phys. **22**, 565(1975).

[52] R.V.F.Jansen and Teng Lek Khoo, Ann.Rev.Nucl.Part.Sci **41**, 321(1991).

Wydanie I *Zam. 38/92* *Nakład 90 egz.*