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ON GALILEAN COVARIANT QUANTUM MECHANICS

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Abstract:

Formalism exhibiting the Galilean covariance of wave mechanics is proposed. A new notion of quantum mechanical forces is introduced. The formalism is illustrated on the example of the harmonic oscillator.

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I. Introduction

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Galilean covariance is the most fundamental property of low energy physics. In spite of that there are many branches of low energy physics which lack from Galilean covariance formulation. This, highly undesirable, situation happened for example in classical mechanics of a single interacting particle where the usual way of defining the acting forces through some force laws breaks down the Galilean covariance. In order to prevent this catastrophe we have recently proposed [1] to determine the acting forces from Galilean covariant differential equations derived from the non-covariant force laws. The most important result of Part I consists in finding the correct transformation rule for the total energy of the interacting particle which is different from the known rule for its kinetic energy.

Part I of our paper, apart from its pedagogical value, opened also a new way of understanding the Galilean covariance in quantum mechanics. There exists a general proof [2] that quantum mechanics based on the Schrödinger equations is Galilean covariant but this proof contains the assumptions that the Hamiltonian of the interacting particle is constructed from a scalar potential. In view of Part I such an assumption must be considered as wrong because already in classical mechanics the potentials are not scalars with respect to the Galilean transformation and the question of a Galilean covariant formulation of quantum mechanics becomes open for a new discussion.

Our paper provides a new approach to Galilean covariant quantum mechanics. A central role in it, in analogy to classical mechanics, is played by the acting forces. As a matter of fact, in our approach we introduce a new notion of a quantum mechanical force which is independent from the classical force laws. Our formalism is not a quantization of the classical canonical formalism in which the non-covariant force laws are used at the very beginning. This fact implies some

essential differences between our appr
one. In particular, it turns out that
restricted class of states and this pro
relations in quantum physics.

We shall work in the framework
space-time [3] because, in contradisti
it admits a fundamental non-singular
metric is the fact that its maximal
group but de Sitter group SO(4,1) [
and Lorentz groups. Therefore, havi
wave mechanics for the interacting p
obtain a unitary representation of t
information on the dynamics of the p
an irreducible representation of the w
the de Sitter group obtained in this w
S0(3,1) which is isomorphic to the Lor
of the Lorentz group will contain th
energy dynamics of the interacting par
consider this way as one of the possibl
of particles without an explicit use of
the first step in that direction.

II. The non-covariance of clas

Through this paper we shall consi
chanics for scalar particles. The gener
and does not change our results.

III. Galilean covariant formulation of wave mechanics.

We shall develop our new approach to wave mechanics in the framework of the five-dimensional model of the Galilean space-time [3,6]. In this model, in addition to the usual space and time coordinates $\vec{x}$ and t, with the transformation rules (2.2), we use also a fifth coordinate θ with the transformation rule

$$\theta \to \theta' = \theta + R\vec{x}\cdot\vec{u} + \frac{1}{2}\vec{u}^{\,2}t + \varphi \tag{3.1}$$

where φ realizes the one-parameter central extension of the Galilei group [6]. The advantage of using the eleven parametric extended Galilei group instead of the usual ten parametric Galilei group primarily consists in the fact that the unitary projective representation (2.1) for the latter is replaced by the ordinary unitary representation

$$(U(R,\vec{u},\vec{a},b,\varphi)\Psi)(\vec{x}',t',\theta') = \Psi(\vec{x},t,\theta) \tag{3.2}$$

for the former but now the new wave functions depend on five coordinates $(\vec{x},t,\theta)$. It has been shown in ref. [7] that the five-dimensional approach to the Galilean covariant wave equations reproduces the results of the four-dimensional approach. Unfortunately all these discussion in the case of wave mechanics again applies only in the case of free particles.

Another advantage of the five-dimensional formulation of wave mechanics is connected with the Galilean transformation rules for quantum mechanical observables. In particular, identifying the operators $-i\hbar\frac{\partial}{\partial x_j}$ with the components of the momentum operator $\hat{\vec{p}}_j$ of the free particle, in the four-dimensional formulation of wave mechanics we get the transformations rule

$$\tilde{\alpha} = 0 \tag{3.20}$$

Quite similarly as for operators $\hat{X}_j$ we shall see that for the interacting particles $\tilde{\beta}$ does not vanish. From that it follows that the operator $-i\hbar\frac{\partial}{\partial x_j}$ is the momentum operator $\hat{P}_j$ only for free particles.

To proceed further let us now concentrate our attention on the Hamiltonian $\hat{H}$. From the Schrödinger equation (2.5) it follows that under Galilean transformations we should have

$$\hat{H} \to \hat{H}' = \hat{H} - i\hbar\vec{u}\cdot R\vec{\nabla} + \frac{i\hbar}{2}\vec{u}^{\,2}\frac{\partial}{\partial\theta} \tag{3.21}$$

Comparing this with the transformation rule [1]

$$E \to E' = E + \vec{u}\cdot R\vec{P} + \frac{1}{2}m\vec{u}^{\,2} \tag{3.22}$$

for the total energy of the particle, we see that all physical states must be eigenstates of the mass operator $i\hbar\frac{\partial}{\partial\theta}$ and that the operator $-i\hbar\vec{\nabla}$ has to be interpreted as the quantum mechanical counterpart of the time independent quantity $\vec{P}$ present in the transformation rule (3.22) and formed from the momentum of the particle $\vec{p}(t)$ and quantities describing the interaction. Since only for free particles

$$\vec{P} = \vec{p}(t) = \overrightarrow{\text{const}} \tag{3.23}$$

we must conclude that only for free particles the operator $-i\hbar\vec{\nabla}$ is the momentum operator.

In Galilean physics time has an absolute meaning and therefore the total classical time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} + \frac{1}{2}\vec{v}^{\,2}\frac{\partial}{\partial\theta} \tag{3.24}$$

is invariant under the change of the reference frame. Translating this condition into the quantum mechanical language we must require that

$$\left[\hat{H}, \hat{H}'\right] = 0 \tag{3.25}$$

because otherwise the time derivatives defined by the operators $\hat{H}$ and $\hat{H}'$, respectively, will be incompatible. The time evolution of two observables in two different inertial reference frames should then have different physical meaning. Condition (3.25) is just that condition which is usually not taken into account in all discussion of the Galilean covariance of wave mechanics.

From (3.21) and (3.25) it once again follows that

$$\left[\frac{\partial}{\partial x_j}, \hat{H}\right] = 0 \tag{3.26}$$

and therefore the Hamiltonian $\hat{H}$ may be of the following general form

$$\hat{H} = \sum_{j=1}^{3}\left\{A(f^2)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} + B(f^2)\frac{\partial}{\partial x_j}\frac{\partial}{\partial f_j} + C(f^2)f_j\frac{\partial}{\partial x_j} + \right.$$
$$\left. + D(f^2)f_j\frac{\partial}{\partial f_j} + E(f^2)\frac{\partial}{\partial f_j}\frac{\partial}{\partial f_j}\right\} + V(f^2) \tag{3.27}$$

where all coefficients are arbitrary functions of the indicated argument. Under the Galilean transformation this expression behaves according to the rule

$$\hat{H} \to \hat{H}' = \hat{H} = A(f^2)\left(-2\vec{u}\cdot R\vec{\nabla} + \vec{u}^2\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial\theta} - \\ - \left[B(f^2)\vec{u}\cdot R\frac{\partial}{\partial\vec{f}} + C(f^2)\vec{u}\cdot R\vec{f}\right] \tag{3.28}$$

Comparing this with the required transformation rule (3.21) and taking into account that all states must be eigenstates of the mass operator $i\hbar\frac{\partial}{\partial\theta}$ we get

$$A\left(\vec{f}^2\right) = -\frac{\hbar^2}{2m} \tag{3.29}$$

$$B\left(\vec{f}^2\right) = C\left(\vec{f}^2\right) = 0 \tag{3.30}$$

From (3.10), (3.12), (3.18) and (3.27) we get the conditions

$$D\left(\vec{f}^2\right) = 0 \tag{3.31}$$

$$\tilde{\beta} = \frac{2imE\left(\vec{f}^2\right)\alpha}{\hbar} \tag{3.32}$$

from which it follows that the function $E\left(f^2\right)$ is a constant. From (3.11), (3.13) and (3.27) we finally get the relation

$$F_j\left(\vec{f}^2\right) = \frac{2mE\alpha}{\hbar^2}\frac{\partial V\left(\vec{f}^2\right)}{\partial f_j} = \frac{4mE\alpha}{\hbar^2}V\left(\vec{f}^2\right)f_j \tag{3.33}$$

Summarizing up and changing the notation from E to β we get the following representation for our basic quantum mechanical observables

$$\hat{X}_j = x_j + \alpha f_j \tag{3.34}$$

$$\hat{P}_j = -i\hbar\frac{\partial}{\partial x_j} + \frac{2im\alpha\beta}{\hbar}\frac{\partial}{\partial f_j} \tag{3.35}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + \beta\sum_{j=1}^{3}\frac{\partial}{\partial f_j}\frac{\partial}{\partial f_j} + V\left(\vec{f}^2\right) \tag{3.36}$$

$$\hat{F}_j = \frac{4m\alpha\beta}{\hbar^2}V\left(\vec{f}^2\right)f_j \tag{3.37}$$

where α and β are two dimensional constants which in each particular case have to be determined from the characteristic constants of the problem and the function $V\left(\vec{f}^2\right)$ plays the role of "potential' which determine the shape of the acting forces.

Since for the free particle there are no constants which may give the right dimension to the constants α and β we must conclude that in this case we have

$$\alpha = \beta = 0 \tag{3.38}$$

and our formalism reduces to the standard formalism of wave mechanics for free particles.

In part I we have shown that the non-covariant force laws may be replaced by covariant differential equations for the acting forces. Similarly, we may form such equations also in quantum mechanics where they will have the form

$$-\frac{1}{\hbar^2}\left[\hat{H},\left[\hat{H},\hat{F}_j\right]\right] = \varrho_j(\hat{F}) \tag{3.39}$$

where the right-hand side is determined from the corresponding differential equation for the classical force. In this way we get the correspondence between the quantum and classical problems.

IV. The physical interpretation of the formalism.

Before going to a particular example we would like to discuss some aspects of the physical interpretation of the new wave functions.

From the shape of the Hamiltonian (3.36) we get the following representation of the wave functions

$$\psi(\vec{x},t,\theta,\vec{f}) = \left(exp\,\frac{-im\theta}{\hbar}\right) \int d^3x\, exp\left[\frac{i\vec{q}\cdot\vec{x}}{\hbar} - i\frac{q^2}{2m\hbar}t\right]\psi(\vec{q},t,\vec{f}) \qquad (4.1)$$

where the function $\psi(\vec{q},t,\vec{f})$ satisfies the wave equation

$$i\hbar\frac{\partial\psi(\vec{q},t,\vec{f})}{\partial t} = \left[\beta\Delta_f + V(\vec{f})\right]\psi(\vec{q},t,\vec{f}) \qquad (4.2)$$

with

$$\Delta_f = \sum_{j=1}^{3}\frac{\partial^2}{\partial f_j^2} \qquad (4.3)$$

The dependence of $\psi(\vec{q},t,\vec{f})$ on the argument $\vec{q}$ is not determined by the dynamical equation (4.2) but by the initial form of the wave packet represented by the wave function $\psi(\vec{x},t,\theta,\vec{f})$. During the time this wave packet as usually is spreading and after a long period of time it may be quite different from the initial packet.

Since the variable $\vec{x}$ does not represent the position of the particle described by the wave function $\psi(\vec{x},t,\theta,\vec{f})$ we cannot interpret the absolute square of the wave function as the probability density of finding the particle at some point. However, the wave functions properly normalized should serve for the evaluation of mean values of quantum mechanical observables. To see what the normalization

condition should be let us define the mean value of any observable $O(\vec{x},\vec{\nabla},\vec{f},\partial_{\vec{f}})$ in the state $\psi(\vec{x},t,\theta,\vec{f})$ by the usual formula

$$\bar{O}(\psi) = \int d^3x\, d^3f\, \psi^*(\vec{x},t,\theta,\vec{f})O(\vec{x},\vec{\nabla},\vec{f},\partial_{\vec{f}})\psi(\vec{x},t,\theta,\vec{f}) \qquad (4.4)$$

In particular, the mean value of the j-th component of the acting force is given by

$$\bar{F}_j(\psi) = \int d^3x\, d^3f\, \psi^*(\vec{x},t,\theta,\vec{f})\hat{F}_j\psi(\vec{x},t,\theta,\vec{f}) = \int d^3x\, w(\vec{f},t)F_j(\vec{f}) \qquad (4.5)$$

where $F_j(\vec{f})$ is the function in (3.13) and

$$w(\vec{f},t) = \int d^3x|\psi(\vec{x},t,\theta,\vec{f})|^2 = (2\pi)^3\int d^3q|\psi(\vec{q},t,\vec{f})|^2 \qquad (4.6)$$

serves as a statistical weight in the integral (4.5). From that it follows that the integrals in (4.6) may be interpreted as the probability density that at time t on the considered particle acts the force $\vec{F} = (F_1(\vec{f}), F_2(\vec{f}), F_3(\vec{f}))$. For this purpose the density $w(\vec{f},t)$ should be normalized to unity and we get the normalization condition for our new wave functions in the form

$$\int d^3x\, d^3f|\psi(\vec{x},t,\theta,\vec{f})|^2 = 1 \qquad (4.7)$$

As we have seen the wave functions, in general, do not have the probabilistic interpretation in the position space. The situation changes, however, if at the end of calculations we restrict our theory to the subspace of the space $R^3 \times R^3$, spanned by the arguments $\vec{x}$ and $\vec{f}$ defined by the classical force law

$$\vec{F}(t) = \vec{\mathcal{F}}(\vec{x}(t)) \qquad (4.8)$$

i.e., to the subspace given by the equations

$$F_j(\vec{f}) = \mathcal{F}_j(\vec{x}) \tag{4.9}$$

where $F_j(\vec{f})$ is the function from (3.13). On this subspace, the functions $F_j(\vec{f})$ which realize our quantum notion of force are connected with the variable $\vec{x}$ by the same relation (4.8) which determines the classical force law and our "force picture" has an image in the $\vec{x}$-space. In this sense we may restore the usual space - time picture of the wave function. It must be however remembered that this procedure breaks down the Galilean covariance of the formalism because all classical force laws for a single particle are non-covariant.

It is also interesting to look on the restoration of the standard force laws in terms of observables $\hat{x}_j$ and $\hat{F}_j$. From (3.34) and (3.37) it is clear that a relation of the form

$$\hat{F}_j = \mathcal{F}_j(\hat{x}_1, \hat{x}_2, \hat{x}_3) \tag{4.10}$$

may be satisfied only for such states for which the multiplication by the variables x_j gives zero. Such states are described by wave functions $\psi(\vec{x}, t, \theta, \vec{f})$ which as functions of the variable $\vec{x}$ have the properties of Dirac δ-function. This can be however achieved only for the initial time because any wave packet is spreading in time. In fact, taking in (4.1) the functions $\psi(\vec{q}, t, \vec{f})$ as independent from the variable $\vec{q}$ we have

$$\psi(\vec{x}, 0, \theta, \vec{f}) = (2\pi\hbar)^3 \delta(\vec{x}) \psi(\vec{f}, t) \tag{4.11}$$

where $\Psi(\vec{f}, t)$ is some solution of the equation (4.2). Therefore, the classical force

laws may be translated into quantum mechanics only as an initial relation between force and position.

In classical physics the force laws determine also the time derivative of the force. A different situation happens in wave mechanics. In fact, from (3.36) and (3.37) it is seen that the time derivative of the force is given by the operator

$$\begin{aligned} \frac{d}{dt}\hat{F}_j \equiv \hat{G}_j &= \frac{i}{\hbar}[\hat{H}, \hat{f}_j] = \\ &= \frac{i\beta}{\hbar}\sum_{k=1}^{3}\left(\frac{\partial^2 F_j}{\partial f_k^2} + 2\frac{\partial F_j}{\partial f_k}\frac{\partial}{\partial f_k}\right) \end{aligned} \tag{4.12}$$

and this operator may be related to the operators $\hat{P}_j$ only on that states for which the wave functions do not depend on the variable $\vec{x}$. This may be achieved provided we choose the initial conditions in such a way that

$$\psi(\vec{q}, t, \vec{f}) \sim \delta(\vec{q})\tilde{\psi}(\vec{f}, t) \tag{4.13}$$

where again the function $\tilde{\psi}(\vec{f}, t)$ is a solution of the equation (4.2).

Clearly, the class of states for which we may reproduce the classical force laws on the operator level being specified by (4.11) and by the class of states for which the time derivative of the force is fixed by the force law being specified by (4.13) are disjoint and therefore it is impossible reproduce simultaneously the force law and its time derivative. The operator versions of force laws carry therefore only part of the information contained in classical force laws and clearly this is a consequence of the quantum mechanical uncertainty relation.

Both classes of states specified either by (4.11) or (4.13) contain only non-normalizable states in the sense of (4.70 and therefore our probabilistic interpretation does not apply to them. For normalizable states the content of quantum

mechanical force laws is different from their classical counterparts. The only common property they have is that they always break down the Galilean covariance.

Among all normalizable states a special role is played by the stationary states for which

$$\psi(\vec{q},t,\vec{f}) = e^{-i\frac{\Omega t}{\hbar}}\psi(\vec{q},\vec{f}) \tag{4.14}$$

where $\psi(\vec{q},\vec{f})$ satisfies the eigenvalue equation

$$\left[\beta\Delta_f + V(\vec{f}^2)\right]\psi(\vec{q},\vec{f}) = \Omega\psi(\vec{q},\vec{f}) \tag{4.15}$$

Substituting (4.14) into (4.1) we get the wave functions in the form

$$\psi(\vec{x},t,\theta,\vec{f}) = \int d^3x\ exp\frac{i}{\hbar}\left[\vec{q}\cdot\vec{x} - \left(\frac{q^2}{2m}+\Omega\right)t - m\theta\right]\psi(\vec{q},\vec{f}) \tag{4.16}$$

and we easily recognize them as elements of the carrier space of the unitary representation of the extended Galilei group. From the general theory of such representations [6] it is known that each of them is characterized by three numbers; the spin s, which we have assumed to be zero; the mass of the particle m, which we have fixed at the beginning and the value of the rest energy Ω, which is the only parameter not fixed a priori by our formalism. Since the eigenvalue problem (4.15) has solution only for restricted quantized values of Ω we see that our formalism produces representations of the Galilei group with quantized values of the rest energy, which are determined by the acting force. The picture just described shows the essential differences between classical and quantum Galilean covariant mechanics from one side and between our covariant formalism and the standard one from the other. In classical physics the acting forces influence the motion of

the particle without changing its internal structure. In our formalism the acting forces change the internal structure of the particle because they determine its internal rest energy while the space-time picture is modelled by the initial form of wave packet and the spreading phenomenon. The standard formalism of quantum mechanics, like classical mechanics, does not allow to describe the changes of internal structure of the particle on which the force acts and restricts its role to the description of the influence of the acting forces on the spreading wave packets. As we have already explained, we can also describe this phenomenon by restricting our general covariant formalism to a non-covariant subspace in the combined $\vec{x}$ and $\vec{f}$ spaces which is selected by a given classical force law.

Since the $\vec{q}$-dependence of the functions $\psi(\vec{q},\vec{f})$ is determined by the initial conditions, the eigenfunctions $\psi_n(\vec{q},\vec{f})$ in (4.15) which belong to a non-degenerate eigenvalue Ω_n are of the form

$$\psi_n(\vec{q},\vec{f}) = \psi_n(\vec{q})\,\psi_n(\vec{f}) \tag{4.17}$$

where n denotes the whole collection of quantum numbers. For such states the probability density $w(\vec{f},t)$ defined by (4.6) is time independent and equal to

$$w_n(\vec{f},t) = c_n|\psi_n(\vec{f})|^2 \equiv w_n(\vec{f}) \tag{4.18}$$

where

$$c_n = (2\pi)^3\int d^3q|\psi_n(\vec{q})|^2 \tag{4.19}$$

Because the functions $\psi_n(\vec{q})$ are determined by the initial shape of the wave packet we may always normalize the integrals in (4.19) to unity and then we simply have

$$w_n(\vec{f}) = |\psi_n(\vec{f})| \tag{4.20}$$

similarly to the standard wave mechanics probability density for the coordinate.

Among non-diagonal matrix elements of observables a physical meaning have only elements taken between states for which the eigenfunctions $\psi_n(\vec{f})$ are different but the functions $\psi_n(\vec{q})$ are the same because only transitions between such states initially prepared identically are connected with the interaction. For such states the matrix elements of the position operators (3.34) and momentum operators (3.35) have the form

$$x_j(n_1, n_2) = A_j \delta_{n_1 n_2} + \alpha \int d^3 f \psi^*_{n_1}(\vec{f}) f_j \psi_{n_2}(\vec{f}) \tag{4.21}$$

and

$$p_j(n_1, n_2) = B_j \delta_{n_1 n_2} + \beta \int d^3 f \psi^*_{n_1}(\vec{f}) \frac{\partial}{\partial f_j} \psi_{n_2}(\vec{f}) \tag{4.22}$$

where A_j and B_j are contributions from the first terms in (3.34) and (3.33), respectively, which obviously are diagonal in the space of the functions $\psi_n(\vec{f})$. From this it easily follows that in the space of $\psi_n(\vec{f})$ functions, the operators $\hat{x}_j$ and $\hat{p}_j$ satisfy the standard commutation relations.

V. The example of the harmonic oscillator.

We shall now apply our general approach to the simple case of the harmonic oscillator. The more complicate cases, in particular the Kepler problem we plan to consider in separate papers.

Assuming that the function $F_j(\vec{f})$ in (3.13) is simply equal to f_j and neglecting the unessential integration constant in (3.37) we get the following expression for the function $V(\vec{f}^2)$ in (3.36):

$$V(\vec{f}^2) = \frac{\hbar^2}{4m\alpha\beta} \vec{f}^2 \tag{5.1}$$

Calculating the second time derivative of the force according to the formula

$$\ddot{\hat{F}}_j = -\frac{1}{\hbar^2} \left[\hat{H}, \left[\hat{H}, \hat{F}_j\right]\right] \tag{5.2}$$

we come to the equation of motion for the acting force

$$\ddot{\hat{F}}_j - \frac{1}{m\alpha} \hat{F}_j = 0 \tag{5.3}$$

Identifying now the constant α to

$$\alpha = -\frac{1}{m\omega^2} \tag{5.4}$$

we see that equation (5.3) coincides with the equation for the harmonic oscillator force [1] where ω is the frequency of the oscillator.

The total energy of the classical harmonic oscillator is given by [1]

$$E = \frac{1}{2m} \left(\vec{p}(t) + \frac{\vec{G}(t)}{\omega^2}\right)^2 + \frac{1}{2m\omega^2} \left(\vec{F}^2(t) + \frac{\vec{G}^2(t)}{\omega^2}\right) \tag{5.5}$$

and the quantity $\vec{P}$ in (3.22) is given by

$$\vec{P} = \vec{p}(t) + \frac{\vec{G}(t)}{\omega^2} \tag{5.6}$$

We have earlier explained that the quantum mechanical counterpart of this quantity is simply given by the operator $-i\hbar\vec{\nabla}$. From that and (3.35) we get the quantum mechanical counterpart of the quantity $\vec{G}$ in the form

$$\hat{G}_j = \frac{2i\beta}{\hbar}\frac{\partial}{\partial f_j} \tag{5.7}$$

Substituting this and (5.6) into (3.36) we come to the following form of the Hamiltonian

$$\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} + \frac{\hat{\vec{G}}}{\omega^2}\right)^2 - \frac{\hbar^2}{4\beta}\hat{\vec{G}}^2 - \frac{\hbar^2\omega^2}{4\beta}\hat{\vec{F}}^2 \tag{5.8}$$

This quantity will be the quantum mechanical counterpart of the classical total energy (5.5) provided

$$\beta = -\frac{\hbar^2 m\omega^4}{2} \tag{5.9}$$

In our formalism the harmonic oscillator is therefore determined by the following basic quantum mechanical observables

$$\hat{X}_j = x_j - \frac{1}{m\omega^2}f_j \tag{5.10}$$

$$\hat{p}_j = -i\hbar\frac{\partial}{\partial x_j} + i\hbar m\omega^2\frac{\partial}{\partial f_j} \tag{5.11}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta_{\vec{x}} - \frac{\hbar^2 m\omega^4}{2}\Delta_{\vec{f}} + \frac{1}{2m\omega^2}\vec{f}^2 \tag{5.12}$$

$$\hat{F}_j = f_j \tag{5.13}$$

It is easy to verify that these observables satisfy the basic quantum mechanical relations (3.10) - (3.12) and have the correct Galilean transformation properties.

It is also easy to see that on states for which the condition (4.11) is satisfied we have the operator relation

$$\hat{F}_j = -m\omega\hat{X}_j \tag{5.14}$$

while on states which satisfy (4.13) we have

$$\hat{G}_j = \omega^2\hat{p}_j \tag{5.15}$$

Obviously the relations (5.14) and (5.15) coincide with the classical force law for the harmonic oscillator.

Solving the Schrödinger equation (2.5) with the Hamiltonian (5.12) we get the following expressions for the functions $\psi(\vec{q},\vec{f},t)$ in (4.1)

$$\psi(\vec{q},\vec{f},t) = exp\left(-\frac{\vec{f}^2}{2\hbar m\omega^2}\right) \sum_{n_1,n_2,n_3}\psi_{n_1,n_2,n_3}(\vec{q})e^{i\omega_{n_1,n_2,n_3}t}\prod_{j=1}^{3}H_{n_j}\left(-\frac{f_j}{\sqrt{\hbar m\omega^3}}\right) \tag{5.16}$$

where $\psi_{n_1,n_2,n_3}(\vec{q})$ are arbitrary functions to be determined from the initial shape of the wave packet, (n_1,n_2,n_3) is a triple of integers, H_n are Hermite polynomials and

$$\omega_{n_1,n_2,n_3} = (n_1+n_2+n_3+\frac{3}{2})\omega \tag{5.17}$$

are the usual frequencies of the quantum mechanical oscillator.

Obviously, restricting the theory on the subspace of the classical force law given by the equations

$$f = -m\omega^2 x \tag{5.18}$$

the function $\psi(\vec{q}, \vec{f}, t)$ reproduces the wave function of the standard wave mechanics but we loose the Galilean covariance because (5.18) is not a covariant relation.

VI. Conclusions.

We have developed a new formalism which allows to exhibit the Galilean covariance of wave mechanics. The essential point of our approach is connected with a new notion of quantum mechanical force. The force, like in the classical version of our approach, is subjected to its own equation of motion but, in addition, it is realized as an independent observable. The wave functions of our formalism depend therefore on more variables than tha wave functions of the standard formalism do and a particular modification of the probabilistic interpretation was needed.

The standard approach to quantum mechanics implicitly assumes that the measurement of the position of the particle gives also a complete information on the forces acting on it. We have followed here an independent quantization of the force because it seems to us that this way is closer to the physical reality. Our approach may lead to new uncertainty relations connected with the force laws but this problem waits for more detailed investigation.

Like a situation was at the earliest stage of quantum mechanics [10] we have verified our formalism only on the simplest example of the harmonic oscillator and we showed that it works. This creates the hope that in the near future we will be able to work out the Kepler problem, the most fundamental problem of any quantum theory.

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