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A ONE-DIMENSIONAL SPHERICAL MODEL WITH A PHASE TRANSITION

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The one-dimensional spherical model is generalized to include long-range interactions. A phase transition is shown to occur for a certain type of interaction. The partition function is explicitly calculated, and the critical behavior of the model is examined.

1. Introduction

In order to obtain a phase transition in one dimension, we shall generalize the conventional spherical model which involves only nearest neighbor interactions, to include long-range interactions. First, following Baxter[1], let us briefly review the spherical model solved by Berlin and Kac[2] in 1952. In this model, one assigns a real spin variable σ_j to each lattice site, subject to the constraint $\sum_{j=1}^{N}\sigma_j^2 = N$. Thus, the partition function for the spherical model can be written as

$$Z_N = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} d\sigma_1\cdots d\sigma_N \exp\{K\sum_{(j,l)}\sigma_j\sigma_l + H\sum_j\sigma_j\}\delta[N-\sum_j\sigma_j^2] \qquad (1)$$

where $K = J/kT$, $H = h/kT$, and the summation (j,l) ranges over nearest neighbors only. However, in our modification, the interaction term is replaced by $K\sum_{j,l}\sigma_j J_{jl}\sigma_l$ where J_{jl} is the generic matrix element of $\mathbf{J}$ in equation (3) below, and depends only upon the distance between spins σ_j and σ_l. Thus, the partition function of our model becomes

$$Z_N = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} d\sigma_1\cdots d\sigma_N \exp\{K\sum_{(j,l)}\sigma_j J_{jl}\sigma_l + H\sum_j\sigma_j\}\delta[N-\sum_j\sigma_j^2].$$

With a suitable choice of the matrix $\mathbf{J}$, this formula can express the partition function a spherical model on an arbitrary lattice in any dimension. However, since we are n considering the one-dimensional case with periodic boundary conditions, and, as mention above, J_{jl} depends on the distance between σ_j and σ_l, the matrix $\mathbf{J}$ is cyclic, as follows

$$\mathbf{J} = \begin{pmatrix} J(0) & J(1) & \cdots & J(N-1) \\ J(N-1) & J(0) & \cdots & J(N-2) \\ \vdots & & \ddots & \vdots \\ J(1) & J(2) & \cdots & J(0) \end{pmatrix} \qquad ($$

To ensure translational invariance, i. e., for a finite circular model, rotational invariance, assume $J(N-n) = J(n)$. Hence, the above matrix is also symmetric. We shall evalua the thermodynamic functions of systems defined by choosing certain specific expressio for J_{jl}.

2. Evaluation of the Partition Function

We will first follow the derivation by Berlin and Kac[2] as presented by Baxter[1], and th modify this procedure as appropriate for our examples. The partition function of equati (2) can be rewritten as

$$Z_N = (2\pi)^{-1}\int_{-\infty}^{\infty} d\sigma_1\cdots d\sigma_N \int_{-\infty}^{\infty} ds \exp\{K\sum_{(j,l)}\sigma_j J_{jl}\sigma_l + H\sum_j\sigma_j + (a+is)N - (a+is)\sum_j\sigma$$

($$

where we have used the Fourier transform of the delta function and also added an ext term $0 = aN - a\sum_j\sigma_j^2$ in the exponent, where a is an arbitrary real constant. Now, define the matrix $\mathbf{V}$ by

$$\mathbf{V} = (a+is)\mathbf{I} - K\mathbf{J} \qquad ($$

where I is the $N \times N$ unit matrix. Moreover, defining $\vec{H}$ to be an N-dimensional vector with all components equal to H, and $\vec{\sigma}$ as an N-dimensional vector with elements σ_j, one can write the partition function as

$$Z_N = (2\pi)^{-1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\vec{\sigma} \int_{-\infty}^{\infty} ds \exp\{-\vec{\sigma}^T \mathbf{V} \vec{\sigma} + \vec{H}^T \vec{\sigma} + (a + is)N\}. \tag{6}$$

After diagonalizing the matrix $\mathbf{V}$ and integrating over $\vec{\sigma}$, Z_N becomes

$$Z_N = \frac{1}{2}\pi^{N/2-1} \int_{-\infty}^{\infty} ds [\det \mathbf{V}]^{-1/2} \exp\{(a + is)N + \frac{1}{4}\vec{H}^T \mathbf{V}^{-1} \vec{H}\} \tag{7}$$

From the definition of the matrix $\mathbf{V}$ in equation (5), since $\mathbf{J}$ is a symmetric real cyclic matrix, the eigenvalues of $\mathbf{V}$ are[3]

$$\lambda_k = a + is - K[\sum_{l=1}^{N} J(l) \cos(2\pi k l/N)]. \tag{8}$$

Thus,

$$\ln \det \mathbf{V} = \sum_{k=1}^{N} \ln \lambda_k = \sum_{k=1}^{N} \ln\{(a + is) - K[\sum_{l=1}^{N} J(l) \cos(2\pi k l/N)]\}, \tag{9}$$

and defining $\omega_k = 2\pi k/N$, this becomes

$$\sum_k \ln[(a + is) - K(J(1)\cos\omega_k + J(2)\cos 2\omega_k + \cdots + J(N)\cos N\omega_k)]$$

The above summation over ω_k can replaced by integration when N is large, and we shall write

$$\ln \det \mathbf{V} = N[\ln K + g(z)] \tag{10}$$

where

$$g(z) = (2\pi)^{-1} \int_0^{2\pi} d\omega \ln[z + \xi - (J(1)\cos\omega + J(2)\cos 2\omega + \cdots)], \tag{11}$$

$$z = (a + is - K\xi)/K, \tag{12}$$

and $\xi = \sum_{l=1}^{N} J(l)$.

Since $\mathbf{J}$ is cyclic, $\vec{1} = (1, \ldots, 1)$ is an eigenvector corresponding to the eigenvalue Kz. Hence, $\vec{H}$ is the eigenvector of $\mathbf{V}$ corresponding to the eigenvalue $a + is - K\xi = Kz$ and

the second term in the exponent of equation (7) becomes $NH^2/4Kz$. Substituting this and the above equations (11) and (12) into (7), we can now write the partition function as

$$Z_N = (K/2\pi i)(\pi/K)^{N/2} \int_{c-i\infty}^{c+i\infty} dz e^{Nf(z)}, \tag{13}$$

where

$$f(z) = Kz + K\xi - \frac{1}{2}g(z) + \frac{H^2}{4Kz}, \tag{14}$$

and $c = (a - K\xi)/K$. Evaluating the above integral by the method of steepest descent, the partition function for sufficiently large N becomes

$$Z_N = \frac{K}{2\pi}(\frac{\pi}{K})^{\frac{N}{2}} e^{Nf(z_0)} \tag{15}$$

where z_0 is the value of z at which $f'(z) = 0$, and the arbitrary constant a has been chosen to be $K(z_0 + \xi)$ (i. e. $c = z_0$). From the above equation (14), the condition $f'(z_0) = 0$ is equivalent to

$$K - \frac{H^2}{4Kz_0^2} = \frac{1}{2}g'(z_0). \tag{16}$$

3. Evaluation of Thermodynamic Functions

From the expression (15) for the partition function, the free energy per spin ψ in the thermodynamic limit becomes

$$-\frac{\psi}{kT} = \lim_{N\to\infty} N^{-1} \ln Z_N = \frac{1}{2}\ln(\frac{\pi}{K}) + f(z_0). \tag{17}$$

Hence, the magnetization, magnetic susceptibility, internal energy, and specific heat per spin are

$$M = -\frac{d}{dH}(\frac{\psi}{kT}) = \frac{H}{2Kz_0} = \frac{h}{2Jz_0}, \tag{18}$$

$$\chi = \frac{\partial M}{\partial h} = \frac{1}{2Jz_0} - \frac{h}{2Jz_0^2}(\frac{\partial z_0}{\partial h}), \tag{19}$$

$$U = -T^2 \frac{\partial}{\partial T}(\frac{\psi}{T}) = \frac{1}{2}kT - J(z_0 + \xi) - \frac{h^2}{4Jz_0}, \tag{20}$$

and

$$C = \frac{\partial U}{\partial T} = \frac{1}{2}k - J(\frac{\partial z_0}{\partial T}) + \frac{h^2}{4Jz_0^2}(\frac{\partial z_0}{\partial T}). \tag{21}$$

z_0 is a function of K and H (in particular, a function of T), and is determined from equations (11) and (16). We shall examine a special case and evaluate z_0 as a function of K and H in the following section.

4. A Special Choice of J

In the previous sections we have described the derivation of the partition function and some thermodynamic functions for an arbitrary translation-invariant one-dimensional spherical model. However, we have not specified the generic matrix element $J(l)$ of $\mathbf{J}$, i. e., the interaction of spins σ_j and σ_{j+l}. Thus, we have as many one-dimensional spherical models with distance-dependent couplings as choices of $\mathbf{J}$. If we choose $J(l) = \exp(-l^2/4)$, the interaction becomes Gaussian, and if we choose $J(1) = 1$ and $J(l) = 0$ $(l > 1)$, we recover the nearest neighbor interaction model, and so on.

However, we must suitably restrict our choice of $\mathbf{J}$ in order to obtain a phase transition. From equation (11), the right hand side of equation (16) becomes

$$\frac{1}{2}g'(z_0) = \frac{1}{2}(\frac{\partial g(z)}{\partial z})_{z=z_0} = (4\pi)^{-1}\int_0^{2\pi} d\omega[z_0 + \xi - \sum_{l=1}^{N} J(l)\cos(l\omega)]^{-1}. \tag{22}$$

Following Baxter[1], there exists a critical point $K_c = \frac{1}{2}g'(0)$ only if this integral converges. Thus, if we define $j(\omega)$, for large N, to be

$$j(\omega) = \sum_{l=1}^{N} J(l)\cos(l\omega), \tag{23}$$

then, in order to obtain a finite non-zero critical temperature, $j(\omega)$ must be of the form ω^α as $\omega \downarrow 0$, where $0 < \alpha < 1$. Although the above integral also converges for $\alpha \leq 0$, the value of α must be positive since negative values of α correspond to negative temperatures. Hence, for the simplest choice, each element of the generic matrix $J(l)$ becomes a Fourier coefficient of ω^α, that is

$$J(l) = -\frac{1}{2\pi}\int_0^{2\pi} \omega^\alpha \cos(l\omega)d\omega, \tag{24}$$

or, by a known formula[4],

$$J(l) = \frac{1}{4\pi l^{1+\alpha}}\{e^{i(1+\alpha)\pi/2}\Gamma(1+\alpha, -2l\pi i) + e^{-i(1+\alpha)\pi/2}\Gamma(1+\alpha, 2l\pi i) - 2\cos[(1+\alpha)\pi/2]\Gamma(1+\alpha) \tag{25}$$

where $\Gamma(x, y)$ denotes the incomplete gamma function. From the above equation (24), $J(l)$ is a positive, monotonic decreasing function of the lattice distance l. The behavior of $J(l)$, i. e., the strength of the coupling, for various values of α ($\alpha = \frac{1}{5}, \frac{1}{2}, \frac{4}{5}$), is shown in Fig. 1. The α dependence of the critical point $K_c = \frac{1}{2}g'(0)$ can be obtained by substituting $j(\omega) = \omega^\alpha$ into equation (22) above, which yields

$$K_c = \frac{(2\pi)^{-\alpha}}{2(1-\alpha)}. \tag{26}$$

Hence, the maximum critical temperature $T_{c\,MAX}$ is obtained for $\alpha \cong 0.46$, and the value of $T_{c\,MAX}$ is

$$T_{c\,MAX} = \frac{J(2\pi)^{1/2}}{k}. \tag{27}$$

$T_c \to 0$ for both limiting cases, $\alpha \to 0$ and $\alpha \to 1$, since $J(l) = 0$ in the limit $\alpha \to 0$. The behavior of K_c as a function of α is shown in Fig. 2.

The explicit expression for the partition function can be obtained by evaluating the integral in equation (11), and substituting the result into equation (15), which yields

$$\begin{aligned} Z_N = {} & \frac{K}{2\pi}(\frac{\pi}{K})^{\frac{N}{2}}(z_0 + (2\pi)^\alpha)^{-\frac{1}{2}}\exp\{N[K(z_0 + \frac{(2\pi)^\alpha}{1+\alpha}) \\ & + \frac{1}{2}(\alpha - {}_2F_1(1, \alpha^{-1}; 1+\alpha^{-1}; -z_0^{-1}(2\pi)^\alpha) + \frac{H^2}{2Kz_0})]\} \end{aligned} \tag{28}$$

where ${}_2F_1(a, b; c; z)$ denotes the hypergeometric function.

5. Behavior of Thermodynamic Functions

Now, we shall examine the behavior of the thermodynamic functions in the limit $h \to 0$. First, consider the temperature range $T > T_c$ ($K < K_c$). Then, it follows from equations (16) and (22) that z_0 approaches a non-zero value w in the limit $h \to 0$ ($H \to 0$). Also as in Baxter[1] (p. 67), w is a monotonic function of T for $T \geq T_c$, and approaches zero a

$T \to T_c$. Thus, the thermodynamic functions in the limit $h \to 0$ become

$$M = 0, \tag{29}$$

$$\chi = \frac{1}{2Jw}, \tag{30}$$

$$U = \frac{1}{2}kT - J(w + \xi), \tag{31}$$

and

$$C = \frac{1}{2}k - J(\frac{\partial w}{\partial T}) \tag{32}$$

where $\xi = (2\pi)^\alpha/(1+\alpha)$. No spontaneous magnetization occurs above the critical temperature, and since $w \to 0$ as $T \to T_c$, the magnetic susceptibility diverges at the critical point.

For $T < T_c$, $z_0 \to 0$ as $h \to 0$. Hence, from equation (16),

$$\lim_{H \to 0} \frac{H}{z_0} = [4K(K - K_c)]^{\frac{1}{2}} = 2K(1 - \frac{T}{T_c})^{\frac{1}{2}}. \tag{33}$$

Hence,

$$M = (1 - \frac{T}{T_c})^{\frac{1}{2}}, \tag{34}$$

$$\chi \to \infty, \tag{35}$$

$$U = \frac{1}{2}kT - J\xi, \tag{36}$$

and

$$C = \frac{1}{2}k. \tag{37}$$

The behavior of the magnetization and magnetic susceptibility are shown in Fig. 3.

6. Conclusion

The critical behavior of this model is similar to that of the conventional spherical model in dimensions greater than two. This follows from the fact that the integral in equation (22) converges for dimensions greater than two in the conventional spherical model. However, in the present model, the introduction of long-range interactions eliminates the need to consider higher dimensions in order to exhibit critical behavior.

Dyson's hierarchical model[5] is a well known one-dimensional spin model with a phase transition, but is rather artificial in lacking translational invariance. By contrast, the interaction in the present model is translation invariant, and decreases monotonically with increasing distance between spins. Application of general theorems on Fourier series[6] shows that the series $\sum_{l=0}^{\infty} J(l)$ is absolutely convergent, in particular, $J(l) = o(l^{-1})$ as $l \to \infty$. Thus, the coupling introduced in the present case appears quite natural and physically plausible.

Needless to say, one may expect that the introduction of suitable long range interactions in a two-dimensional spherical model would also yield a phase transition, and moreover, that the required coupling strength would decrease even more rapidly than in the preceding one-dimensional case. The two-dimensional case will be discussed in a subsequent paper.

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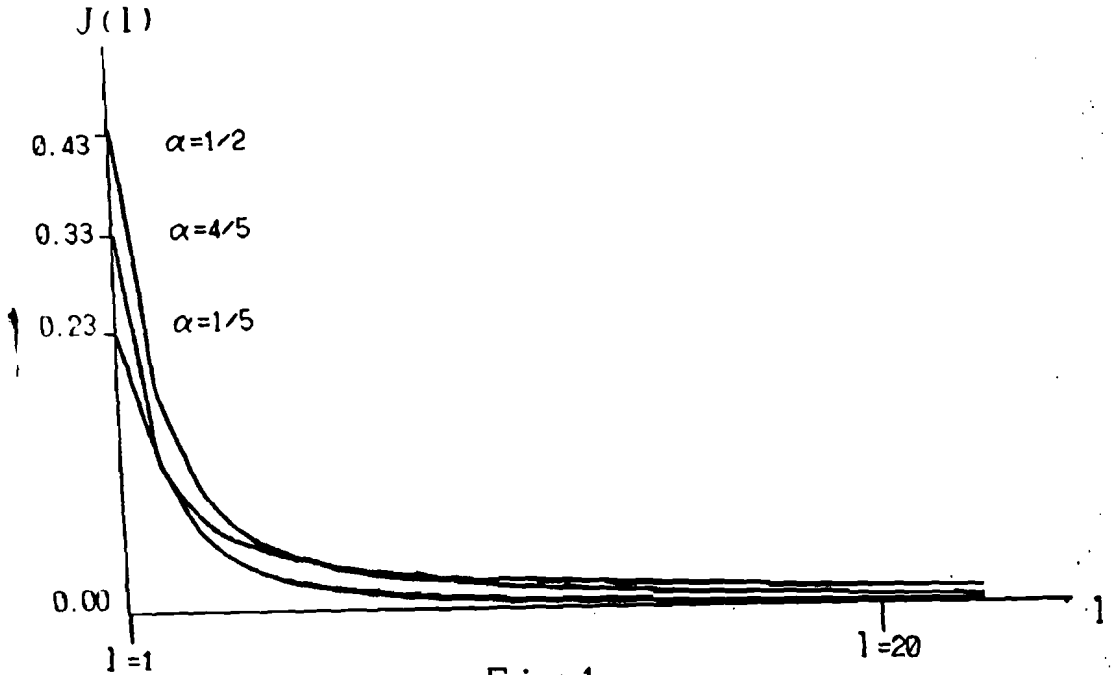

Fig.1

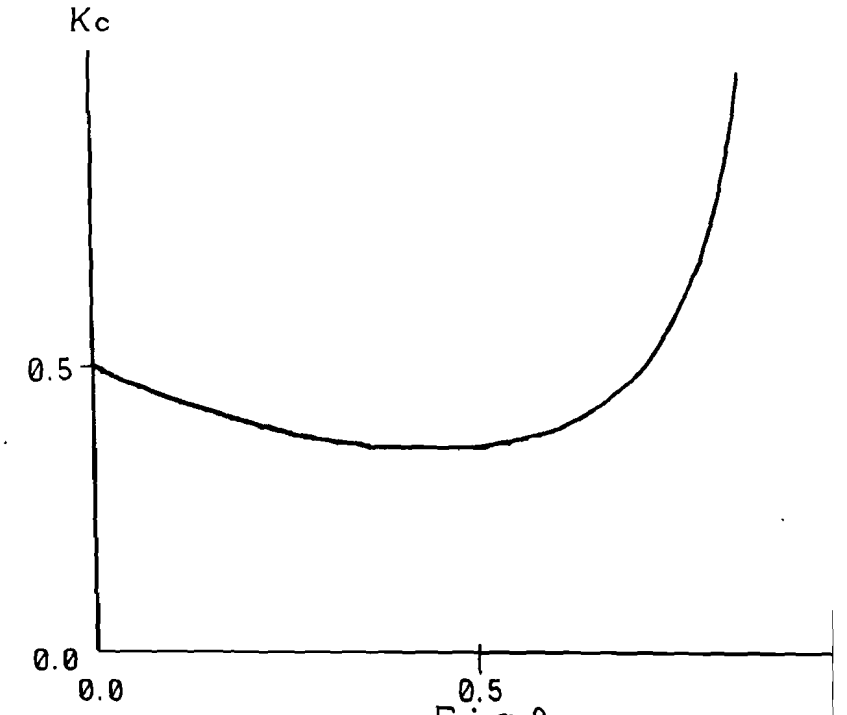

Fig.2

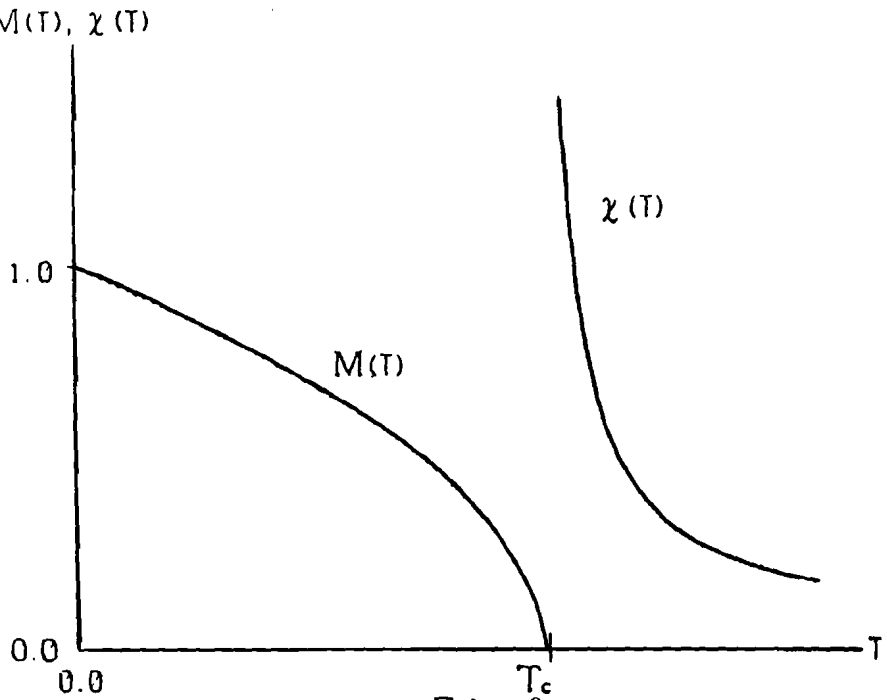

Fig.3