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Topological Variants of Lattice Field Models

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The conventional Ising model is generalised so as to introduce a topological frustration for both ferromagnetic and antiferromagnetic cases. This generalisation is extended to a broad class of lattice field models, including the conventional XY model and classical Heisenberg model as special cases. Twisted external fields are also introduced in a manner symmetrical with respect to the intrinsic fields. The partition function and the two-point correlation function are explicitly determined for several varieties of twisted 1-dimensional chains with zero external field. Cohomological and fibre bundle interpretations are discussed.

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1. INTRODUCTION

With a view to certain topological generalisations, we first reformulate the conventional Ising model in the following manner. Consider any lattice formed by the 1-dimensional skeleton of a polyhedral subdivision (triangular, cubical or otherwise) of any conected polytope K, which will generally (but not necessarily) be a closed manifold. Fix an arbitrary lattice point x_0, and assign a number $\tau(L) = \pm 1$ in Z_2 to each unoriented 1-simplex (link) L in the lattice. In addition, choose a number $\sigma = \pm 1$. Any such choice $(\sigma, \tau(L) \mid L \in K^1)$ will be referred to as a configuration. To any closed polygonal curve C based at the point x_0, assign the ordered product $\tau(C)$ of the numbers $\tau(L_j)$ corresponding to the successive links L_j of the curve:

$$\tau(C) = \prod_j \tau(L_j) \tag{1}$$

We now subject the possible configurations τ to the constraint which requires that the correspondence

$$C \longrightarrow \tau(C)$$

determines some fixed representation $\mathcal{R}$ of the fundamental group $\pi_1(K)$ in the group Z_2. In particular, this implies that each configuration satisfies the local commutativity condition i. e., the product of the link value $\tau(L)$ around the boundary of any 2-cell (or "plaquett") is equal to 1.

Further, let $T(K, \mathcal{R})$ denote the set of all link functions $\tau(L)$ satisfying the above condition. Then we define the Hamiltonian of a configuration $\tau \in T(K, \mathcal{R})$ by the formula

$$\mathcal{H}(\tau, \sigma) = -J \sum_i \tau_i \tag{2}$$

where J is a real constant. Thus, the Hamiltonian is independent of the variable σ.

Consider first the case where the representation chosen is the trivial representation

$$\pi_1(K) \longrightarrow 1$$

Then, the model is equivalent to the conventional Ising model with zero external field. This is readily seen from the identification

$$\tau(L_{ij}) = \sigma_i \sigma_j \quad , \tag{3}$$

where σ_i and σ_j denote the spin variables assigned to the endpoints x_i and x_j of the link L_{ij}. The variable σ in the configuration (τ, σ) could be identified with the direction of the spin at one arbitrarily fixed lattice point.

As an illustration, considered a subdivision of the circle S^1 into N segments $L_1, L_2, \ldots, L_N$. The fundamental group $\pi_1(S^1)$ has two representations in Z_2. For the trivial representation, as already observed, the resulting model is equivalent to the 1-dimensional closed Ising chain. In this case, the configurations are sequences of numbers $\tau_1, \tau_2, \ldots, \tau_N$ $(\tau_i = \pm 1)$ subject to the constraint

$$\prod_i \tau_i = 1 \quad . \tag{4}$$

However, for the nontrivial representation, the resulting model is not equivalent to conventional closed Ising chain, the constraint being

$$\prod_i \tau_i = -1 \quad . \tag{5}$$

The Hamiltonian assumes the same form in both cases, and involves only uncoupled variables. Thus, in the present formulation, the Hamiltonian is free of explicit interactions, and all the dynamics has been included in the constraints. Moreover, the twofold degeneracy associated with the up-down symmetry has been isolated in the variable σ. In the case of zero external field, the effect of this variable is simply to double the partition function, and in fact one may disregard σ in all the ensuing zero field calculations. The result of eliminating σ from the model will be referred to as the *reduced model*, and the associated partition function will be distinguished by the superscript r.

2. SOLUTION OF THE TWISTED ISING CHAIN PROBLEM

The above-mentioned problem corresponding to the nontrivial representation of $\pi_1(S^1)$, which we shall refer to as the twisted chain problem, may be physically visualized as follows. Consider a long, narrow Mobius band, divided into a sequence of N small rectangles $r_1, r_2, \ldots, r_N$. In each rectangle, choose an orientation which may also be regarded as a spin vector perpendicularly directed toward one or the other side of the band, and to each pair of successive rectangles r_i, r_{i+1}, assign a number τ_i equal to ± 1 according as the orientations of this pair of rectangles are coherent or incoherent.

The partition function can be determined by the following recursion process, utilizing the known solution of the conventional 1-dimensional Ising chain. Denote the partition functions of the conventional (untwisted) and twisted chains of length N by $Z_u(N)$ and $Z_t(N)$, respectively. Then, by summing over the first variable, one obtains the following recursion relation for the reduced functions,

$$Z_t^r(N) \;=\; e^{\beta J} Z_t^r(N-1) \;+\; e^{-\beta J} Z_u^r(N-1) \tag{6}$$

where, as mentioned above, $Z_u^r = Z_u/2$. This relation is readily solved, yielding the result

$$Z_t^r(N) \;=\; \frac{1}{2}\left[\,(2\cosh(\beta J))^N - (2\sinh(\beta J))^N\,\right] \tag{7}$$

One can easily verify that free energy per spin in the thermodynamic limit is identical with that of the conventional untwisted Ising chain (closed or open), as might have been expected.

The following transfer matrix form of the recursion relation is interesting in that one can thereby simultaneously calculate the partition function for the twisted and untwisted Ising chains.

$$\begin{pmatrix} Z_u^r(N) \\ Z_t^r(N) \end{pmatrix} = \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} \begin{pmatrix} Z_u^r(N-1) \\ Z_t^r(N-1) \end{pmatrix} \tag{8}$$

The two-point correlation function is defined for $r \leq N/2$ by

$$< \tau_{i,i+r} >_t \;\equiv\; < \tau_i \tau_{i+1} \cdots \tau_{i+r} >_t \tag{9}$$

which is consistent with the conventional definition for the untwisted case, since

$$< \sigma_i \sigma_{i+r} >_u = < \sigma_i \sigma_{i+1} \sigma_{i+2} \cdots \sigma_{i+r-1} \sigma_{i+r} >_u = < \tau_i \tau_{i+1} \cdots \tau_{i+r-1} >_u \tag{10}$$

and can be similarly determined by the following recursion,

$$\begin{aligned} Z_t^r(N) < \tau_{1,r} >_t (N) \;=\; & e^{\beta J} Z_t^r(N-1) < \tau_{2,r} >_t (N-1) \\ & - e^{-\beta J} Z_u^r(N-1) < \tau_{2,r} >_u (N-1) \end{aligned} \tag{11}$$

or by the same transfer matrix procedure as the above equation (8),

$$\begin{pmatrix} Z_t^r(N) < \tau_{1,r} >_t \\ Z_u^r(N) < \tau_{1,r} >_u \end{pmatrix} = \begin{pmatrix} e^{\beta J} & -e^{-\beta J} \\ -e^{-\beta J} & e^{\beta J} \end{pmatrix}^r \begin{pmatrix} Z_t^r(N-r) \\ Z_u^r(N-r) \end{pmatrix} \tag{12}$$

with the result

$$< \tau_{1,r} >_t (N) = \frac{[\tanh(\beta J)]^r - [\tanh(\beta J)]^{N-r}}{1 - [\tanh(\beta J)]^N} \tag{13}$$

Notice that for odd N, the twisted ferromagnetic chain with coupling constant J is isomorphic to the untwisted antiferromagnetic chain with coupling constant $-J$, in the sense that the transformation $\tau_i \rightarrow -\tau_i$ defines a bijection of configurations which preserves the Hamiltonian. In particular, these two models have the same partition function. However, the models are *not* equivalent, since the two-point correlation functions are all positive for the twisted ferromagnetic chain but alternate in sign for the untwisted antiferromagnetic chain.[1] Similarly, the twisted antiferromagnetic chain is isomorphic although not equivalent to that of the untwisted ferromagnetic chain. However, this is not true for even N. In both the odd and even cases, the ground state of the ferromagnetic twisted chain is $2N$-fold degenerate (simply N-fold degenerate in the reduced model), whereas for even N the ground state of the antiferromagnetic untwisted chain is degenerate of order 2 (or nondegenerate in the reduced version).

Although a rigorous proof will not be attempted in the present paper, it is intuitively clear that the partition function for the 2-dimensional Ising model twisted in, say, one direction, can be obtained by using antiferromagnetic coupling constants and an odd number $(2M+1)$ of spins in the twisted direction, then analytically continuing the domain of definition of the resulting partition function Z_{2M+1}, to even integers $2M$, and similarly for twisting in both directions. The expressions required, may be found, e. g., in the treatise of McCoy and Wu.[2]

Although the Möbius band has been mentioned above for purposes of graphic illustration, the case in point was essentially a circle, and not a Möbius band. The present twisting construction is applicable to a lattice of completely arbitrary form on any compact polytope, orientable or not. For example, the fundamental group of the 2-dimentional torus is $Z \times Z$, while that of the Klein bottle is $Z \times Z_2$. In both cases, there are three essentially different representations of the fundamental group into Z_2, i. e., trivial, twisted in one direction, and twisted in both directions. Thus, considering both the ferromagnetic and antiferromagnetic cases, one obtains a total of six essentially different Ising-type models on each of these two manifolds.

For 1-dimensional models, the only possible finite critical temperature is presumably zero, and hence consideration of the zero-temperature continuum limits of the above models could serve to display their topological differences independently of the finite length N. This can be achieved, for example, by applying the following renormalization procedure to the correlation functions.

First, define

$$\gamma(\alpha, \beta) \;=\; \lim_{N \to \infty} [\tanh(\beta N \tanh^{-1} \alpha^{1/N})]^N \tag{14}$$

where $0 < \alpha < 1$ and $0 < \beta < \infty$. This function possesses the following properties. First, for all $N > 1/\beta$,

$$[\tanh(\beta N \tanh^{-1} \alpha^{1/N})]^N \geq [\tanh(\tanh^{-1} \alpha^{1/N})]^N = \alpha \tag{15}$$

Hence, $1 \geq \gamma(\alpha,\beta) \geq \alpha > 0$. Moreover, $\gamma(\alpha,\beta)$ is clearly monotonic increasing with respect to both α and β.

For purposes of comparison, we first indicate the results for the conventional (untwisted) cases.

1). Closed ferromagnetic chain

Consider a closed ferromagnetic Ising chain with N spins and coupling constant J. As is well known,[3] the 2-point correlation is given by

$$< \sigma_i \sigma_{i+r} > = \frac{[\tanh(\beta J)]^r + [\tanh(\beta J)]^{N-r}}{1 + [\tanh(\beta J)]^N}. \tag{16}$$

Now, choose any number α with $0 < \alpha < 1$, allow N and r to approach at a fixed ratio, $r = sN$, $(0 < s \leq 1/2)$, and let

$$J \equiv J_N \equiv N \tanh^{-1}(\alpha^{1/N}) \tag{17}$$

Hence, if we define $\varphi(\alpha,\beta,s)$ as the limiting value of the 2-point correlation function, then/par

$$\varphi_{CF}(\alpha,\beta,s) = \frac{[\gamma(\alpha,\beta)]^s + [\gamma(\alpha,\beta)]^{1-s}}{1 + \gamma(\alpha,\beta)} \tag{18}$$

2). Open ferromagnetic chain

For the open ferromagnetic Ising chain, the 2-point correlation function is given by

$$< \sigma_i \sigma_{i+r} > = [\tanh(\beta J)]^r \tag{19}$$

and the corresponding limiting process gives

$$\varphi_{OF}(\alpha,\beta,s) = [\gamma(\alpha,\beta)]^s \tag{20}$$

3). Twisted ferromagnetic chain

Applying the same limiting process to the 2-point correlation of the twisted ferromagnetic chain given by formula (13) above for $r \leq N/2$, one obtains

$$\varphi_{TF}(\alpha,\beta,s) = \frac{[\gamma(\alpha,\beta)]^s - [\gamma(\alpha,\beta)]^{1-s}}{1 - \gamma(\alpha,\beta)} \tag{21}$$

In the foregoing formulae, the subscripts CF, OF, and TF indicate closed ferromagnet, open ferromagnet, and twisted ferromagnet respectively. Notice that the twisted ferromagnetic chain provides a well-defined and finite limit in contrast with the untwisted antiferromagnetic case, where the limit is undefined.

3. GENERALIZATIONS

The preceding construction can be generalised to an arbitrary matrix group G, and matrix representation ρ of G, subject only to the restriction that $\Re$ (Tr $\rho(g)$) = $\Re$ (Tr $\rho(g^{-1})$) for any $g \in G$. The assignment of an element g_L in G to each oriented link L in the 1-skeleton K^1 of the polytope K under consideration, in such a manner that the inverse element $(g_L)^{-1}$ is assigned to the reversed link L^{-1}, is required to define a representation

$$\mathcal{R} : \pi_1(K) \longrightarrow G$$

in precisely the same manner as described above for the special case $G = Z_2$.

Similarly, the Hamiltonian of a configuration is defined by

$$-J \times \sum_{\text{all links } L \in K^1} \Re(Tr\rho(g_L)) \tag{22}$$

where the orientations of the links in the sum are arbitrary chosen. For example, choosing $G = U(1)$, and $\mathcal{R}$ as the trivial representation, one obtains the reduced XY model. Again, by reduced, we mean that the global symmetry (or degeneracy) has been factored out. This degeneracy could be restored by simply adjoining one varible $\varphi \in S^1$ (analogous to the above-mentioned $\sigma \in Z_2$), which does not enter into the Hamiltonian, thereby recovering the conventional (unreduced) XY model. Thus, if the polytope K is defined by a subdivision of the circle S^1, one obtains, corresponding to the totality of representations of the cyclic group

$$\mathcal{R} : \pi_1(K) \cong Z \longrightarrow U(1) \tag{23}$$

a continuum of twisted XY models labelled by the angular parameter θ, $0 \leq \theta < 2\pi$.

Letting $Z_{\mathcal{R}}(\beta, J)$ denote the partition function associated with the representation $\mathcal{R}$ of $\pi_1(K)$ in G, one can also define a *holocyclic* partition function $\hat{Z}(\beta, J, G)$ by summing (or integrating) the twisted partition function corresponding to all possible representations $\mathcal{R}$, i. e.,

$$\hat{Z}(\beta, J, G) = \sum_{\mathcal{R}:\pi_1(K)\to G} Z_{\mathcal{R}}(\beta, J). \tag{24}$$

Thus, the holocyclic partition function embraces all possible link configurations satisfying the local commutativity condition, i. e., such that the ordered product of the link variables around the boundary of any 2-cell is equal to the identity element of G. If K is 1-dimensional, this merely amounts to eliminating all couplings, resulting in a trivial product of sums over individual link variables. In the case of the aforementioned Ising models,

$$\hat{Z}(N) = Z_u^r + Z_t^r = [2\cosh(\beta J)]^N \tag{25}$$

i. e., the effect of the summation is to eliminate all but the maximum eigenvalue of the transfer operator. Moreover, this is also true for the XY model on s^1, as follows immediately by integration of formula (39) below with respect to h. That is, in these cases, the maximum

value of the transfer operator is associated with the trivial character of the group G. At present, the author does not know whether this is merely a curious coincidence or is true in some more general context.

4. CIRCULAR MODELS

In this section, the model defined in the preceding section will be specialized to the case where the polytope K is a circle and the twisted partition function will be explicitly evaluated for G = U(1) and $\rho(g) = g$ as well as G = SU(2) and ρ the spin 1 representation.

First, consider a circle divided into n segments, and assign link variables $g_1, g_2, \cdots, g_n$ ($g_j \in G$) to the succcessive segments. These link variables g_i are subject to the constraint

$$\prod_{i=N}^{1} g_i \equiv g_N g_{N-1} \cdots g_1 = h \tag{26}$$

for some fixed $h \in G$, which uniquely determines the representation $\mathcal{R}_h$ of $\pi_1(S^1) \cong Z$ defined by $\mathcal{R}_h(a) = h$, where a is a generator of $\pi_1(S^1)$. Now, define a Hamiltonian by

$$\mathcal{H}(N,\rho) = -J\sum_{i=1}^{N} \Re\chi_\rho(g_i) \tag{27}$$

where $\chi_\rho(g) = \mathrm{Tr}\rho(g)$ is the trace of the representation ρ of the group G. Letting $\nu = \beta J$, the partition function is then

$$Z_N(h) = \int_{[\prod_{i=N}^{1} g_i = h]} dg_N \cdots dg_1 e^{\nu \sum_{i=1}^{N} \Re\chi_\rho(g_i)} \tag{28}$$

and by integrating over the Nth variable, one obtains the following recursion formula,

$$Z_N(h) = \int dg_N e^{\nu\chi_\rho(g_N)} Z_{N-1}(g_N^{-1}h). \tag{29}$$

The expression on the right-hand side is a convolution. Moreover, since $Z_N(h)$ obviously depends only upon the conjugacy class of h, the Peter-Weyl theorem as applied to the present case provides a generalized Fourier expansion in terms of the irreducible characters $\chi_j(g)$, and one obtains

$$c_j(N) = [\int \chi_j^*(g) e^{\nu\chi_\rho(g)} dg][c_j(N-1)]d_\rho^{-1} \tag{30}$$

where c_j is the jth Fourier component of the partition function Z_N, and d_ρ is the dimension of ρ. Hence, by induction,

$$c_j(N) = [\int \chi_j^*(g) e^{\nu\chi_\rho(g)} dg]^N d_\rho^{-N}. \tag{31}$$

In the spetial case where the trace of every irreducible character $\chi_j(g)$ is real (e g , G = SU(2)), this Fourier component c_j can be expressed as follows

$$c_j = d_\rho^{-1} \int \chi_j^*(g) e^{\nu\chi_\rho(g)} dg = \sum_{n=0}^{\infty} \frac{\nu^n}{n!} \int \chi_j(g)\chi_\rho^n(g) dg \tag{32}$$

Therefore,

$$c_j(N) = d_j^{-N}\{\sum_{n=0}^{\infty} \frac{\nu^n}{n!} n_s(\rho_j \otimes \rho \otimes \cdots \otimes \rho)\}^N \tag{33}$$

where n copies of ρ appear in the tensor product in the nth term, and $n_s(\rho_j \otimes \rho \otimes \cdots \otimes \rho)$ denotes the number of times the singlet representation occurs in the Clebsch-Gordan decomposition of $\rho_j \otimes \rho \otimes \cdots \otimes \rho^{(n)}$. Hence, we obtain the partition function for the circular model with respect to an arbitrary compact Lie group in the form

$$Z_N^\rho(h) = \sum_j c_j(N)\chi_j(h) = \sum_j [\int \chi_j^*(g) e^{\nu\chi_\rho(g)} dg]^N d_\rho^{-N}, \tag{34}$$

and, in particular, if all the irreducible characters are real, then

$$Z_N^\rho(h) = \sum_j c_j(N)\chi_j(h) = \sum_j [d_j^{-1} \sum_{n=0}^{\infty} \frac{\nu^n}{n!} n_s(\rho_j \otimes \rho \otimes \cdots \otimes \rho)]^N \chi_j(h) \tag{35}$$

However, the above formulae are not necessarily the most convenient ones for specific calculations.

As an example, consider first G = U(1). In this case, the partition function corresponding to the above equation (28) is,

$$Z_N(h) = \int_{[\sum_{i=1}^{N} \varphi_i = h]} \frac{d\varphi_N}{2\pi} \cdots \frac{d\varphi_1}{2\pi} e^{\nu \sum_{i=1}^{N} \cos\varphi_i} \tag{36}$$

where $\varphi_i = \theta_{i+1} - \theta_i$ may be regarded as the angular difference between two successive spins in the X-Y plain. Clearly, for the trivial representation $h = 0$, one obtains the partition function for the conventional XY model in reduced form, i e., divided by a factor of 2π. The recursion formula is

$$Z_N(h) = \int_0^{2\pi} \frac{d\varphi_N}{2\pi} e^{\nu\cos\varphi_N} Z_{N-1}(h - \varphi_N). \tag{37}$$

Hence, the j-th Fourier coefficient is

$$c_j(N) = [I_j(\nu)]^N \tag{38}$$

where I_j is the Bessel function of imaginary argument. Writing the Fourier expansion, one obtains the partition function for the twisted XY model as follows

$$Z_N(h) = \sum_{j=-\infty}^{\infty} [I_j(\nu)]^N e^{i(jh)}. \tag{39}$$

Thus, the twisted partition function is essentially the generating function for the Nth powers of the Bessel function. In the thermodynamic limit, the effect of the non-trivial representation (i. e., $h \neq 0$) will clearly disappear since $I_0 > I_k$ for all $k \neq 0$.

The two-point correlation function[5] for this model can also be calculated in exactly the same manner, with the result

$$C_N(r,h) = \sum_{j=-\infty}^{\infty} [I_{j-1}(\nu)]^r [I_j(\nu)]^{N-r-1} e^{i(jh)} \tag{40}$$

where C_N is defined by

$$C_N(r,h) \equiv Z_N(h) < e^{-i(\varphi_0+\cdots+\varphi_{r-1})} >, \tag{41}$$

and $r \leq N/2$ is the distance between the two spins.

Next, consider G = SU(2). The character of the spin 1 representation is

$$\chi_\rho(g) = 1 + 2\cos(\tau) \tag{42}$$

where τ is the angle of rotation, and in general the spin j character $\chi_j(g)$ is given by[6]

$$\chi_j(g) = \sum_{m=-j}^{j} e^{im\tau} = \frac{\sin[(j+\frac{1}{2})\tau]}{\sin(\frac{\tau}{2})} \tag{43}$$

where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$. Substituting these results into the above equation (34), one obtains

$$Z_N(h,\tau) = \sum_{j=0,1/2,1,\cdots} \{[\frac{e^\nu}{2j+1} \sum_{m=-j}^{j} \tilde{I}_m(2\nu)]^N (\frac{\sin[(j+\frac{1}{2})\tau]}{\sin(\frac{\tau}{2})})\} \tag{44}$$

where $\tilde{I}_m(z)$ is defined by

$$\tilde{I}_m(z) \equiv \frac{1}{2\pi}\int_0^{2\pi} e^{z\cos\theta + im\theta} d\theta \tag{45}$$

and coincides with the Bessel function $I_m(z)$ for integral m.

5. ALTERNATIVE HAMILTONIANS

The preceding constructions include the conventional Ising and XY models as special cases, but not include the classical Heisenberg model. To obtain a generalization of the latter, the definition of the Hamiltonian must be modified as follows.

Let K be a connected polytope, fix a vertex x_0 as base point, choose a maximal tree[4] $\mathcal{T}$ in the 1-skeleton K^1 of K, and regard $\mathcal{T}$ as a set of links L, oriented from the root x_0

toward the tips of $\mathcal{T}$. If $\mathcal{S} \subset \mathcal{T}$, let $|\mathcal{S}|$ denote the cardinality (number of links) of $\mathcal{S}$. For $L \in \mathcal{T}$, define

$$\delta(L) = L_1 \cdots L_n \,; \quad L_n = L$$

as the unique minimal path from x_0 which includes L, and

$$\gamma(L) = L_1 \cdots L_{n-1}$$

(or the trivial path at x_0 if n = 1), i. e., $\delta(L) = \gamma(L)L$.

Next, let G be a Lie group, let $G^{\mathcal{T}}$ denote totality of mappings from $\mathcal{T}$ to G, i. e., the set of all configulations $g = \{g_L \mid L \in \mathcal{T}\}$ and for $g \in G^{\mathcal{T}}, L \in \mathcal{T}$, define

$$g_{\delta(L)} = g_{L_n} \cdots g_{L_1}$$

$$g_{\gamma(L)} = g_{L_{n-1}} \cdots g_{L_1}$$

where the L_j are related to L_n as indicated above, and $g_{\gamma(L)}$ is the identity element of G if n = 1.

Let H be a Lie subgroup of G, let G/H denote the left coset space, and let $(G/H)^{K^0}$ be the set of all configulations $\{\varphi(x) \in G/H \mid x \in K^0\}$, where K^0 denotes the 0-skeleton (i. e., vertex set) of K. Fix an element $u_0 \in G/H$ (e. g., $u_0 = H$) and define

$$\Phi_0 = \{\varphi \in (G/H)^{K^0} \mid \varphi(x_0) = u_0\} \tag{46}$$

Proposition. $G^{\mathcal{T}}$ is a principal fiber bundle over Φ_0, with a Lie group $H^{\mathcal{T}}$ as fiber. If G is compact, then $H^{\mathcal{T}}$ is compact, moreover, the Haar measure on $G^{\mathcal{T}}$, considered as a direct product of groups $G_L = G$, $L \in \mathcal{T}$, is locally a product of the usual measure on Φ_0, considered as a cartesian product of coset spaces, and the Haar measure on the fiber $H^{\mathcal{T}}$.

This proposition is proved in Appendix A.1.

Now, consider any link L which is *not* contained in the maximal tree, arbitrarily choose an orientation (x,x') of L, and extend the foregoing definition of $\gamma(L)$ and $\delta(L)$ in the obvious manner by defining $\gamma(L) = \delta(\tilde{L})$, where $\delta(\tilde{L})$ is the minimal path in $\mathcal{T}$ from the base point x_0 to the initial point x of L, and $\delta(L) = \gamma(L)L$.

Next, given a representation $\mathcal{R} : \pi_\infty(K) \to G$, let $G_{\mathcal{R}}^{K^1}$ denote the totality of link configuarations $\{g_L \mid L \in K^1\}$ defined on the entire 1-skeleton K^1 and associated with the representation $\mathcal{R}$ as indicated in §3 above. Observe that any $g \in G_{\mathcal{R}}^{K^1}$ is uniquely determined by the restriction $g_{|\mathcal{T}}$, and conversely that any $g \in G^{\mathcal{T}}$ uniquely determines an element $g \in G_{\mathcal{R}}^{K^1}$. Hence, $G_{\mathcal{R}}^{K^1}$ can be identified with $G^{\mathcal{T}}$, and accordingly $G_{\mathcal{R}}^{K^1}$ may be regarded as a principal fiber bundle over Φ_0, with a compact fiber if G is compact.

Next, given a representation $\mathcal{R} : \pi_1(K) \to G$, and assuming that G/H is equipped with a G-invariant inner product, define a Hamiltonian on $G_{\mathcal{R}}^{K^1}$ by

$$\tilde{\mathcal{H}}_{\mathcal{R}}(g,u_0) = -J \sum_{L \in K^1} (g_{\delta(L)}u_0, g_{\gamma(L)}u_0). \tag{47}$$

In general, this model will depend upon the choice of the maximal tree $\mathcal{T}$ and the initial spin u_0 as well as the orientations of the links $L \in K^1 \backslash \mathcal{T}$, although this dependence

could be removed by averaging the partition function over all possible choice. However, in the case where $\mathcal{R}$ is the trivial representation $\mathcal{R}_0$, one can readily verify that the above Hamiltonian is already independent of the orientations of the links $L \in K^1\backslash\mathcal{T}$, and that

$$\tilde{\mathcal{H}}_{\mathcal{R}_i}(g, u_0) = \mathcal{H}_{Hei}(pg), \tag{48}$$

where pg is the image of g under the bundle projection $p : G^{\mathcal{T}} \to \Phi_0$ of the above proposition, and $\tilde{\mathcal{H}}_{Hei}$ is the Hamiltonian of the conventional classical Heisenberg type model. Also, in this case, the partition function is clearly independent of the choice of u_0. Since the measure on $G^{K^1}_{\mathcal{R}_i}$ is locally a product of the conventional measure on Φ_0 and the fibre measure induced by the Haar measure on $G^{\mathcal{T}}$, the partition function defined by $\tilde{\mathcal{H}}_{\mathcal{R}_i}(g, u_0)$ will coincide with that of the conventional Heisenberg type model, except for an immaterial constant factor corresponding to the global symmetry involved in the choice of the initial spin u_0. In particular, for the trivial representation $\mathcal{R}_0$, the partition function will also be independent of the choice of the maximal tree $\mathcal{T}$.

Note that the rather tedious construction given above is necessary only if G is non-abelian. For abelian G, the procedure employed in §1 and §2 is sufficient to construct models which include, e. g., the conventional Ising and X-Y models as special cases.

As a specific example, consider the twisted Heisenberg model on a circle S^1, with $G = SO(3)$ and $G/H = S^2$, and a representation $\mathcal{R}_h$ defined by an arbitrary element $h \in SO(3)$, i. e., $\mathcal{R}_h(\alpha) = h$, where α is a generator of $\pi_1(S^1) \cong Z$. Subdivid S^1 into N consecutive segments $L_1, \cdots, L_N$; in the present case, there exists essentially only one maximal tree $\mathcal{T}$, say, $\mathcal{T} = \{L_1, \cdots, L_{N-1}\}$, and the indeterminacy in the choice of the orientation of L_N corresponds to replacement of the element h by h^{-1}, which in the case of SO(3) is conjugate to h. Hence, if the partition function is averaged over the conjugacy class of h, then the ambiguity corresponding to the orientation of L_N will also be eliminated.

Applying the above construction, one obtains the partition function

$$\tilde{Z}_N(h, u_0) = \int dg_N \cdots dg_1 \exp\{\nu \sum_{i=1}^{N} (g_{i-1} \cdots g_1 u_0, g_i \cdots g_1 u_0)\} \tag{49}$$

where u_0 is the initial unit spin vector. Using the G-invariance of the Haar measure and the inner product, this can be rewritten as

$$\tilde{Z}_N(h, u_0) = \int_{[\prod_{i=N}^{1} \hat{g}_i = h]} d\hat{g}_N d\hat{g}_{N-1} \cdots d\hat{g}_1 \exp\{\nu \sum_{i=1}^{N} (\hat{g}_i u_0, u_0)\} \tag{50}$$

where the new link variable $\hat{g}_i$, is defined by

$$\hat{g}_i \equiv g_1^{-1} \cdots g_{i-1}^{-1} g_i \cdots g_1. \tag{51}$$

Hence, one obtains the recursion relation

$$\tilde{Z}_N(h, u_0) = \int d\hat{g}_N e^{\nu(\hat{g}_N u_0, u_0)} \tilde{Z}_{N-1}(\hat{g}_N^{-1} h) \tag{52}$$

If $u_0 = \hat{z} = (0, 0, 1)$ in cartesian coordinates, then $(\hat{g}u_0, u_0) = (\hat{g}\hat{z}, \hat{z}) = \cos\theta$, the cosine of the second Euler angle of $\hat{g}$ in the usual parametrization of SO(3). Hence, omitting the circumflex over g,

$$\tilde{Z}_N(h, \hat{z}) = \int dg_N e^{\nu \cos\theta_N} \int_{[\prod_{i=N-1}^{1} g_i = h]} \prod_{i=1}^{N-1} dg_i e^{\sum_{i=1}^{N-1} \cos\theta_i} \tag{53}$$

Here again, the partition function $\tilde{Z}_N$ can be calculated by using the FPW (Fourier-Peter-Weyl) expansion, with the result

$$\tilde{Z}_N(h, \hat{z}) = \sum_l [\frac{1}{2}(2l+1)^{1/2} \tilde{P}_l(\nu)]^N \varphi_l \tag{54}$$

where

$$\begin{aligned} \tilde{P}_l(\nu) \equiv & \sum_{s=0}^{[l/2]} \{(-1)^s \frac{(2l-2s)!}{2^l s!(l-s)!(l-2s)!} [(-1)^{l-2s} \frac{2(l-2s)!}{\nu^{l-2s+1}} \sinh\nu \\ & + \frac{1}{\nu}(e^{\nu} - (-1)^{l-2s} e^{-\nu}) \sum_{t=0}^{l-2s-1} (-\frac{1}{\nu})^t \frac{(l-2s)!}{(l-2s-t)!}]\}, \end{aligned} \tag{55}$$

$$\varphi_l = (2l+1)^{1/2} P_l(\cos\theta), \tag{56}$$

and θ is the second Euler angle of h.

However, $\tilde{Z}_N$ is *not* invariant under conjugation of h, hence, in order to obtain an invariant partition function Z_N, the above $\tilde{Z}_N$ must be integrated over the entire conjugacy class of h, i. e.,

$$Z_N(h) = \int_{k \in G} \tilde{Z}_N(khk^{-1}, \hat{z}) dk \tag{57}$$

Performing this integration, one obtains the partition function for the twisted 1-dimentional classical Heisenberg model as follows.

$$Z_N(h_r) = 2\pi \sum_l \{(2l+1)^{1/2} [\frac{1}{2}(2l+1)^{1/2} \tilde{P}_l(\nu)]^N \int P_l(\cos\theta)(1 - \cos\tau_k) \sin\alpha d\alpha d\tau_k\} \tag{58}$$

where

$$\cos\theta = 1 - \sin^2\alpha(1 - \cos\tau_k)(2 - \sin^2\alpha)(1 - \cos\tau_k)(1 - \cos\tau_k) \tag{59}$$

The explicit calculation of the foregoing results is given in Appendix 2.

Other Hamiltonians may naturally arise if noncompact groups G are considered. For example, let $G = R^n$, considered as an ablian group with respect to vector addition. For a link configuration $\{\vec{v} \mid L \in K^1\}$, where $\vec{v} \in R^n$, define a Hamiltonian by

$$\mathcal{H} = J \sum_L |\vec{v}|^2. \tag{60}$$

Again, consider a circle $K = S^1$, subdivided into N links, and the representation $\mathcal{R}$ determined by a fixed $\vec{v} \in R^n$, i. e., $\mathcal{R}(\alpha) = \vec{v}$, where α is a generator of $\pi_1(S^1) \cong Z$. In this case, the recursion relation assumes the form

$$Z_N(\vec{v}) = \int_{R^n} d\vec{v}_N e^{-\nu|\vec{v}_N|^2} Z_{N-1}(\vec{v} - \vec{v}_N), \tag{61}$$

hence, a straightforward application of the conventional Fourier transform in R^n and an induction yields the result

$$Z_N(\vec{v}) = [(\frac{\pi}{\nu})^{\frac{N}{2}}(\frac{2\nu}{N})^{\frac{1}{2}}]^n e^{-\frac{\nu}{N}|\vec{v}|^2}, \tag{62}$$

already familiar from the theory of Brownian motion, [7] and (with a proper normalization) representing the probability density, after N steps, of particles departing from the origin and executing an unbiased random walk on a hypercubical lattice with spacing $d = (1/4\nu)^{1/2}$.

6. EXTERNAL MAGNETIC FIELDS

A twisted external magnetic field can be introduced by means of another configuration $h(L) \in G$, defined on the same polytope K and determining the same representation $\mathcal{R}$ of $\pi_1(K)$ as the foregoing set of configurations g_L. Let x be any vertex in K, and let Γ denote an arbitrary polygonal path from a fixed base point x_0 to x. Define

$$g(\Gamma) = g_{L_r} g_{L_{r-1}} \cdots g_{L_1} \tag{63}$$

$$h(\Gamma) = h_{L_r} h_{L_{r-1}} \cdots h_{L_1} \tag{64}$$

where $L_1, L_2, ..., L_r$ denote the successive links in the path Γ. Then, $\mathcal{H}_I$, the portion of the Hamiltonian describing the interaction between the external field h and the spins g, can be defined as

$$\mathcal{H}_I^r(g, h, H) = -H \sum_x \Re(Tr[h(\Gamma_x) g^{-1}(\Gamma_x)]) \tag{65}$$

where H represents the strength of the applied field, the sum extends over all vertices (lattice points) x of K, and Γ_x denotes any polygonal path from x_0 to x. By virtue of the constraints imposed upon g and h, this definition is independent of the particular selection of these paths. Thus, the partition function is given by

$$Z^r(\beta, J, K, H) = \sum_g \exp \beta[J \sum_L \Re(Trg_L) + K \sum_L \Re(Trh_L) + \mathcal{H}_I^r(g, h, H)] \tag{66}$$

where the sum (or more precisely, in the case of a continuous group G, the integral) is performed over all configurations g associated with the given representation $\mathcal{R}$, with h

remaining fixed. The energy term corresponding to h alone has been introduced merely to emphasize the complete symmetry between the roles of the spins and external fields in the Hamiltonian. Thus, one could equally fix well g and sum over all configurations h. The magnetization and susceptibility with respect to any external field configuration h can then be defined in the conventional manner by differentiation with respect to the field strength H.

The above model, although interesting *per se*, is reduced in the sense that the Zeeman splitting effect on the global degeneracy is absent (hence the use of the superscript r in (65) and (66)). This splitting may be recovered by the following modification. Assume that the group G acts upon the coset space $X = G/H$ with a real G-invariant inner product. Introduce the auxiliary variables u, $v \in X$, replace the interaction Hamiltonian (65) by

$$\mathcal{H}_I(g, u; h, v; H) = -H \sum_x (g(\Gamma_x)u, h(\Gamma_x)v) \tag{67}$$

and extend the summation in the partition function (66) over both g and u, keeping both h and v fixed, i. e.,

$$Z(\beta, J, K, H, h, v) = \sum_{g,u} \exp \beta[J \sum_L \Re(Trg_L) + K \sum_L \Re(Trh_L) + \mathcal{H}_I(g, u; h, v, H)] \tag{68}$$

As a special case, if one chooses the trivial representation of $\pi_1(K)$, the trivial external field configuration $h_L = 1$ (i. e., the identity of G), and v an arbitrary fixed element of X, then one recovers the conventional XY model for $G = U(1)$ and $X = S^1$, and, the conventional Ising model for $G = Z_2$ and $X = S^0 = Z_2$. If ,in addition, the first two terms of the Hamiltonian in the above equation (68) are modified as indicated in §5, then one recovers the conventional classical Heisenberg model for $G = SO(3)$ and $X = S^2$.

As a specific example, consider again the twisted Ising model on $K = S^1$, subdivided into N links $L_1, \ldots, L_N$, and let $(H, h^{\pm}(N))$ denote the external fields defined by the intensity H and the link configurations $h^{\pm}(N)$ such that

$$h^{\pm}_{L_1}(N) = \pm 1, \quad h^{\pm}_{L_2} = \cdots = h^{\pm}_{L_N}(N) = 1.$$

The partition function in the general formula (68) is clearly independent of the choice of v, which can be arbitrary fixed at $v = 1$, say. The summation variable u in (68) will be denoted by σ in the following calculation ($\sigma = \pm 1$). Letting $B = \beta H$, define

$$Z_u^\sigma = \sum_{r_1 \cdots r_N = +1} e^{\nu \sum_{i=1}^N r_i + \sigma B \sum_{i=1}^N r_1 \cdots r_i}, \tag{69}$$

referring to the reduced untwisted model, and similarly,

$$Z_t^\sigma = \sum_{r_1 \cdots r_N = -1} e^{\nu \sum_{i=1}^N r_i + \sigma B \sum_{i=1}^N r_1 \cdots r_i}, \tag{70}$$

referring to the reduced twisted model. These expressions correspond to the sums over the link variables g, prior to summation over u (denoted here by σ) in formula (68),

for the untwisted and twisted models, respectively, with the external fields $(H, h^+(N))$ and $(H, h^-(N))$, respectively (also, the constant term with coefficient K in (68) has been omitted). Then, the following recursion relations, written in matrix form, are readily verified.

$$\begin{pmatrix} Z_u^\sigma(N) \\ Z_t^\sigma(N) \end{pmatrix} = \begin{pmatrix} e^{\nu+\sigma B} & e^{-\nu+\sigma B} \\ e^{-\nu-\sigma B} & e^{\nu-\sigma B} \end{pmatrix} \begin{pmatrix} Z_u^\sigma(N-1) \\ Z_t^{-\sigma}(N-1) \end{pmatrix} \tag{71}$$

Moreover, a direct calculation for $N = 2$ yields

$$\begin{pmatrix} Z_u^\sigma(2) \\ Z_t^{-\sigma}(2) \end{pmatrix} = \begin{pmatrix} e^{2\nu+2\sigma B} + e^{-2\nu} \\ 1 + e^{-2\sigma B} \end{pmatrix} \tag{72}$$

The eigenvalues λ_1 and λ_2 of the matrix in (71) are identical with those occurring in the evaluation of the partition function for the conventional (untwisted) closed Ising chain with an external field.[3] Hence, one can evaluate $Z_t^\sigma(N)$ by a straight forward diagonalization procedure, and the final result follows, i. e.,

$$Z_t(N) = Z_t^+(N) + Z_t^-(N). \tag{73}$$

However, the explicit expression so obtained is rather complicated, and will be omitted.

7. COHOMOLOGICAL INTERPRETATION

In the case where the space underlying the polytope K is a group manifold, or the group G is abelian, one may restrict the treatment to representations of the 1st homology group $H_1(\mathrm{K})$. Each representation $\mathcal{R}$ thus defines a cohomology class $\hat{\mathcal{R}} \in H^1(\mathrm{K}; \mathrm{G})$. For the trivial representation, the above-mentioned link configurations $\{\tau\}$ are exact 1-cocycles, defining the trivial cohomology class, while a nontrivial representation corresponds to the link configurations defining a specified nontrivial cohomology class. This is analogous to abelian gauge theory, where the fields arising from globally defined gauge potentials are exact 2-cocycles over the (abelian) Lie algebra of G, while the closed inexact 2-cocycles are known as monopoles. Thus, the above construction may be regarded as an extension of the conventional spin configurations to those in cohomologically nontrivial soliton sectors.

If the underlying space polytope $|\mathrm{K}|$ is a manifold M, then, in the continuum limit, the configurations defined in §3 may be identified with smooth 1-forms ω on M with values in the Lie algebra $\mathcal{G}$ of G, possibly nonabelian, and the path-ordered exponentials[4],[8] of ω computed around smooth loops with a fixed basepoint x_0 are required to define a given representation $\mathcal{R}$: $\pi_1(\mathrm{M}) \to \mathrm{G}$, i. e.,

$$P\exp\left(\oint_c \omega\right) = \mathcal{R}([\mathrm{C}]), \tag{74}$$

where $[\mathrm{C}]$ denotes the element of $\pi_1(\mathrm{M})$ represented by C. In particular, every such form ω is closed in the sense that the path-ordered exponential of ω around any sufficiently

small loops is equal to the identity element of G. Note that these forms ω are essentially different from the local connexion forms of gauge fields, which are *not* closed except for flat connexions, and need not be globally defined. However, if $\mathcal{R}$ is the trivial representation, then the correspondence

$$x \to g(x) = P\exp\left(\int_{C_x} \omega\right), \tag{75}$$

where C_x denotes an arbitrary path from x_0 to x, uniquely defines a mapping from M to G, and then ω is merely the pullback with respect to this mapping of the Maurer-Cartan form on G (i. e., a globally defined pure gauge potential).

Assuming that M is equipped with a pseudoriemannian metric, the most plausible candidate for the Hamiltonian in this case appears to be the usual

$$\mathcal{H}(\omega) = J \int_{\mathrm{M}} Tr(\omega_\rho \wedge *\omega_\rho) \tag{76}$$

where ω_ρ is the image of ω under the Lie algebra homomorphism induced by the representation ρ of G, and $*\omega$ denotes the Hodge dual,[9] although numerous variants are conceivable.

When $\mathrm{G} = \mathrm{U}(1)$, so that $\mathcal{G} = i\mathbf{R}$, the forms ω may be identified with real-valued forms (ω/i), and if ρ is the identity representation $\mathrm{G} \to \mathrm{G}$, then a variation of the above Hamiltonian (i. e., action) within a fixed cohomology class Ω, that is

$$\omega \to \omega + \delta\omega = \omega + \mathrm{d}f \tag{77}$$

where f is an arbitrary locally defined real function, conbined with an application of Stokes' theorem, immediately yields the equation

$$\mathrm{d} * \omega = \Delta\omega = 0. \tag{78}$$

Hence, the minimal energy configuration in each soliton sector (i. e., cohomology class) Ω is just the unique harmonic 1-form in Ω. In the discrete toroidal lattice version, the corresponding conclusion is obvious from symmetry considerations.

In analogy with the proof of the Proposition in §5, for each $\omega \in \Omega_0$, the space of $\mathcal{G}$-valued 1-forms ω associated with the trivial representation $\mathcal{R}_0$ of $\pi_1(\mathrm{M})$ in G, one obtains a continuous spin field on M by the assignment

$$x \to \varphi_\omega(x) \equiv P\exp\left(\int_{C_x} \omega\right) u_0, \tag{79}$$

where u_0 is a fixed spin in G/H and C_x denotes an arbitrary path from x_0 to x. This presumably exhibits Ω_0 as a principal fibre bundle, now infinite-dimensional, over the space

$$\Phi_0 = \{\varphi \in (\mathrm{G/H})^{\mathrm{M}} \mid \varphi(x_0) = u_0\}, \tag{80}$$

where as usual $(G/H)^M$ denotes the totality of smooth spin fields $\{\varphi(x) \in G/H \mid x \in M\}$. However, a rigorous proof of this assertion, generalising the proof of the aforesaid proposition, would appear to present some nontrivial technical difficulties.

If the representation $\mathcal{R}$ is nontrivial, then let $\Omega_{\mathcal{R}}$ denote the space of smooth $\mathcal{G}$- valued 1-forms on M associated with $\mathcal{R}$, and let $\tilde{M}_{\mathcal{R}}$ denote the regular covering space[10] of M corresponding to the kernel $\tilde{\pi}$ of $\mathcal{R}$. Then, any $\omega \in \Omega_{\mathcal{R}}$ lifts under the pullback of the covering map to a 1-form $\tilde{\omega}$ associated with the trivial representation of $\pi_1(\tilde{M}_{\mathcal{R}}) \cong \tilde{\pi}$, and $\tilde{\omega}$ in turn defines a spin configuration on $\tilde{M}_{\mathcal{R}}$ as described above. In this manner, $\Omega_{\mathcal{R}}$ can be described as a principal fibre bundle over a certain subspace of the spin configuration space

$$\tilde{\Phi}_0 \;=\; \{\tilde{\varphi} \in (G/H)^{\tilde{M}_{\mathcal{R}}} \mid \tilde{\varphi}(\tilde{x}_0) = u_0\}, \tag{81}$$

where $\tilde{x}_0$ is an arbitrary base point in $\tilde{M}_{\mathcal{R}}$. Since the covering map is a local homeomorphism, locally defined Hamiltonians for the spin configurations on M can be lifted to $\tilde{M}_{\mathcal{R}}$ and then from $\tilde{\Phi}_0$ to $\Omega_{\mathcal{R}}$; however, the calculation of partition functions using such Hamiltonians may involve technical complications if $\tilde{M}_{\mathcal{R}}$ is noncompact.

In the 1-dimensional case, i.e., $M = S^1$, a probability measure, the continuum analogue of a Haar product measure, can be rigorously constructed on the space Ω_c of continuous forms ω of the above mentioned type. Details will be deferred to a subsequent paper.

8. OPEN PROBLEMS

Problems arising in connection with extensions of conventional lattice model theory in the foregoing context include the following.

1. Explicit calculations of magnetization and susceptibility
2. Establishment of correlation inequalities
3. Derivation of low- and high-temperature expansions
4. Formulation of duality in two dimensions
5. Application of the conventional concepts of mean field approximation and renormalisation group (i. e., Kadanoff transformations, etc.) to twisted models
6. Rigorous construction of continuum limits
7. Adaptation of the above constructions to quantum models, e. g., the quantum Heisenberg model
8. Definition of interactions with gauge fields

9. Creation of efficient algorithms for Monte Carlo computation

9. CONCLUSION

Finally, a few quasiphilosophical comments may be in order

In the literature concerning conventional Ising and related lattice field models, one sees frequent references to lattices "wrapped on a torus", etc However, this phrase merely indicates periodic boundary conditions, which only involve the topology of the 1-skeleton, and the presence of the 2-cells of the torus is irrelevant. any or all of the 2-cells (or "plaquettes") could be removed without any effect upon the model. On the other hand, such topological changes would radically affect the twisted models defined above Thus, the present formulation, unlike the conventional one, relates the spin configurations to the 2-dimensional topology of the ambient space.

The models discussed above have been introduced in the spirit expounded by Baxter[11], i. e., that any solvable or potentially solvable lattice models are worth investigating As pointed out in §6, the present constructions are essentially of a soliton type, and as such their introduction requires no more or less justification than the introduction of Dirac or nonabelian monopoles, etc., the physical existence or importance of which still remains to be established.

In view of the fact that, for nonabelian groups, the configurations introduced above constitute fibre bundles over the conventional configuration spaces, as indicated in the Proposition of §5, they could be regarded as more fundamental entities than the latter. just as spinor bundles are of a more basic character than tensor bundles. Of course the ultimate justification for this view in the present case could only come from future experimental evidence.

In any event, the material discussed in the present paper merely sets the stage for detailed numerical computations as well as more interesting examples in two or higher dimensions. These will be pursued in subsequent papers.

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APPENDIX

Appendix 1.

The proposition in §5 is proved as follows.

Using the notation introduced in §5, define a product $g \circ g'$ in $G^{\mathcal{T}}$ by the rule

$$(g \circ g')_L = g_L g_{\gamma(L)} g'_L g^{-1}_{\gamma(L)} ,$$

where the product on the right-hand side is calculated in G. Note the identities

$$(g \circ g')_{\delta(L)} = g_{\delta(L)} g'_{\delta(L)}$$

and

$$(g \circ g')_{\gamma(L)} = g_{\gamma(L)} g'_{\gamma(L)}$$

where again, in each case, the product on the right-hand side is calculated in G. One can readily verify that $G^{\mathcal{T}}$ becomes a Lie group with respect to the product $g \circ g'$ and the cartesian product topology, and that $H^{\mathcal{T}}$ form a Lie subgroup of $G^{\mathcal{T}}$. Note that, in the case where $\mathcal{T}$ consists of just two links, this group is merely the usual semi-direct product of two copies of H with respect to inner automorphisms.

For each $g \in G^{\mathcal{T}}$, define an element $\varphi_g \in \Phi_0$ by the rule $\varphi_g(x) = g_{\delta(L)} u_0$, where $\delta(L)$ is a minimal path in $\mathcal{T}$ from x_0 to x if $x \neq x_0$, and $\varphi_g(x_0) = u_0$. Clearly, the mapping p: $G^{\mathcal{T}} \to \Phi_0$ defined by $pg = \varphi_g$ is surjective. Moreover, for any point $x \in K^0$ and $h \in H^{\mathcal{T}}$,

$$(\varphi_{g \circ h})(x) = (g \circ h)_{\delta(L)} u_0 = [g_{\delta(L)} h_{\delta(L)}] u_0 = g_{\delta(L)} u_0 = \varphi_g(x)$$

Hence, the projection p is constant on the left cosets of $H^{\mathcal{T}}$ in $G^{\mathcal{T}}$. Furthermore, if $pg = pg'$, i.e., $\varphi_g = \varphi_{g'}$, then, for all links $L \in \mathcal{T}$, $g_{\delta(L)} u_0 = g'_{\delta(L)} u_0$, hence $g_{\delta(L)} = g'_{\delta(L)} h_L$ for some element $h_L \in H$. Define h'_L inductively by $h'_L = h_L$ for $|\gamma(L)| = 1$, and $h'_L = h_L (h'_{\gamma(L)})^{-1}$ for $|\gamma(L)| > 1$. Then,

$$h'_{\delta(L)} = h'_L h'_{\gamma(L)} = h_L (h'_{\gamma(L)})^{-1} h'_{\gamma(L)} = h_L ,$$

hence,

$$g_{\delta(L)} = g'_{\delta(L)} h'_{\delta(L)} = (g' \circ h')_{\delta(L)}$$

for all L, therefore, $g_L = (g' \circ h')_L$ for all L, and $g = g' \circ h'$. Since $h \in H^{\mathcal{T}}$, we can conclude that each fibre $p^{-1}\varphi$ constitutes exactly one left coset in $G^{\mathcal{T}}/H^{\mathcal{T}}$. Therefore, Φ_0 can be identified with the left coset space $G^{\mathcal{T}}/H^{\mathcal{T}}$, in particular, $G^{\mathcal{T}}$ is a principal fibre bundle over Φ_0, with the Lie group $H^{\mathcal{T}}$ as fibre.[12] Clearly, if G is locally compact, then the Haar measure μ on $G^{\mathcal{T}}$, considered as a direct product of groups $G_L = G$ $(L \in \mathcal{T})$, is also invariant under the multiplication $(g, g') \to g \circ g'$, and therefore μ also constitutes the unique Haar measure on the group $G^{\mathcal{T}}$ with respect to the product $g \circ g'$. Furthermore, the left coset corresponding to the element $\{g_{\delta(L)} \phi_0 \mid L \in \mathcal{T}\}$ in Φ_0 is mapped under

left multiplication by $h' \in G^{\mathcal{T}}$ into the element $\{g'_{\delta(L)} g_{\delta(L)} \phi_0 \mid L \in \mathcal{T}\}$, and the usual measure on Φ_0, considered as a cartesian product of coset spaces $(G/H)_L$, $L \in \mathcal{T}$, is obviously invariant under this mapping. This demonstrates the assertion concerning the local product measure in the proposition.

Appendix 2.

Let G be a compact Lie group, let $f \in L^2(G)$, and let ${}^f c^{ij}_u$ denote the FPW coefficient of f corresponding to the normalized (i,j)-th matrix element $\phi^u_{ij}(g)$ of the irreducible representation $D^{(u)}$ of G. If $f, g \in L^2(G)$, then the FPW coefficients of the convolution $f * g$ can be calculated as follows.

$$\begin{aligned}
{}^{f*g}c^{ij}_u &= \int dh \phi^{u\,*}_{ij}(h) \int dg f(g) g(hg^{-1}) \\
&= \int dg f(g) \int dh \phi^{u\,*}_{ij}(h) \sum_{v,k,l} {}^g c^{kl}_v \phi^v_{kl}(hg^{-1}) \\
&= \int dg f(g) \int dh \phi^{u\,*}_{ij}(h) \sum_{v,k,l,m} {}^g c^{kl}_v \phi^v_{km}(h) \phi^v_{ml}(g^{-1}) \\
&= \sum_{v,k,l,m} {}^g c^{kl}_v \int dg f(g) \phi^{v\,*}_{lm}(g) \int dh \phi^{u\,*}_{ij}(h) \phi^u_{km}(h) \\
&= \sum_{v,l,m} {}^g c^{kl}_v \, {}^f c^{lm}_v \delta_{uv} \delta_{ij,km} = \sum_l {}^g c^{il}_u \, {}^f c^{lj}_u
\end{aligned}$$

Now, if $G = SO(3)$, parametrized by the Euler angles (ϕ, θ, ψ) (frequently referred to as (α, β, γ)), then, in the special case where $f(\theta)$ is a function of θ alone, the coefficients ${}^f c^{ij}_u$ are zero unless $i = j = 0$, and the expansion of $f(\theta)$ becomes [4]

$$f(\theta) = \sum_{l=0}^{\infty} c_l P_l(\cos\theta) ,$$

where

$$c_l \equiv c^{00}_l = \frac{1}{2}(2l+1) \int_0^{\pi} f(\theta) P_l(\cos\theta) \sin\theta d\theta$$

Applying the above result to the convolution on the RHS of (53), we observe that the indicated recursion is equivalent to successive multiplication by the FPW coefficient c_l of the function $f(\theta) = e^{\nu\cos\theta}$. The Haar measure dg_i of equation (53), expressed in terms of the Euler angles is

$$\int dg_i = \int_0^{2\pi} d\psi_i \int_0^{2\pi} d\varphi_i \int_0^{\pi} d\theta_i \sin\theta_i .$$

Substituting this and $f(\theta) = e^{\nu\cos\theta}$ into the above equation, one obtains

$$c_l = \frac{1}{2}(2l+1)^{1/2} \int_0^{\pi} e^{\nu\cos\theta} P_l(\cos\theta) \sin\theta d\theta = \frac{1}{2}(2l+1)^{1/2} \tilde{P}_l(\nu),$$

where $\tilde{P}_l(\nu)$ is given by equation (55). Thus, one obtains equation (54).

Substituting $\tilde{Z}_N$ of equation (53) into equation (57), and using the G-invariance of the Haar measure and the inner product, one obtains

$$
\begin{aligned}
Z_N(h) &= \int_{k\in G} dk \int_{[\prod_{i=N}^{1} \dot{g}_i = k^{-1}hk]} \prod_{i=N}^{1} d\dot{g}_i e^{\nu \sum_{i=1}^{N} (\dot{g}_i u_0, u_0)} \\
&= \int_{k\in G} dk \int_{[\prod_{i=N}^{1} g'_i = h]} \prod_{i=N}^{1} dg'_i e^{\nu \sum_{i=1}^{N} (g'_i k u_0, k u_0)} \\
&= \int_{k\in G} dk \int_{[\prod_{i=N}^{1} g'_i = h]} \prod_{i=N}^{1} dg'_i e^{\nu \sum_{i=1}^{N} (g'_i k \hat{z}, k \hat{z})} \\
&= \int_{k\in G} dk \int_{[\prod_{i=N}^{1} g'_i = k^{-1}hk]} \prod_{i=N}^{1} d\dot{g}_i e^{\nu \sum_{i=1}^{N} (\dot{g}_i \hat{z}, \hat{z})} \\
&= \int_{k\in G} dk \int_{[\prod_{i=N}^{1} g'_i = k^{-1}hk]} \prod_{i=N}^{1} d\dot{g}_i e^{\nu \sum_{i=1}^{N} \cos\theta_i}
\end{aligned}
$$

where, since the k integration ranges over all G, the initial vector u_0 has for convenience been set equal to $\hat{z}$, and again, θ_i denotes the second Euler angle in the customary parametrization of SO(3). Having evaluated $\tilde{Z}_N$, we must next evaluate the integral in equation (57). Expressing the Haar measure dk in terms of the angle τ and the axis Ω of the rotation,[13] the partition function can be expressed as follows.

$$\int_k \tilde{Z}(khk^{-1}) = 2\int_k \tilde{Z}(\cos\theta_{khk^{-1}})(1-\cos\tau_k)d\tau_k d\Omega_k$$

where $d\Omega_k = \sin\alpha d\alpha d\beta$, and α, β are the polar coordinate angles. Expressing $\cos\theta$ in terms of τ and Ω, (i. e., in terms of α, β, and τ), one obtains equation (59). Substituting equations (54) and (59) into the above equation, we finally obtain equation (58) for the partition function.

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