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DISCRETE TRANSFORMATION AND EXPLICIT FORM OF EQUATIONS OF TWO-DIMENSIONAL HEISENBERG HIERARCHY

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Abstract

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The equations of $(1+2)$ integrable systems belonging to the Heisenberg hierarchy which are invariant with respect to discrete transformations of the corresponding integrable mapping are represented in the explicit form.

Аннотация

Лезнов А.Н., Юзбашьян Э.А. Дискретное преобразование и явная форма уравнений двумерной Гайзенберговской иерархии: Препринт ИФВЭ 95-29. – Протвино, 1995. – 6 с., библиогр.: 6.

В статье представлены в явной форме уравнения (2+1)-мерных интегрируемых систем, принадлежащих к Гайзенберговской иерархии и являющиеся инвариантными по отношению к дискретному преобразованию соответствующей интегрируемой подстановки.

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1. Introduction

In this paper we will continue investigating two-dimensional integrable systems [1,2] in terms of properties of the group of discrete integrable mappings [3,4,5].

This program was formulated in [3] and consists in the following construction. With each local invertible substitution of the form

$$\overleftarrow{u}^{\equiv}\phi(u) = \phi(u, u', u'', ...) \tag{1.1}$$

(where u is s-dimensional vector function , $u''^{\cdots}$ are its derivatives of arbitrary order with respect to independent arguments) it is connected with the Frechet derivative operator

$$\phi'(u) = \frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial u'}D + \frac{\partial\phi}{\partial u''}D^2 + ... \quad . \tag{1.2}$$

As it follows from definition (1.2) $\phi'(u)$ is $s \times s$ matrix operator. Then it is necessary to consider the following equation:

$$\overleftarrow{F} \equiv F(\phi(u)) = \phi'(u)F(u), \tag{1.3}$$

where F is s–dimensional vector, components of which depend on vector function u and its derivatives up to some limited order.

Equation (1.3) always possesses one (trivial) solution $F(u) = u'$ in which it is possible to get convinced by differentiation (1.1) with respect to some of independent arguments of the problem. If (1.3) possesses some other solution different from the trivial one such substitution was called in [3] the integrable substitution or integrable mapping. With each of solutions of (1.3) it is possible to connect the evolution type equation

$$u_t = F(u), \tag{1.4}$$

which is obviously invariant relative to transformation (1.1). In [4,5] hope was expressed that the background of future theory of integrable systems is the theory of representations of groups.

The goal of the present paper is to investigate two–dimensional Heisenberg integrable mapping and construct on this base the explicit form of integrable systems belonging to this hierarchy.

2. Two-dimensional Heisenberg substitution

Under this term we will understand the direct ($\overleftarrow{f}$) and inverse ($\overrightarrow{f}$) transformations of two functions (u, v) of the form

$$\begin{aligned} \overleftarrow{u} &= v^{-1}, \qquad \frac{1}{1+\overleftarrow{u}v} = \frac{1}{1+uv} + \frac{\phi_{xy}}{\phi_x\phi_y}, \quad \phi = \ln v, \\ \overrightarrow{v} &= u^{-1}, \qquad \frac{1}{1+\overrightarrow{u}v} = \frac{1}{1+uv} + \frac{\psi_{xy}}{\psi_x\psi_y}, \quad \psi = \ln u. \end{aligned} \tag{2.1}$$

Denoting $t_1 = \frac{u_y v_x}{(1+uv)^2}$ and $t_2 = \frac{u_x v_y}{(1+uv)^2}$ one can get convinced that

$$(\ln t_1)_{xy} = \overleftarrow{t_1} + \overrightarrow{t_1} - 2t_1 = \Delta_1, \quad (\ln t_2)_{xy} = \overleftarrow{t_2} + \overrightarrow{t_2} - 2t_1 = \Delta_2. \tag{2.2}$$

The explicit form of the Frechet derivative operator in this case is the following:

$$\phi'(u) = \begin{pmatrix} 0 & -v^{-2} \\ (\frac{v\overleftarrow{R}}{R})^2 & -(1+(\frac{uv\overleftarrow{R}}{R})^2 + (\overleftarrow{R})^2\delta(\phi_x^{-1}D_x + \phi_y^{-1}D_y - \frac{v}{v_{xy}}D_{xy}) \end{pmatrix}. \tag{2.3}$$

In (2.3) D_x, D_y are operators of differentiation with respect to two independent space coordinates of the problem,

$$\delta = \frac{vv_{xy}}{v_x v_y}, \quad R = 1 + uv, \quad \overleftarrow{R} = 1 + \overleftarrow{u}v.$$

By a short calculation it is possible to show that equation (1.3) possesses nontrivial solution $F_1 = u, F_2 = -v$ and so substitution (2.2) by definition is integrable.

Now we can rewrite equation (1.3) in more observable form. Let us denote $F_1 = uB, F_2 = vA$. From the first equation (1.3) we obtain immediately $B = -\overrightarrow{A}$. The second equation after some transformations may be rewritten in the form of single equation for function A

$$(\frac{\overleftarrow{u}v}{(1+uv)^2})(\overleftarrow{A} - A) - \frac{uv}{(1+uv)^2}(A - \overrightarrow{A}) = (\phi_x\phi_y)^{-1}(\frac{\phi_{xy}}{\phi_x}A_x + \frac{\phi_{xy}}{\phi_y}A_y - A_{xy}). \tag{2.4}$$

As we know from introduction main equation (1.3) always possesses the trivial solution $F_1 = u_x, (u_y); F_2 = v_x, (v_y)$ or $A = \phi_x, (\phi_y)$. Let us look for the solution of (2.4) in the form $A = \phi_x\alpha$. Instead of (2.4) we obtain equation for α

$$(\frac{u_x\overleftarrow{v}_x}{(1+uv)^2})(\overleftarrow{\alpha} - \alpha) - \frac{u_x v_x}{(1+uv)^2}(\alpha - \overrightarrow{\alpha}) = (\frac{\alpha_y}{\theta})_x, \quad \theta = \frac{\phi_y}{\phi_x}. \tag{2.5}$$

Resolving (2.5) by the substitution

$$\frac{\alpha_y}{\theta} = \overleftarrow{s} - s$$

we obtain the equation for determining s

$$s_x = s_0 \int dy[\theta(\overleftarrow{s} - s) - \overrightarrow{\theta}^{1}(s - \overrightarrow{s})], \tag{2.6}$$

where

$$s_0 = \frac{u_x v_x}{(1+uv)^2}.$$

3. Solution of the main equation

First of all let us notice that one of solutions of (2.6) is exactly the function s_0 by itself that can be checked up by direct substitutions of all values in (2.6) using the explicit formulas of Heisenberg substitution (2.1). We have

$$\overleftarrow{s_0} - s_0 = 2\phi_x(\frac{1}{1+uv})_x + 2\phi_{xy}\frac{\phi_x}{\phi_y}\frac{1}{1+uv} + \phi_x(\frac{\phi_{xy}}{\phi_x\phi_y})_x - \phi_{xy}\frac{\phi_x}{\phi_y} + \frac{\phi_{xy}^2}{\phi_y^2}. \tag{3.1}$$

With the help of (3.1) it is not difficult to see that s_0 is indeed the solution of (2.6). Let us seek for the solution of (2.6) in the form $s = s_0 \int dy \alpha_0$. After substituting this into (2.6) and some not complicated computations we find the following equation for α_0 (this equation of course is only sufficient but not necessary)

$$(\alpha_0)_x + \alpha_0 \int dy[\theta\overleftarrow{s_0} - \theta s_0 - \overrightarrow{\theta} s_0 + \overrightarrow{\theta s_0}] = \theta\overleftarrow{s_0} \int dy(\overleftarrow{\alpha_0} - \alpha_0) + \overrightarrow{\theta s_0} \int dy(\overrightarrow{\alpha_0} - \alpha_0). \tag{3.2}$$

Denoting $\Delta = -(s_0 - \overrightarrow{s_0})\overrightarrow{\theta} + (\overleftarrow{s_0} - s_0)\theta$ it is easy to see that $\Delta = \overrightarrow{t_2} - t_2 - t_1 + \overleftarrow{t_1}$ and we can rewrite (3.2) in the form

$$(\alpha_0)_x + \alpha_0 \int dy\Delta = \overleftarrow{t_1} \int dy(\overleftarrow{\alpha_0} - \alpha_0) + \overrightarrow{t_2} \int dy(\overrightarrow{\alpha_0} - \alpha_0). \tag{3.3}$$

Now let us substitute in (3.3) α_0 in the form $\alpha_0 = \overleftarrow{t_1}\alpha_1 + \overrightarrow{t_2}\beta_1$. Grouping terms with m and terms with n and equating expression for m and for n to zero we find for α_1 and β_1

$$\begin{aligned} (\alpha_1)_x + \alpha_1 \int dy(\Delta + \overleftarrow{\Delta_1}) &= \int dy(\overleftarrow{\alpha_0} - \alpha_0), \\ (\beta_1)_x + \beta_1 \int dy(\Delta + \overrightarrow{\Delta_2}) &= \int dy(\overrightarrow{\alpha_0} - \alpha_0), \end{aligned} \tag{3.4}$$

where $\alpha_0 = \overleftarrow{t_1}\alpha_1 + \overrightarrow{t_2}\beta_1$. Using $\Delta + r\Delta_1 = (\Delta + \overrightarrow{\Delta_2})$ and summarizing the first equation with the right transformed equation for β_1 we obtain

$$(\alpha_1 + \overleftarrow{\beta_1})_x + (\alpha_1 + \overleftarrow{\beta_1}) \int dy(\Delta + \overleftarrow{\Delta_1}) = 0.$$

This equation may be integrated up to the end, but for us it will be sufficient to use the obvious solution $\alpha_1 + \overleftarrow{\beta_1} = 0$. In that case for α_1 we have

$$(\alpha_1)_x + \alpha_1 \int dy(\Delta + \overleftarrow{\Delta_1}) = \int dy[(\overleftarrow{t_1}^2 \alpha_1 - t_2\overleftarrow{\alpha_1}) - (\overleftarrow{t_1}\alpha_1 - \overrightarrow{t_2}\overrightarrow{\alpha_1})]. \tag{3.5}$$

Equation (3.5) possesses the following trivial (with respect to this equation but not to our problem as a whole) solution $\alpha_1 = 1$. From this with the help of the previous formulas we find the new solution of (2.6)

$$s = s_1 = s_0 \int dy(\overleftarrow{t_1} - \overrightarrow{t_2}),$$

which gives us the next system of equations in the Heisenberg hierarchy. Substituting in (3.5) instead α_1 $\int dy\alpha_1$, after simple calculations we have

$$(\alpha_1)_x + \alpha_1 \int dy(\Delta + \overleftarrow{\Delta_1}) = \overleftarrow{t_1}^2 \int dy(\overleftarrow{\alpha_1} - \alpha_1) + \overrightarrow{t_2} \int dy(\overrightarrow{\alpha_1} - \alpha_1). \tag{3.6}$$

This equation is analogous to (3.5) and we can use the same way representing α_1 as $\alpha_1 = \overleftarrow{t_1}^2 \alpha_2 + \overrightarrow{t_2}\beta_2$.

In the general case by inducting the following formulae one may prove

$$(\alpha_n)_x + \alpha_n \int dy D_n = \int dy[(\overleftarrow{t_1}^{n+1}\overleftarrow{\alpha_n}^1 - t_2\alpha_n) - (\overleftarrow{t_1}^n \alpha_n - \overrightarrow{t_2}^1\overrightarrow{\alpha_n}^1)], \tag{3.7}$$

or if we take instead α_n $\int dy\alpha_n$

$$(\alpha_n)_x + \alpha_n \int dy D_n = \overleftarrow{t_1}^{n+1} \int dy(\overleftarrow{\alpha_n}^1 - \alpha_n) + \overrightarrow{t_2}^1 \int dy(\overrightarrow{\alpha_n}\alpha_n), \tag{3.8}$$

$$D_n = \Delta + \overleftarrow{\Delta_n}^1 + \ldots + \overrightarrow{\Delta_m}^n = \overrightarrow{t_2}^1 - t_2 + \overleftarrow{t_1}^{n+1} - \overleftarrow{t_1}^n,$$

$$\alpha_k = \overleftarrow{t_1}^{k+1} \int dy\alpha_{k+1} - \overrightarrow{t_2}^1 \int dy\overrightarrow{\alpha_{k+1}}1$$

Each equation (3.7) has the trivial solution $\alpha_n = 1$. This solution gives us the sequence of solutions of (2.6). For example if we want to find s_2 we should take $\alpha_2 = \int dy\alpha_2 = 1$, from which it follows that

$$\alpha_1 = \overleftarrow{t_1}^2 - \overrightarrow{t_2}^1,$$

$$s_2 = s_0 \int dy\alpha_0 = s_0 \int dy[\overleftarrow{t_1}^1 \int dy(\overleftarrow{t_1}^2 - \overrightarrow{t_2}^1) - \overrightarrow{t_2}^1 \int dy(\overleftarrow{t_1}^1 - \overrightarrow{t_2}^2)].$$

One can also prove by induction the expression for s_n in the following symbolical form

$$s_n = s_0 \prod_{i=1}^{n} (1 - L_i \exp[-(i+1)d_i - \sum_{k=i+1}^{n} d_k]) \int dy \overleftarrow{t_1}^{1} \int dy \overleftarrow{t_1}^{2} ..... \int dy \overleftarrow{t_1}^{n}, \tag{3.9}$$

where symbol $\exp d_s$ means the shift by unity of the argument of s-repeated integral $(.... \int dy \overleftarrow{t_1}^{h} ... \rightarrow ... \int dy \overleftarrow{t_1}^{h+1} ...)$ in (3.7) and symbol L_m means the exchange of $\overleftarrow{t_1}^{m}$ for $\overleftarrow{t_2}^{m}$ in the p-repeated integral $... \int dy \overleftarrow{t_1}^{m} ... \rightarrow ... \int dy t_2 m...$

4. Integrals of motion

We can suppose the following general formulae for integrals of motion:

$$I_n = \theta s_n, \quad I_n = \overleftarrow{\theta} s_n. \tag{4.1}$$

Up to now we have only checked (4.1) for sufficiently low values of n=0,1,2,3 and so this proposition may be considered only as a hypothesis.

5. Examples

In this section we represent the simplest integrable systems for unknown functions u,v corresponding to functions s_n with n=0,1.

5.1. n=0

$$v_t = -v_{xx} + 2v_x \int dy (\frac{uv_y}{1+uv})_x, \quad -u_t = -u_{xx} + 2u_x \int dy (\frac{vu_y}{1+uv})_x.$$

5.2. n=1

$$v_t + v_{xxx} - 3v_{xx} \int dy (\frac{uv_y}{1+uv})_x + 3v_x [\int dy (\frac{uv_y}{1+uv})_x]^2 +$$
$$+3v_x \int dy (\frac{u_x v_y}{(1+uv)^2})_x - -3v_x \int dy (\frac{uv_y}{1+uv})_{xx} = 0,$$
$$u_t + u_{xxx} - 3u_{xx} \int dy (\frac{vu_y}{1+uv})_x + 3u_x [\int dy (\frac{vu_y}{1+uv})_x]^2 +$$
$$+3u_x \int dy (\frac{v_x u_y}{(1+uv)^2})_x - 3u_x \int dy (\frac{vu_y}{1+uv})_{xx} = 0.$$

6. Conclusion

The main result of the present paper is contained in formulae (3.9) which gives the explicit form of two-dimensional integrable equations of Heisenberg hierarchy. We emphasize that in this case by contrast to the case of Darboux-Toda hierarchy the solutions s_n are not the conserved quantities : they are not shifted on divergence with respect to the transformation of discrete symmetry. This may be the reflection of the fact that the Heisenberg substitution is not a canonicial one. The invariant Hamiltonian structure is absent in this two-dimensional case.

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