STATE RESEARCH CENTER OF RUSSIA

INSTITUTE FOR HIGH ENERGY PHYSICS

IHEP 95-28

A.N.Leznov and E.A.Yuzbashyan

TWO-DIMENSIONAL INTEGRABLE DARBOUX–TODA SUBSTITUTION AND DAVEY–STEWARTSON HIERARCHY OF INTEGRABLE SYSTEMS

Submitted to *Lett. Math. Phys.*

Protvino 1995

UDK 539.1.01 M–24

Abstract

Leznov A.N. and Yuzbashyan E.A. Two-Dimensional integrable Darboux–Toda substitution and Davey–Stewartson hierarchy of integrable systems: IHEP Preprint 95-28. – Protvino, 1995. – p. 7, refs.: 7.

The equations of $(1+2)$ integrable systems belonging to Davey-Stewartson hierarchy are represented in an explicit form.

Аннотация

Лезнов А.Н. и Юзбашьян Э.А. Двумерная интегрируемая подстановка Дарбу-Тоды и иерархия интегрируемых систем Дэви-Стюартсона: Препринт ИФВЭ 95-28. – Протвино, 1995. – 7 с., библиогр.: 7.

В статье представлены в явной форме уравнения (2+1)-мерных интегрируемых систем, принадлежащих к иерархии Дэви-Стюартсона и являющиеся инвариантными по отношению к дискретному преобразованию двумерной интегрируемой подстановки Дарбу-Тоды.

1. Introduction

In the case of $(1+1)$ integrable systems the Hamiltonian formalism has played the fundamental (background role for many years). However the situation in $(1+2)$ case changes essentially. In this case the Hamiltonian formalism (in the usual sense) does not work at all and many alternative methods to construct integrable systems and their solutions [1], [2] have been proposed.

The goal of the present paper is to demonstrate that the condition of equations invariance with respect to two-dimensional integrable mapping (discrete substitution, transformation) [3,4,5] allows one to construct equations of the corresponding hierarchy in an explicit form. Moreother it is possible to obtain a wide class of exact solutions, including soliton ones [3].

The fundamental role in our construction will play the functions which, with respect to discrete transformation, are shifted on two-dimensional divergence with respect to space coordinates of the problem. We conserve for these functions (for a while) the term Hamiltonian keeping in mind that in one-dimensional limit they indeed go over to the corresponding Hamiltonians (up to nonessential divergence) of the $(1+1)$ integrable system.

We restrict ourselves in this paper by the case of the Darboux–Toda two–dimesional integrable mapping which is directly connected with $(1+2)$ Davey–Stewartson hierarchy [7].

2. Discrete Darboux–Toda (D–T) transformation

The explicit form of the direct and inverse D–T integrable substitution is the following:

$$\overleftarrow{u} = \frac{1}{v}, \quad \overleftarrow{v} = v(uv - (\ln v)_{xy}), \tag{2.1}$$

$$\overrightarrow{v} = \frac{1}{u}, \quad \overrightarrow{u} = u(vu - (\ln u)_{xy}).$$

The functions on variables $f(u,v)$ after j-times direct transformation will be denoted $\overset{\leftarrow j}{f}$, after j-times inverse transformation as $\overset{j\rightarrow}{f}$ ($\overset{\leftarrow(-j)}{f} \equiv \overset{j\rightarrow}{f}$).

The condition of invariance with respect to transformation (2.1) of evolution type equation

$$u_t = F(u) \tag{2.2}$$

(u is vector function (u,v), $F(u)$ is also two–dimensional vector function (F_1, F_2) each component of which depends on (u,v) and their space derivatives up to some definite order) may be written as [3]

$$\overleftarrow{F} \equiv F(\phi(u)) = \phi'(u)F(u), \tag{2.3}$$

where $\phi'(u)$ is corresponding to (2.1) Frechet derivative [6]

$$\phi'(u) = \begin{pmatrix} 0 & -\frac{1}{v^2} \\ v^2 & 2uv - \frac{v_x v_y}{v^2} + \frac{v_x}{v}D_y + \frac{v_y}{v}D_x - D_{xy} \end{pmatrix}, \tag{2.4}$$

where $D_y \equiv \frac{\partial}{\partial y}, D_x \equiv \frac{\partial}{\partial x}$.

System (2.3) in the concrete case of D–T substitution may be rewritten as

$$\overleftarrow{F_1} = -\frac{1}{v^2}F_2 \quad \overleftarrow{F_2} = v^2 F_1 + (2uv - \frac{v_x v_y}{v^2} + \frac{v_x}{v}D_y + \frac{v_y}{v}D_x - D_{xy})F_2. \tag{2.5}$$

It is not difficult to check by the direct computations that $F_0 = (u, -v)$ is the solution of the last equation and so substitution (2.1) is integrable in the sense of [3].

After introducing the new functions $F_1 = uf_1, F_2 = vf_2$ system (2.5) takes the form of a single equation for only one unknown function f_2

$$(\overleftarrow{uv})(\overleftarrow{f_2} - f_2) - (uv)(f_2 - \overrightarrow{f_2}) = -D_{xy}f_2, \quad f_1 = -\overrightarrow{f_2}. \tag{2.6}$$

The meaning of notations in the last equation is explained after formula (2.1).

In further transformations of (2.6) we will use the fact that condition of invariance of some function with respect to the discrete transformation $\overleftarrow{F} = F$ is equivalent to the $F \equiv const.$ This is in some sense the analogue of Liouville theorem in the theory of analicity functions. Using this fact for function $f_2 = \int dy(\overleftarrow{T} - T)$ we obtain the Toda-like chain equation

$$-T_{xy} = T_0 \int dy(\overleftarrow{T} - 2T + \overrightarrow{T}), \quad T_0 = uv. \tag{2.7}$$

In terms of solution of (2.7) evolution type equation (2.2) (invariant with respect to D–T substitution (2.1)) takes the form

$$v_t = v\int dy(\overleftarrow{T} - T), \qquad u_t = u\int dy(\overrightarrow{T} - T).) \tag{2.8}$$

3. Solution of the main equation

Using (2.1) one can get convinced that $T = T_0$ is the solution of (3.1). Let us seek solution of (2.8) in the form

$$T = T_0 \int dy \alpha_0. \tag{3.1}$$

After substitution of this expression into (3.1) and some trivial computations we find the equation for α_0

$$-\alpha_0' = \overleftarrow{T_0} \int dy(\overleftarrow{\alpha_0} - \alpha_0) + \overrightarrow{T_0} \int dy(\overrightarrow{\alpha_0} - \alpha_0) - \alpha_0 \int dy(\overleftarrow{T_0} - 2T_0 + \overrightarrow{T_0}). \tag{3.2}$$

Let us try to solve the last equation with the help of the following substitution (with notation $\Delta \equiv (\overleftarrow{T_0} - 2T_0 + \overrightarrow{T_0}))$

$$\alpha_0 = \overleftarrow{T_0}\alpha_1 + \overrightarrow{T_0}\beta_1.$$

At first we will assume that α, β are some numerical constants. From (3.2) we immediately obtain $\alpha_1 = -\beta_1 = 1$ and as corollary we find a partial solution of our main equation (3.1) in the form

$$T_1 = T_0 \int dy(\overleftarrow{T_0} - \overrightarrow{T_0}). \tag{3.3}$$

Now let us consider α_1, β_1 in the previous expression for α_0 as unknown functions representing them in the integral form: (we keep the same letters for convenience)

$$\alpha_0 = \overleftarrow{T_0} \int dy \alpha_1 + \overrightarrow{T_0} \int dy \beta_1. \tag{3.4}$$

Keeping in mind that $-(\overleftarrow{T_0})_x = \overleftarrow{T_0} \int dy \overleftarrow{\Delta}$ we can rewrite equation (3.2) (equating to zero coefficients before $\overleftarrow{T_0}$ and $\overrightarrow{T_0}$ terms) as

$$-\alpha_1' = \overleftarrow{T_0}^2 \int dy(\overleftarrow{\alpha_1} - \alpha_1) + T_0 \int dy(\overleftarrow{\beta_1} + \alpha_1) - \overrightarrow{T_0} \int dy(\beta_1 + \alpha_1) - \alpha_1 \int dy(\overleftarrow{\Delta} + \Delta),$$

$$-\beta_1' = \overrightarrow{T_0}^2 \int dy(\overrightarrow{\beta_1} - \beta_1) + T_0 \int dy(\overrightarrow{\alpha_1} + \beta_1) - \overleftarrow{T_0} \int dy(\beta_1 + \alpha_1) - \beta_1 \int dy(\overrightarrow{\Delta} + \Delta).$$

Summarizing equation for β_1 shifted by direct transformation with equation for α_1, after simple calculations we find

$$\int dy(\beta_1 + \overrightarrow{\alpha_1}) = \frac{c}{T_0\overrightarrow{T_0}}, \qquad \int dy(\overleftarrow{\beta_1} + \alpha_1) = \frac{c}{(\overleftarrow{T_0}T_0)},$$

where c is an arbitrary constant. Substituting these expressions into the last equation for α_1 we obtain (below we put $c = 0$)

$$-\alpha_1' = -\alpha_1 \int dy(\Delta + \overleftarrow{\Delta}) + \overleftarrow{T_0}^2 \int dy(\overleftarrow{\alpha_1} - \alpha_1) + \overrightarrow{T_0} \int dy(\overrightarrow{\alpha_1} - \alpha_1). \tag{3.5}$$

This equation is absolutely the same as equation (3.2) with obviously changing $\overleftarrow{T_0}$ on $\overset{\leftarrow 2}{T_0}$. Let us take α_1 in the form

$$\alpha_1 = \overset{\leftarrow 2}{T_0}\alpha_2 + \overrightarrow{T_0}\beta_2.$$

Assuming that α_2, β_2 are the numerical parameters, from (3.5) we obtain immediately $\alpha_2 = -\beta_2 = 1$ and finally the next partial solution of our main equation is

$$T_2 = T_0 \int dy[\overleftarrow{T_0} \int dy \overset{\leftarrow 2}{T_0} - \overleftarrow{T_0} \int dy \overrightarrow{T_0} - \overrightarrow{T_0} \int dy \overleftarrow{T_0} + \overrightarrow{T_0} \int dy \overset{2\rightarrow}{T_0}]. \tag{3.6}$$

In the case when α_2, β_2 are unknown functions, using the same trick (as in the case with α_1, β_1) we will (after substituting $\int dy\alpha_2$ instead α_2 and the corresponding expression for β_2) get

$$-\alpha_2' = -\alpha_2 \int dy(\overset{\leftarrow 2}{\Delta} + \overleftarrow{\Delta} + \Delta) - \overset{\leftarrow 3}{T_0} \int dy(\overleftarrow{\alpha_2} - \alpha_2) + \overrightarrow{T_0} \int dy(\overrightarrow{\alpha_2} - \alpha_2)$$

and the corresponding equation for β_2.

In the general case it can be proved by induction using every time substitution:

$$\alpha_k = \overset{\leftarrow k+1}{T_0} \int dy\alpha_{k+1} + \overrightarrow{T_0} \int dy\beta_{k+1}$$

on the n-s step we will have the following equation for α_n

$$-\alpha_n' = -\alpha_n \int dy(\overrightarrow{T_0} - T_0 - \overset{\leftarrow n}{T_0} + \overset{\leftarrow n+1}{T_0}) + \overset{\leftarrow n}{T_0} \int dy(\overleftarrow{\alpha_n} - \alpha) + \overrightarrow{T_0} \int dy(\overrightarrow{\alpha_n} - \alpha_n). \tag{3.7}$$

This equation possesses the obvious partial solution of the form (compare with the corresponding solutions in the cases $\alpha_{1,2}$)

$$\alpha_n = \overset{\leftarrow n+1}{T_0} - \overset{1\rightarrow}{T_0}. \tag{3.8}$$

After this explicit expression for T_n may be reconstructed in the form of the sum of 2^n terms, which can be written in following symbolical form:

$$T_n = T_0 \prod_{i=1}^{n}(1 - \exp[-(i+1)d_i - \sum_{k=i+1}^{n} d_k]) \int dy \overset{\leftarrow 1}{T_0} \int dy \overset{\leftarrow 2}{T_0} \ldots\ldots \int dy \overset{\leftarrow n}{T_0}, \tag{3.9}$$

where symbol $\exp d_p$ means the shift by unity of the argument of p-repeated integral $(\ldots. \int dy \overset{\leftarrow p}{T_0} \ldots \rightarrow \ldots \int dy \overset{\leftarrow p+1}{T_0} \ldots)$ in (3.9).

Substituting this expression for T_n in (2.8) we'll find the equation of order $(n+1)$ of the Davey-Stewartson hierarchy.

4. Recurrent formulas for T_n-functions

We may assume that the following recurrent pure algebraic relation connecting Hamiltonian functions T_n takes place

$$T_{n+1} = \frac{1}{2}\sum_{k=0}^{n+1}[F_1^k F_2^{n-k+1} + F_1^{n-k+1} F_2^k] + \sum_{k=0}^{n-1} T_k \int dy T_{n-k-1}. \tag{4.1}$$

We have checked this relation by direct computations up to $n = 4$. But we don't know now if this relation is a direct corollary of our general formulae (4.1) for T_n or not.

5. Conserved values

We have checked also that all the constructed Hamiltonian functions T_n (3.9) up to $n = 3$ are the conserved quantities which in one dimensional limit ($\partial_x = \partial_y$) up to all unessential derivatives go to H_{n+1} –hamiltonian functions of equations of the one-dimensional nonlinear Schrödinger hierarchy. All functions T_n are shifted with respect to D–T discrete transformations (2.1) on divergence in two-dimensional space.

6. Examples

In this section we represent the simplest integrable systems for unknown functions u, v corresponding to Hamiltonian functions T_n with $n = 0, 1, 2, 3, 4$

6.1. n=0

$$T_0 = uv, \quad u_t = au_x + bu_y, \quad v_t = av_x + bv_y.$$

In examples below we shall choose $a = 1, b = 0$ keeping in mind that it is always possible to add the term (with arbitrary numerical coefficient) in which x is changed by y and vise versa.

6.2. n=1

$$T_1 = vu_x - v_x u,$$

$$u_t = u_{xx} - u\int dy(uv)_x, \quad -v_t = v_{xx} - v\int dy(uv)_x.$$

This is the Davey-Stewartson equation in its original form [7].

6.3. n=2

$$T_2 = (uv)_{xx} - 3u_x v_x - 3uv \int dy(uv)_x,$$
$$u_t = u_{xxx} - 3u_x \int dy(uv)_x - 3u \int dy(u_x v)_x,$$
$$v_t = v_{xxx} - 3v_x \int dy(uv)_x - 3v \int dy(v_x u)_x.$$

This is the equation of Veselov-Novikov.

6.4. n=3

$$T_3 = -(T_1)_{xx} - 2(u_x v_{xx} - v_x u_{xx}) + 2uv \int dy(T_1)_x + 4T_1 \int dy(uv)_x,$$
$$v_t = -v_{xxxx} + 4v_{xx} \int dy(uv)_x - 2v_x(\int dy(T_1)_x - 2\int dy(uv)_{xx}) + 2v(\int dy(uv)_{xxx} -$$
$$\int dy(u_x v_x)_x + \int (uv_{xx})_x - ([\int dy(uv)]^2)_{xx} - [\int dy(uv)_x]^2).$$

The equation for u may be obtained from the equation for v under changing $u \to v, v \to u, t \to -t$.

7. Conclusion

The main concrete result of the present paper consists in the explicit form of equations of the $(1+2)$ dimensional Davey–Stewarson hierarchy which are invariant with respect to transformation of corresponding integrable mapping (2.1). We want to emphasize that finally result uses knowledge only of T_0 Hamiltonian function and some general properties of discrete transformation (2.1). Our calculations were purely technical and we are very far from the understanding of their connection with the theory of representations of discrete mapping, which as it follows from the results of the present paper plays the fundamental role in the problem under consideration. We invite all mathematicians to help us in the solution of this interesting problem.

For one of the authors (A.N.L.) the research of this paper partially was possible by Grant N RMM000 of International Science Foundation.

References

[1] Boiti M., Konopelchenko B.G. and Pempinelli F.//*Inverse Probl.* 1 (1985) 33.

[2] Konopelchenko B.G.//*Phys. Lett. A* 156 (1991) 221.

[3] Fairlie D.B., Leznov A.N. – Preprint DTP/93/33, Durham (1993); (to be published in Phys.Lett.B).

[4] Leznov A.N. – Preprint IHEP 130–94, Protvino, 1994 (to be published in Physics D).

[5] Fairlie D.V., Leznov A.N. – Preprint IHEP-95-30, Protvino, 1995.

[6] Olver P.J. Application of Lie Groups to Differential Equations. Springer, Berlin, 1986.

[7] Davey A. and Stewartson K.//Proc. Roy. Soc. A 338, 1974, 101-110.

Received February 14, 1995

А.Н.Лезнов, Э.А.Юзбашьян
Двумерная интегрируемая подстановка Дарбу-Тоды и иерархия интегрируемых систем Дэви-Стюартсона .

Оригинал-макет подготовлен с помощью системы LaTeX.
Редактор Е.Н.Горина. Технический редактор Н.В.Орлова.

Подписано к печати 16.02.1995 г. Формат 60 × 84/8. Офсетная печать.
Печ.л. 0,87. Уч.-изд.л. 0,67. Тираж 250. Заказ 317. Индекс 3649.
ЛР №020498 06.04.1992 г.

ГНЦ РФ Институт физики высоких энергий
142284, Протвино Московской обл.

Индекс 3649

ПРЕПРИНТ 95-28, ИФВЭ, 1995