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TWO-DIMENSIONAL LOTKY-VOLTERRA INTEGRABLE MAPPING AND CORRESPONDING INTEGRABLE HIERARCHY IN (1+2) SPACE

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Abstract

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An explicit form of hierarchy of integrable systems in (1+2) dimensions is represented. These equations are invariant with respect to discrete transformation described by the Lotky-Volterra integrable substitution

Аннотация

Дерягин В.Б., Лезнов А.Н. Двумерная интегрируемая подстановка Лотки-Вольтерра и соответствующая иерархия в (2+1)-пространстве: Препринт ИФВЭ 95-27. – Протвино, 1995. – 5 с., библиогр.: 5.

В статье представлена явная форма иерархии интегрируемых систем в (2+1)-престранстве. Эти уравнения являются инвариантными по отношению к дискретному преобразованию, описываемому интегрируемой подстановкой Ланки-Вольтерра.

1. Introduction

In papers [1,2,3] a new approach to the theory of integrable systems when the main subject of investigation became the discrete integrable substitution with respect to which the concerned equations are invariant was proposed. This approach in principle is independent of the dimension of the space under consideration. The criteria of the choice of integrable mapping among the arbitrary possible ones is the condition of resolution of functional equation with shifted arguments [3]. This equation always possesses some trivial solution (and so is self-consistent). It allows one to answer two main questions of the theory: to choose the integrable substitution by itself and to construct the whole hierarchy of evolution type equations for a given integrable mapping. These equations are invariant with respect to this transformation.

The aim of the present paper is to solve the second part of the problem: to construct (1+2) equations which all are invariant with respect to the Lotky-Volterra two-dimentional integrable substitution.

2. Lotky-Volterra Integrable Mapping and the Main Equation

Under this term we will understand the direct and inverse mapping of two functions u,v of independent variables x,y

$$\overleftarrow{u} = u + (lnv)_x \quad \overleftarrow{v} = v + (ln\overleftarrow{u})_y,$$
$$\overrightarrow{u} = u - (ln\overrightarrow{v})_x \quad \overrightarrow{v} = v - (lnu)_y \tag{2.1}$$

s-times application of the direct (inverse) discrete transformation to some function $f(u,v)$ we will denote as $\overset{\leftarrow s}{f}$ and $\overset{\leftarrow -s}{f} \equiv \overset{s \rightarrow}{f}$, if $s=1$ we will omit index 1. As a direct corollary of (2.1) we have Toda-like recurrent relations for functions $t_1 = uv$, $t_2 = \overset{\leftarrow 1}{u} v$

$$(lnt_m)_{xy} = \overleftarrow{t_m} - 2t_m + \overrightarrow{t_m} \qquad (m = 1, 2). \tag{2.2}$$

Corresponding to (2.1) the Frechet derivative operator calculated by usual rules [4], takes the form

$$\phi' = \begin{pmatrix} 1 & D_x v^{-1} \\ D_y \overleftarrow{u}^{-1} & 1 + D_y \overleftarrow{u}^{-1} D_x v^{-1} \end{pmatrix}. \tag{2.3}$$

Each solution of the equation

$$\overleftarrow{F} \equiv F(\varphi(u)) = \varphi'(u) F(u) \tag{2.4}$$

is connected with the evolution type equation

$$u_t = F(u), \tag{2.5}$$

which is invariant with respect to transformation $\varphi(u)$, in our case (2.1).

Now let us rewrite equation (2.4) with $\varphi'(u)$ given by (2.3) in more observable form. We have successively

$$\overleftarrow{F_1} - F_1 = D_x(v^{-1}F_2) \quad \overleftarrow{F_2} - F_2 = D_y((\overleftarrow{u})^{-1}\overleftarrow{F_1}),$$

the second equality may be resolved by substitution

$$\overleftarrow{F_1} = \overleftarrow{u}(\overleftarrow{S} - S), \quad F_2 = D_y(S), \tag{2.6}$$

after that we have for unknown function S from the first one

$$D_y S = v \int dx[\overleftarrow{u}(\overleftarrow{S} - S) - u(S - \overrightarrow{S})]. \tag{2.7}$$

3. Solution of the main equation

First of all let us notice that equation (2.7) has solution $S_0 = v$. This became obvious after substituting this expression into (2.7) and using (2.1).

Now let us seek the solution of (2.7) in the form $S = v \int dx(\alpha_0)$, where α_0 is unknown function. After substituting this expression into (2.7) and some trivial transformations we come to the equation for α_0

$$D_y \alpha_0 + \alpha_0 \int dx[\overleftarrow{t_1} - t_1 + \overrightarrow{t_2} - t_2] = \overleftarrow{t_1} \int dx(\overleftarrow{\alpha_0} - \alpha_0) + \overrightarrow{t_2} \int dx(\overrightarrow{\alpha_0} - \alpha_0). \tag{3.1}$$

It is necessary to emphasize that this construction is correct in the direction: if α_0 is the solution of (3.1) then S is the solution of (2.7). All operations are well defined only in this direction. Let us try to seek the solutions of (3.1) by anzats

$$\alpha_0 = \overleftarrow{t_1}\alpha_1 + \overrightarrow{t_2}\beta_1, \tag{3.2}$$

where α_1 and β_1 are unknown functions. Having compared coefficients in front of $\overleftarrow{t_1}$,$\overrightarrow{t_2}$ to appear after substitution (3.2) into (3.1), we obtain system of equations for α_1, β_1

$$(\alpha_1)_y + \alpha^1 \int dx[\overleftarrow{t_1}^2 + \overrightarrow{t_2} - t_2 - \overleftarrow{t_1}] = \int dx[\overleftarrow{t_1}^2 \overleftarrow{\alpha_1} + t_2 \overleftarrow{\beta_1} - \overleftarrow{t_1}\alpha_1 - \overrightarrow{t_2}\beta_1],$$

$$(\beta_1)_y + \beta_1 \int dx(\overleftarrow{t}_1 + \overrightarrow{t}_2^{\,2} - \overrightarrow{t}_2 - t_1) = \int dx(t_1\overrightarrow{\alpha}_1 + \overrightarrow{t}_2^{\,2}\overrightarrow{\beta}_1 - \overleftarrow{t}_1\alpha_1 - \overrightarrow{t}_2\beta_1). \tag{3.3}$$

Having shifted the last equation by direct discrete transformation and summarizing with the first equation (3.3) we obtain

$$(\alpha_1 + \overleftarrow{\beta}_1)_y + (\alpha_1 + \overleftarrow{\beta}_1)\int dx[\overleftarrow{t}_1^{\,2} + \overrightarrow{t}_2 - t_2 - \overleftarrow{t}_1] = 0.$$

It is possible to find the general solution for this equation, however it will be sufficient for our purposes to have the partial one for which $\beta_1 = -\overrightarrow{\alpha}_1$. Under this condition the first equation (3.3) takes the form

$$(\alpha_1)y + \alpha_1 \int dx[\overleftarrow{t}_1^{\,2} + \overrightarrow{t}_2 - t_2 - \overleftarrow{t}_1] = \int dx[\overleftarrow{t}_1^{\,2}\overleftarrow{\alpha}_1 - t_2\alpha_1 - \overleftarrow{t}_1\alpha_1 + \overrightarrow{t}_2\overrightarrow{\alpha}_1]. \tag{3.4}$$

Equation (3.4) has the obvious partial solution $\alpha_1 = 1$ and as compared with (3.2)

$$S_1 = v\int dx(\overleftarrow{t}_1 - \overrightarrow{t}_2). \tag{3.5}$$

Let us change in (3.5) the known function α_1 for $\int dx(\alpha_1)$. After that equation (3.4) is changed on

$$(\alpha_1)_y + \alpha_1 \int dx[\overleftarrow{t}_1^{\,2} + \overrightarrow{t}_2 - t_2 - \overleftarrow{t}_1] = \overleftarrow{t}_1^{\,2}\int dx(\overleftarrow{\alpha}_1 - \alpha_1) + \overrightarrow{t}_2\int dx(\overrightarrow{\alpha}_1 - \alpha_1). \tag{3.6}$$

Compare (3.6) with (3.1) we see that they coincide up to the relation of equivalence. So we can represent α_1 in the form

$$\alpha_1 = \overleftarrow{t}_1^{\,2}\alpha_2 + \overrightarrow{t}_2\beta_2$$

The (3.3) has the partial solution

$$\alpha_1 = \int(\overleftarrow{t}_1^{\,2} - \overrightarrow{t}_2) \quad \beta_1 = \int(\overrightarrow{t}_2^{\,2} - \overleftarrow{t}_1).$$

Keeping in mind all the previous changings of variables we obtain for S_2

$$S_2 = v[\int\overleftarrow{t}_1\int\overleftarrow{t}_1^{\,2} - \int\overleftarrow{t}_1\int\overrightarrow{t}_2 - \int\overrightarrow{t}_2\int\overleftarrow{t}_1 + \int\overrightarrow{t}_2\int\overrightarrow{t}_2^{\,2}].$$

Let us assume that equations to determine α^k, β^k have the form

$$(\alpha_k)_y + \alpha_k \int dx[\overleftarrow{t}_1^{\,(k+1)} - \overleftarrow{t}_1^{\,k} - t_2 + \overleftarrow{t}_2] = \int dx(\overleftarrow{\alpha}_k\overleftarrow{t}_1^{\,(k+1)} - \alpha_k\overleftarrow{t}_1^{\,k} + \overleftarrow{\beta}_k t_2 - \beta_k\overrightarrow{t}_2]m,$$

$$(\beta_k)_y + \beta_k \int dx[\overleftarrow{t}_1^{\,k} - \overleftarrow{t}_1^{\,(k-1)} + \overrightarrow{t}_2^{\,2} - \overrightarrow{t}_2] = \int dx[-\alpha_k\overleftarrow{t}_1^{\,k} + \overrightarrow{\alpha}_k\overleftarrow{t}_1^{\,(k-1)} + \overrightarrow{\beta}_k\overrightarrow{t}_2^{\,2} - \beta_k\overrightarrow{t}_2]. \tag{3.7}$$

Then it is possible to verify this assumption by induction. For solutions S_k we have $\alpha_k = -\beta_k = 1$ and the explicit expression for S_k in the formal form

$$S_n = S_0\prod_{k=1}^{n}(1 - L_k\exp -(k+1)d_k - \sum_{j=k+1}^{n} d_j)\int dy\overleftarrow{t}_1\int dy\overleftarrow{t}_1^{\,2}\ldots\ldots\int dy\overleftarrow{t}_1^{\,n}. \tag{3.8}$$

4. The simplest examples

In the case of $S_0 = v$ we obtain the trivial system with the help of (2.6)

$$u_t = u_y \quad v_t = v_y.$$

In the case of $n = 1$

$$S_1 = v \int dx(\overleftarrow{t_1} - \overrightarrow{t_2}) = v_y + v^2 + 2v \int dx(u_y),$$

the corresponding integrable system has the form

$$u_t = -u_{yy} + 2(uv)_y + 2u_y \int dx(u_y) \quad v_t = (v^2 + v_y + 2v \int dx(u_y))_y.$$

In one dimensional case $D_x = D_y$ this system is a partial case of a wider integrable system described in [4].

In the case $n = 2$ we obtain

$$S_2 = v^3 + 3vv_y + v_{yy} + 3vD_x^{-1}(uv)_y + 3(v_y + v^2)D_x^{-1}(u_y) + 3v(D_x^{-1}(u_y))^2.$$

The corresponding integrable system is the following:

$$u_t = D_y(u_{yy} - 3(vu_y) + 3v^2u - 3(u_y - uv)D_x^{-1}(u)_y) +$$
$$D_x + 3D_x^{-1}(u_y)D_x^{-1}(uv)_y + (D_x^{-1}(u_y))^3),$$
$$v_t = D_y(v^3 + 3vv_y + v_{yy} + 3vD_x^{-1}(uv)_y + 3(v_y + v^2)D_x^{-1}(u_y) + 3v(D_x^{-1}(u_y))^2).$$

5. Conclusion

The main result of the present paper is contained in formulas (3.8) which allow one to construct with the help of (2.6) $(2+1)$ dimensional integrable equations of L–V hierarchy. Each of these systems is invariant with respect to transformations of the discrete Lotcky-Volterra substitution. We specially emphasize that our expression for S_k contains only two functions t_1,t_2, shifted multi–times by means of direct or inverse discrete transformations and operation of repeated integrations. From this form we can synonymously conclude that discrete transformation have to play the fundamental role in the whole theory of integrable systems. From the considered examples (section 4) we see that the explicit form of the equations in variables u,v is much more complicated than the corresponding expressions in terms of multi-times shifted background functions t_1, t_2. There arises the question: is it possible to conserve the language of discrete shifts to describe equations by itself and find their solutions? We don't know the answer to such question now.

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