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Vertex Version of the Bazhanov – Baxter Model

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Abstract

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In this paper we formulate a statistical model on the simple cubic lattice. N–valued spin variables of the model belong to the edges of the lattice. Boltzmann weights of the model obey the vertex type Tetrahedron Equation. In the thermodynamic limit our model is equivalent to the Bazhanov–Baxter Model. In the case when $N = 2$ we reproduce the Korepanov and Hietarinta solutions of the Tetrahedron equation as some special cases.

Аннотация

Мангазеев В.В. и др. Вершинная версия модели Бажанова – Бакстера: Препринт ИФВЭ 94-141. – Протвино, 1994. – 14 с., 1 табл., библиогр.: 9.

В данной работе мы формулируем статистическую модель на простой кубической решетке. N–значные спиновые переменные модели лежат на ребрах решетки. Больцмановские веса модели удовлетворяют уравнению тетраэдров. В термодинамическом пределе наша модель эквивалентна модели Бажанова–Бакстера. В случае $N = 2$ мы воспроизводим решения уравнения тетраэдров Корепанова и Хиетаринты как некоторые специальные случаи.

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1. Introduction

Recently two new solutions of the vertex type Tetrahedron equation for the number of colors $N = 2$ [1,2] have been obtained. In our previous paper we tried to generalize these solutions both for $N > 2$ and for general spectral parameters. We succeeded in generalization of the Hietarinta solution for the arbitrary N.

Th Hietarinta solution is some special case of the planar limit of the Bazhanov – Baxter solution [3] in case $N = 2$. Recall that in Bazhanov – Baxter model [4] (BBM for shortness) the N–colored spin variables belong to the vertices of the lattice elementary cubes and Boltzmann weights in the Tetrahedron Equation (TE) are parameterized by the angles of tetrahedron [5]. In the limit when all four vertices of a tetrahedron belong to one plane (the planar limit), it is possible to rewrite the Boltzmann weights using edge variables and as a result to obtain the Hietarinta vertex solution of TE. Note that the $N > 2$ – colored solution from Ref. [3] does not coincide with the planar limit of BBM and seems to be new.

Attempts to remove the planar limit restriction for this solution failed. Instead we have obtained a complete (i.e. depending on three arbitrary angles) vertex solution of TE for the general number of colors N, which in case when $N = 2$ reproduces the solutions by Korepanov and Hietarinta in the static and planar limits, correspondingly. This new model in the thermodynamic limit coincides with BBM. Nevertheless, due to the vertex form, this formulation may be useful for the more careful investigation of the model. Namely, one can try to formulate the Bethe–Ansatz, construct a functional equation for the transfer matrices analogously to the two–dimensional case. One can try to construct a three dimensional generalization of the L operators as well.

This paper is organized as follows. In Section 2 we recall the conventional notations for the functions on Z_N which will be used to construct the Boltzmann weights. In Section 3 we give the explicit form of the vertex weight function and show the equivalence of our vertex model with BBM in the thermodynamic limit. The symmetry properties of the vertex weight we list in Section 4. There we also give some exotic forms of the gauges and write the inversion relation for the weight. The case when the number of colors $N = 2$ is considered in some special gauge in Section 5, where we show the equivalence of our vertex weight in the statistic limit with the Korepanov solution of TE. Section 6 is devoted to

the sketch proof of TE for our vertex weight. At last, in Appendix we present the most useful formulae for ω – hypergeometric series.

2. Notations and Definitions

The list of definitions and notations is chosen to be the shortest. Denote

$$\omega^{1/2} = \exp(\pi i/N). \tag{1}$$

Taking p to be a point on a Fermat curve Υ, so that there defined three complex numbers $x(p), y(p), z(p)$, constrained by the Fermat equation

$$x(p)^N + y(p)^N = z(p)^N \tag{2}$$

and a to be an element of Z_N, define

$$\frac{w(p|a)}{w(p|0)} = \prod_{s=1}^{a} \frac{y(p)}{z(p) - x(p)\omega^s}. \tag{3}$$

The absolute value of $w(p|0)$ we define through

$$\prod_{a=0}^{N-1} w(p|a) = 1. \tag{4}$$

Branches of $y(p)$ and $w(p|0)$ are arbitrary in general, but it is convenient to choose them appropriately. Below all points p-s will be defined so that (when it is possible, we will omit the argument p for the shortness)

$$-2\pi/N < \mathrm{Arg}(x/z) < 0 \text{ and } -\pi/N < \mathrm{Arg}(y/z) < \pi/N. \tag{5}$$

This subregion in Υ we call Υ_0. For $p \in \Upsilon_0$ we define $w(p|0)$ as follows:

$$w(p|0) = \left(\frac{y}{z}\right)^{\frac{N-1}{2}} \frac{1}{d(\omega x/z)} = \left(\frac{x}{y}\right)^{\frac{N-1}{2}} \Phi_0^{-1} d(z/x), \tag{6}$$

where

$$\Phi_0 = \exp\left(\frac{i\pi(N-1)(N-2)}{6N}\right), \tag{7}$$

and

$$d(x) = \exp \sum_{a=1}^{N-1} \frac{a}{N} \log(1 - x\omega^a). \tag{8}$$

It is implied in (8) that

$$-\pi < \mathrm{Im}\log(z) \leq \pi. \tag{9}$$

Defined w functions have the following property:

$$w(p|a)w(Op|-a)\widetilde{\Phi}(a) = 1, \quad a \in Z_N, \tag{10}$$

where automorphism $O : \Upsilon_0 \to \Upsilon_0$ is defined as

$$x(Op) = z(p),\ y(Op) = \omega^{1/2} y(p),\ z(Op) = \omega x(p), \tag{11}$$

and

$$\tilde{\Phi}(a) = \omega^{a(a-N)/2} \exp(\frac{i\pi(N^2-1)}{6N}), \quad \prod_{a \in Z_N} \Phi(a) = 1. \tag{12}$$

It is convenient to define the normalization of w for some other useful subregions of Υ through

$$w(x^n, y, z|0) = w(x, y, z|n), \quad (x, y, z) \in \Upsilon, \tag{13}$$

and

$$\begin{gathered} \frac{w(x, \omega y, z|0)}{w(x, y, z|0)} = -\omega^{1/2}, \\ \frac{w(x, \omega^{-1} y, z|0)}{w(x, y, z|0)} = -\omega^{1/2}, \\ (x, y, z) \in \Upsilon_0. \end{gathered} \tag{14}$$

The reader can find a set of the most useful formulae and identities for the w function in Appendix.

3. The Vertex Weight

For the given spherical triangle with the angles $\theta_1, \theta_2, \theta_3$ and the corresponding linear angles (i.e. sides of the spherical triangle) a_1, a_2, a_3 define four points on Fermat curve (2)

$$\begin{gathered} x(p_1) = \omega^{-1/2} \exp(i\frac{a_3}{N}) \sqrt[N]{\frac{\sin\beta_1}{\sin\beta_2}},\ y(p_1) = \exp(i\frac{\beta_1}{N}) \sqrt[N]{\frac{\sin a_3}{\sin\beta_2}}; \\ x(p_2) = \omega^{-1/2} \exp(i\frac{a_3}{N}) \sqrt[N]{\frac{\sin\beta_2}{\sin\beta_1}},\ y(p_2) = \exp(i\frac{\beta_2}{N}) \sqrt[N]{\frac{\sin a_3}{\sin\beta_1}}; \\ x(p_3) = \omega^{-1} \exp(i\frac{a_3}{N}) \sqrt[N]{\frac{\sin\beta_3}{\sin\beta_0}},\ y(p_3) = \exp(-i\frac{\beta_3}{N}) \sqrt[N]{\frac{\sin a_3}{\sin\beta_0}}; \\ x(p_4) = \omega^{-1} \exp(i\frac{a_3}{N}) \sqrt[N]{\frac{\sin\beta_0}{\sin\beta_3}},\ y(p_4) = \exp(-i\frac{\beta_0}{N}) \sqrt[N]{\frac{\sin a_3}{\sin\beta_3}}; \\ z(p_i) = 1,\ i = 1, 2, 3, 4; \end{gathered} \tag{15}$$

where β_i being usual linear excesses

$$\beta_0 = \pi - \frac{a_1 + a_2 + a_3}{2},\ \beta_i = \pi - \beta_0 - a_i. \tag{16}$$

In this paper we will regard system (15) as the definition of functions $p_i = p_i(a_1, a_2, a_3)$. With these notations the vertex weight function is

$$R^{j_1, j_2, j_3}_{i_1, i_2, i_3} = \delta_{j_2+j_3, i_2+i_3} \omega^{j_3(j_1-i_1)} \frac{w(p_1|i_1 - i_2) w(p_2|j_1 - j_2)}{w(p_3|i_1 - j_2) w(p_4|j_1 - i_2)}. \tag{17}$$

and all five space permutations are given by

$$
\begin{aligned}
F^{-1}\otimes F^{-1}\otimes F^{-1}R(a_1,a_2,a_3)F\otimes F\otimes F &= n_{13}P_{13}R^t(a_3,a_2,a_1)P_{13},\\
J\otimes J\otimes FR(a_1,a_2,a_3)J\otimes J\otimes F^{-1} &= P_{12}R^t(a_2,a_1,a_3)P_{12}.\\
F\otimes 1\otimes 1R(a_1,a_2,a_3)F^{-1}\otimes 1\otimes 1 &= n_{23}P_{23}R^t(a_1,a_3,a_2)P_{23},\\
F^{-1}\otimes F^{-1}\otimes JR(a_1,a_2,a_3)F\otimes F\otimes J &= n_{23}P_{23}P_{12}R(a_3,a_1,a_2)P_{12}P_{23},\\
J\otimes F\otimes FR(a_1,a_2,a_3)J\otimes F^{-1}\otimes F^{-1} &= n_{13}P_{12}P_{23}R(a_2,a_3,a_1)P_{23}P_{12},
\end{aligned}
\tag{28}
$$

where the normalizators are

$$
n_{13}=\left(\frac{\sin a_1}{\sin a_3}\right)^{\frac{N-1}{N}},\quad n_{23}=\left(\frac{\sin a_2}{\sin a_3}\right)^{\frac{N-1}{N}}. \tag{29}
$$

Combining Fourier transformations with diagonal gauge transformations, one can obtain other forms of the R – matrix. Note that these Fourier and combined transformations are the gauge transformations of the lattice, but not the gauge transformations of the Tetrahedron equation in general.

Note that there exists the conservation law $i_2+i_3=j_2+j_3$ for weight (17). Below we write out some combined transformations which lead to the other forms of the spin conservations laws:

$$
\begin{aligned}
&N^{-1}\sum_{\alpha_3,\beta_3}\omega^{\alpha_3 j_3-\beta_3 i_3}\left(\frac{\widetilde{\Phi}(\alpha_3)}{\widetilde{\Phi}(\beta_3)}\right)^{\epsilon}R^{j_1,j_2,\alpha_3}_{i_1,i_2,\beta_3}(a_1,a_2,a_3)=\\
&=\delta(j_3+j_1-i_3-i_1-\epsilon(j_2-i_2))\widetilde{\Phi}(j_2-i_2)^{-\epsilon}\omega^{-i_3(j_2-i_2)}\\
&\quad\frac{w(p_1(a_1,a_2,a_3)|i_1-i_2)w(p_2(a_1,a_2,a_3)|j_1-j_2)}{w(p_3(a_1,a_2,a_3)|i_1-j_2)w(p_4(a_1,a_2,a_3)|j_1-i_2)},
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&N^{-2}\sum_{\alpha_m,\beta_m}\omega^{\alpha_1 j_1-\beta_1 i_1+\alpha_2 j_2-\beta_2 i_2}\left(\frac{\widetilde{\Phi}(\alpha_2)}{\widetilde{\Phi}(\beta_2)}\right)^{\epsilon}R^{\alpha_1,\alpha_2,j_3}_{\beta_1,\beta_2,i_3}(a_1,a_2,a_3)=\\
&=\delta(j_2+j_1-i_2-i_1-\epsilon(j_3-i_3))\widetilde{\Phi}(j_3-i_3)^{\epsilon}\omega^{j_2(i_3-j_3)}\\
&\quad n_{23}\frac{w(p_1(a_1,a_3,a_2)|-j_1-j_3)w(p_2(a_1,a_3,a_2)|-i_1-i_3)}{w(p_3(a_1,a_3,a_2)|-j_1-i_3)w(p_4(a_1,a_3,a_2)|-i_1-j_3)},
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&N^{-3}\sum_{\alpha_m,\beta_m}(\prod_m\omega^{\alpha_m j_m-\beta_m i_m})\left(\frac{\widetilde{\Phi}(\alpha_1)}{\widetilde{\Phi}(\beta_1)}\right)^{\epsilon}R^{\alpha_1,\alpha_2,\alpha_3}_{\beta_1,\beta_2,\beta_3}(a_1,a_2,a_3)=\\
&=\delta(j_1+j_2-i_1-i_2-\epsilon(j_3-i_3))\widetilde{\Phi}(j_3-i_3)^{-\epsilon}\omega^{i_1(i_3-j_3)}\\
&\quad n_{13}\frac{w(p_1(a_3,a_2,a_1)|j_3-j_2)w(p_2(a_3,a_2,a_1)|i_3-i_2)}{w(p_3(a_3,a_2,a_1)|j_3-i_2)w(p_4(a_3,a_2,a_1)|i_3-j_2)},
\end{aligned}
\tag{32}
$$

where $\delta(a)=\delta_{a,0}$, the symbol $\widetilde{\Phi}(a)$ is defined by (12) and ϵ is an arbitrary integer. Other choices of the diagonal Φ factors lead us to complicated nonmultiplicative expressions for the weights without any conservation laws. The exception is the case when $N=2$.

5. Case $N = 2$

In the case when $N = 2$ the list of suitable Fourier transformations enlarges. Namely,

$$N^{-3} n_3 \sum_{\alpha_m, \beta_m} \left(\prod_{m=1}^{3} \omega^{\alpha_m j_m - \beta_m i_m} \frac{\tilde{\Phi}(\beta_m)}{\tilde{\Phi}(\alpha_m)} \right) R^{\alpha_1, \alpha_2, j_3}_{\beta_1, \beta_2, i_3}(a_1, a_2, a_3) =$$
$$= \delta(i_1 + j_1 + i_3 + j_3 - i_2 - j_2) \omega^{(i_3 - j_3)(i_2 - j_3)} \times$$
$$\times \frac{w(r_1| - i_1 + j_2 - j_3) w(r_3 | i_1 - i_2 + j_3)}{w(Or_0 | i_1 - i_2 + i_3) w(Or_2 | - i_1 + j_2 - i_3)} \equiv \overline{R}^{j_1, j_2, j_3}_{i_1, i_2, i_3}, \tag{33}$$

where $\omega = -1$,

$$n_3 = \left(\frac{\sin a_3}{2\sqrt{\sin \beta_0 \ldots \sin \beta_3}} \right)^{\frac{N-1}{N}} \tag{34}$$

and four points r_i are given by

$$x(r_i) = \exp(-i\beta_i/N), \; y(r_i) = \omega^{1/4} \sqrt[N]{2 \sin \beta_i},$$
$$z(r_i) = \exp(i\beta_i/N), \; i = 0, 1, 2, 3. \tag{35}$$

Due to the total symmetry, this transformation is the gauge transformation of the Tetrahedron equation, so this weight $\overline{R}$ obeys the Tetrahedron equation (see Section 6).

When $N = 2$ the function w is very simple:

$$\frac{w(r_i|1)}{w(r_i|0)} = \exp(i\frac{\pi}{4}) \sqrt{\tan \frac{\beta_i}{2}}. \tag{36}$$

Defining for shortness

$$\sqrt{\tan \frac{\beta_i}{2}} = t_i, \; i = 0, 1, 2, 3, \tag{37}$$

we can represent the weights (33) by the almost compact table.

The weight, defined by this table, differs from that defined by (33) in unessential normalization. Using the following property of a spherical triangle:

$$\tan \frac{\beta_i}{2} \tan \frac{\beta_j}{2} = \tan \frac{\alpha_k}{2} \tan \frac{\alpha_l}{2}, \; \{i, j, k, l\} = \{0, 1, 2, 3\}, \tag{38}$$

where α_i are the angle excesses of the spherical triangle, we can easily obtain the static limit of the $\overline{R}$ (the case when $\alpha_0 = 0$). This static limit appears to be the Korepanov solution of Tetrahedron equation [1]. Moreover, in the planar limit when $\beta_2 = 0$ the vertex weight (33) coincides with the $N = 2$ solution by Hietarinta [2,3].

Table 1.

$\overline{R}^{0,0,0}_{0,0,0}$	$=$	$\overline{R}^{0,1,1}_{0,1,1}$	$=$	$\overline{R}^{1,0,1}_{1,0,1}$	$=$	$\overline{R}^{1,1,0}_{1,1,0}$	$=$	1
$\overline{R}^{1,1,1}_{1,1,1}$	$=$	$\overline{R}^{1,0,0}_{1,0,0}$	$=$	$\overline{R}^{0,1,0}_{0,1,0}$	$=$	$\overline{R}^{0,0,1}_{0,0,1}$	$=$	$t_0t_1t_2t_3$
$\overline{R}^{0,1,0}_{0,0,1}$	$=$	$\overline{R}^{0,0,1}_{0,1,0}$	$=$	$-\overline{R}^{1,0,0}_{1.1,1}$	$=$	$-\overline{R}^{1,1,1}_{1,0,0}$	$=$	t_0t_1
$\overline{R}^{1,0,1}_{1,1,0}$	$=$	$\overline{R}^{1,1,0}_{1,0,1}$	$=$	$-\overline{R}^{0,1,1}_{0,0,0}$	$=$	$-\overline{R}^{0,0,0}_{0,1,1}$	$=$	t_2t_3
$\overline{R}^{1,1,1}_{0,1,0}$	$=$	$\overline{R}^{1,0,0}_{0,0,1}$	$=$	$-\overline{R}^{0,0,1}_{1,0,0}$	$=$	$-\overline{R}^{0,1,0}_{1,1,1}$	$=$	it_0t_2
$\overline{R}^{0,0,0}_{1,0,1}$	$=$	$\overline{R}^{0,1,1}_{1,1,0}$	$=$	$-\overline{R}^{1,1,0}_{0,1,1}$	$=$	$-\overline{R}^{1,0,1}_{0,0,0}$	$=$	$-it_1t_3$
$\overline{R}^{0,0,1}_{1,1,1}$	$=$	$\overline{R}^{1,1,1}_{0,0,1}$	$=$	$\overline{R}^{1,0,0}_{0,1,0}$	$=$	$\overline{R}^{0,1,0}_{1,0,0}$	$=$	t_0t_3
$\overline{R}^{1,1,0}_{0,0,0}$	$=$	$\overline{R}^{0,0,0}_{1,1,0}$	$=$	$\overline{R}^{0,1,1}_{1,0,1}$	$=$	$\overline{R}^{1,0,1}_{0,1,1}$	$=$	t_1t_2

6. The Tetrahedron Equation

The vertex form of TE is the following:

$$\sum_{\substack{k_1,k_2,k_3,\\ k_4,k_5,k_6}} R^{k_1,k_2,k_3}_{i_1,i_2,i_3} R'^{j_1k_4k_5}_{k_1i_4i_5} R''^{j_2j_4k_6}_{k_2k_4i_6} R'''^{j_3j_5j_6}_{k_3k_5k_6} =$$
$$= \sum_{\substack{k_1,k_2,k_3,\\ k_4,k_5,k_6}} R'''^{k_3,k_5,k_6}_{i_3,i_5,i_6} R''^{k_2k_4j_6}_{i_2i_4k_6} R'^{k_1j_4j_5}_{i_1k_4k_5} R^{j_1j_2j_3}_{k_1k_2k_3} \quad . \tag{39}$$

A complete solution of this equation is parameterized by six angles of a tetrahedron (five of them are independent):

$$\begin{aligned} R &= R(\theta_1,\theta_2,\theta_3), \\ R' &= R(\theta_1,\theta_4,\theta_5), \\ R'' &= R(\pi-\theta_2,\theta_4,\theta_6) \\ R''' &= R(\theta_3,\pi-\theta_5,\theta_6). \end{aligned} \tag{40}$$

The ordering of the dihedral angles is natural with respect to numeration of the spaces and differs from that in the standard equation (2.2) in Ref. [8].

For each vertex in (39) let a_i be the corresponding planar angles:

$$\begin{aligned}(\theta_1,\theta_2,\theta_3) &\to (a_1,a_2,a_3),\\ (\theta_1,\theta_4,\theta_5) &\to (a_1',a_2',a_3'),\\ (\pi-\theta_2,\theta_4,\theta_6) &\to (a_1'',a_2'',a_3''),\\ (\theta_3,\pi-\theta_5,\theta_6) &\to (a_1''',a_2''',a_3''').\end{aligned} \tag{41}$$

Four vertex weights (17) with the angles defined by (41) obey this Tetrahedron equation. In this section we give a sketch proof of this proposition.

Substitute (17) in (39). Due to the spins conservation laws, six summations in both sides reduce to three summations. It is useful to choose the indices k_1, k_2, k_4 as the independent spins of the summation. The summands in the left and right hand sides are the products of the phase factors $\omega^{\cdots}$ and w functions. Gather in both sides all the factors depending on the spin k_1. The sums over k_1 have the form ${}_2\Psi_2$ (see Appendix). As the first step apply to these ${}_2\Psi_2$ $(\tau\rho)^2$ transformations (see formula 50 from Appendix). As a result there appear extra w functions, depending on $k_2 - k_4$, and we demand the cancellation of these extra factors with the similar w–functions in the left and right hand sides.

After this the summation over k_2 and k_4 becomes independent. Moreover, there are no phase factors, depending on k_2 and k_4, and we can sum over k_2 and k_4 using the "Star – Square" relation (see formula 53 from Appendix). The final sums over k_1 are of the ${}_4\Psi_4$ type and both have the same spin structure, so we regard the equality between the right and left hand sides as tautology. TE is proved.

The restrictions arising when we prove TE (namely, cancellation of the w functions, depending on $k_2 - k_4$, the "Star – Square" applicability conditions and the coincidence of the final expressions) are satisfied automatically by parameterization (15) and (41).

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Appendix

In this Appendix we have collected formulae used in the theory of ω – hypergeometric series. These formulae appeared in many papers devoted to the Chiral Potts Model and to TE. Following conventional notations, define ${}_r\Psi_r$ series as

$$ {}_r\Psi_r\left(\left.\begin{matrix}(p_1,m_1)\dots(p_r,m_r)\\(p'_1,m'_1)\dots(p'_r,m'_r)\end{matrix}\right|n\right)=\sum_{\sigma\in Z_N}\frac{w(p_1|m_1+\sigma)\dots w(p_r|m_r+\sigma)}{w(p'_1|m'_1+\sigma)\dots w(p'_r|m'_r+\sigma)}\frac{\omega^{n\sigma}}{\sqrt{N}}. \tag{42}$$

Spin independent factors in all identities are given for the case when all arguments of the w functions in the left hand sides belong to the region (5) and the phases of ws are chosen according to (6–9). If we abandon restriction (5) then the phases of ws can be chosen so that the formulae would still remain correct.

For the sake of shortness we omit the argument in the components of point p and regard

$$p_i=(x_i,y_i,z_i) \tag{43}$$

for every i. There will appear many new points on the Fermat curve in the right hand sides of formulae in this Appendix. In these cases we have to introduce new letters for y components. They have to be defined by (2) (and by (5) according to our convention).

All the formulae in this Appendix are the formulae of summation (they exist for $r=1,2,3$) and symmetry transformation (they exist for $r=1,2,3,4$).

We begin with the Ramanujan summation formula for $r=1$:

$$\begin{gathered}{}_1\Psi_1\left(\left.\begin{matrix}(p_1,m_1)\\(p_2,m_2)\end{matrix}\right|n\right)=\\=\Phi_0^{1/2}\left(\frac{\xi}{y_1y_2}\right)^{\frac{N-1}{2}}\omega^{-nm_2}\\ \frac{w(z_1y_2,\xi,y_1z_2|-n)w(x_1z_2,\xi,\omega z_1x_2|m_1-m_2)}{w(x_1y_2,\xi,\omega y_1x_2|m_1-m_2-n)}=\\=\Phi_0^{-1/2}\left(\frac{\omega^{-1/2}\xi}{y_1y_2}\right)^{\frac{N-1}{2}}\omega^{-nm_1}\\ \frac{w(y_1x_2,\omega^{-1/2}\xi,x_1y_2|m_2-m_1+n)}{w(y_1z_2/\omega,\omega^{-1/2}\xi,z_1y_2|n)w(z_1x_2,\omega^{-1/2}\xi,x_1z_2|m_2-m_1)},\end{gathered} \tag{44}$$

where $-\pi/2<\mathrm{Arg}(z^{1/2})<\pi/2$.

For the proof of this formula see, for example, Ref. [9]. As the partial case of (44) it is easy to obtain the inversion relation which is to be used when one proves inversion relation (24):

$$\sum_\sigma\frac{w(x,y,z|m_1+\sigma)}{w(x,y,\omega z|m_2+\sigma)}=N\delta_{m_1,m_2}\left(\frac{\omega^{1/2}x}{y}\right)^{N-1}. \tag{45}$$

To obtain symmetry formulae for higher r, we use the following trick. Let g_1 and g_2 be arbitrary functions on Z_N. If

$$\tilde{g}_i(k) = \sum_{\sigma \in Z_N} g_i(\sigma) \frac{\omega^{k\sigma}}{\sqrt{N}}, \tag{46}$$

then

$$\sum_{\sigma \in Z_N} g_1(\sigma) g_2(\sigma) = \sum_{\sigma \in Z_N} \tilde{g}_1(\sigma) \tilde{g}_2(-\sigma). \tag{47}$$

Using this, it is easy to obtain the following symmetry transformation for ${}_2\Psi_2$:

$$\begin{aligned} {}_2\Psi_2 \left(\begin{matrix} (p_1, m_1)(p_3, m_3) \\ (p_2, m_2)(p_4, m_4) \end{matrix} \middle| n \right) &= {}_2\Psi_2 \left(\begin{matrix} (q_1, 0) & (q_3, m_4 - m_3 + n) \\ (q_2, n) & (q_4, m_1 - m_2) \end{matrix} \middle| m_2 - m_3 \right) \\ &\left(\frac{\xi_{12}\xi_{43}}{y_1 y_2 y_3 y_4} \right)^{\frac{N-1}{2}} \omega^{-nm_3} \frac{w(x_1 z_2, \xi_{12}, \omega z_1 x_2 | m_1 - m_2)}{w(z_3 x_4, \xi_{43}, x_3 z_4 | m_4 - m_3)}, \end{aligned} \tag{48}$$

where

$$\begin{aligned} q_1 &= (z_1 y_2, \xi_{12}, y_1 z_2), \\ q_2 &= (y_3 z_4/\omega, \xi_{43}, z_3 y_4), \\ q_3 &= (y_3 x_4, \xi_{43}, x_3 y_4), \\ q_4 &= (x_1 y_2, \xi_{12}, \omega y_1 x_2). \end{aligned} \tag{49}$$

This relation originally appeared as the $(\tau\rho)$ transformation in Ref. [6] for the BB weight function. Note that $(\tau\rho)^6 = 1$. In this paper we have used $(\tau\rho)^2$ transformation:

$$\begin{aligned} &{}_2\Psi_2 \left(\begin{matrix} (p_1, m_1)(p_3, m_3) \\ (p_2, m_2)(p_4, m_4) \end{matrix} \middle| n \right) = \\ &= {}_2\Psi_2 \left(\begin{matrix} (s_1, 0) & (s_3, m_1 - m_4 - n) \\ (s_2, m_2 - m_3) & (s_4, -n) \end{matrix} \middle| m_3 - m_4 \right) \\ &\omega^{-nm_2 - (m_2 - m_3)(m_4 - m_3)} \left(\frac{\Gamma\Gamma'}{\xi_{12}\xi_{43} y_1 y_2 y_3 y_4} \right)^{\frac{N-1}{2}} \\ &\frac{w(x_1 z_2, \xi_{12}, \omega z_1 x_2 | m_1 - m_2)}{w(z_3 x_4, \xi_{43}, x_3 z_4 | m_4 - m_3)} \frac{w(z_1 z_3 y_2 y_4, \Gamma, z_2 z_4 y_1 y_3 | -n)}{w(x_1 x_3 y_2 y_4, \Gamma', \omega x_2 x_4 y_1 y_3 | -\overline{n})}, \end{aligned} \tag{50}$$

where

$$\overline{n} = n - m_1 - m_3 + m_2 + m_4 \tag{51}$$

and

$$\begin{aligned} s_1 &= (y_1 z_2 \xi_{43}, \Gamma, z_3 y_4 \xi_{12}), \\ s_2 &= (y_1 x_2 \xi_{43}, \Gamma', x_3 y_4 \xi_{12}), \\ s_3 &= (x_1 y_2 \xi_{43}, \Gamma', y_3 x_4 \xi_{12}), \\ s_4 &= (z_1 y_2 \xi_{43}, \Gamma, y_3 z_4 \xi_{12}). \end{aligned} \tag{52}$$

We finish the list of the symmetry formulae for ${}_2\Psi_2$ in the $T = (\tau\rho)^3$

$$
{}_2\Psi_2\left(\begin{matrix}(p_1,m_1)(p_3,m_3)\\(p_2,m_2)(p_4,m_4)\end{matrix}\middle|\, n\right) = {}_2\Psi_2\left(\begin{matrix}(\overline{p}_1,-m_3)(\overline{p}_3,-m_1)\\(\overline{p}_2,-m_4)(\overline{p}_4,-m_2)\end{matrix}\middle|\, \overline{n}\right)
$$
$$
\omega^{-nm_2-m_4\overline{n}}\left(\frac{\xi_{12}\xi_{43}\xi_{32}\xi_{41}}{\Gamma\Gamma'}\right)^{\frac{N-1}{2}}\frac{w(x_1z_2,\xi_{12},\omega z_1x_2|m_1-m_2)}{w(z_3x_4,\xi_{43},x_3z_4|m_4-m_3)}
$$
$$
\frac{w(z_1z_3y_2y_4,\Gamma,z_2z_4y_1y_3|-n)}{w(x_1x_3y_2y_4,\Gamma',\omega x_2x_4y_1y_3|-\overline{n})}\frac{w(x_3z_2,\xi_{32},\omega z_3x_2|m_3-m_2)}{w(z_1x_4,\xi_{41},x_1z_4|m_4-m_1)}, \tag{53}
$$

where

$$
\begin{aligned}
\overline{p}_1 &= (z_3\Gamma',\xi_{32}\xi_{43}y_1,x_3\Gamma),\\
\overline{p}_2 &= (z_4\Gamma',\omega\xi_{41}\xi_{43}y_2,\omega x_4\Gamma),\\
\overline{p}_3 &= (z_1\Gamma',\xi_{41}\xi_{12}y_3,x_1\Gamma),\\
\overline{p}_4 &= (z_2\Gamma',\xi_{32}\xi_{12}y_4,\omega x_2\Gamma).
\end{aligned} \tag{54}
$$

Note that in the case when $\frac{x_1x_3}{x_2x_4} = \omega\frac{z_1z_3}{z_2z_4}$, (53) becomes the Star – Star relation for BBM, and $p_i = \overline{p}_i$.

To obtain a summation formula for ${}_2\Psi_2$, consider (48) and suppose $n = 0$ and $q_1 = q_2$. Then applying (44) to the right hand side of (48), we obtain the "Star – Square" relation:

$$
{}_2\Psi_2\left(\begin{matrix}(p_1,m_1)(p_3,m_3)\\(p_2,m_2)(p_4,m_4)\end{matrix}\middle|\, 0\right) = \left(\frac{\omega^{1/2}\Gamma'}{y_1y_2y_3y_4}\right)^{\frac{N-1}{2}}\omega^{-(m_2-m_3)(m_1-m_2)}
$$
$$
\Phi_0^{1/2}w(x_2x_4y_1y_3,\omega^{-1/2}\Gamma',x_1x_3y_2y_4|m_2+m_4-m_1-m_3)
$$
$$
\frac{w(x_1z_2,\xi_{12},\omega z_1x_2|m_1-m_2)}{w(z_3x_4,\xi_{43},x_3z_4|m_4-m_3)}\frac{w(x_3z_2,\xi_{32},\omega z_3x_2|m_3-m_2)}{w(z_1x_4,\xi_{41},x_1z_4|m_4-m_1)}, \tag{55}
$$

where the parameters in the left hand side have to obey the special restriction:

$$
\frac{y_1y_3}{y_2y_4} = \omega\frac{z_1z_3}{z_2z_4} \tag{56}
$$

and the phases in the right hand side are connected by

$$
\frac{\xi_{12}}{\xi_{43}} = \frac{y_1z_2}{y_4z_3},\quad \frac{\xi_{32}}{\xi_{41}} = \frac{y_3z_2}{y_4z_1},\quad \Gamma' = \omega^{-1/2}\xi_{12}\xi_{32}\frac{y_4}{z_2}. \tag{57}
$$

Hence forth we will try to avoid such long notations as in (53) and in (55). Extra w multipliers in all consequent formulae will have the same structure as in the right hand side of (53) and so we will use only ξ_{ij} to denote the whole argument dependence of w.

Consider now $r = 3$. A summation formula can be obtained by summing (53) over n with the help of the Ramanujan formula. The result reads

$$
{}_3\Psi_3\left(\begin{matrix}(p_1,m_1)(p_3,m_3)(q_1,m_2+m_4-\lambda)\\(p_2,m_2)(p_4,m_4)(q_2,m_1+m_3-\lambda)\end{matrix}\middle|\, 0\right) =
$$

$$= \Phi_0^{-1/2} \left(\frac{\xi_{12}\xi_{43}\xi_{32}\xi_{41}}{y_1 y_2 y_3 y_4 \Xi} \right)^{\frac{N-1}{2}}$$

$$\frac{{}_\omega(m_4 - \lambda)(m_1 + m_3 - m_2 - m_4)}{w(x_1 x_3 z_2 z_4, \Xi, \omega^2 x_2 x_4 z_1 z_3 | m_1 + m_3 - m_2 - m_4)}$$

$$\frac{w(\xi_{12}|m_1 - m_2) w(\xi_{32}|m_3 - m_2)}{w(\xi_{43}|m_4 - m_3) w(\xi_{41}|m_4 - m_1)} \frac{w(\overline{p}_1|\lambda - m_3) w(\overline{p}_3|\lambda - m_1)}{w(\overline{p}_2|\lambda - m_4) w(\overline{p}_4|\lambda - m_2)}, \tag{58}$$

where $w(\xi_{ij})$ and $w(\overline{p}_i)$ are the same as in (53) and

$$\begin{aligned} q_1 &= (\omega x_2 x_4 \Gamma, y_2 y_4 \Xi, z_2 z_4 \Gamma'), \\ q_2 &= (x_1 x_3 \Gamma, y_1 y_3 \Xi, \omega z_1 z_3 \Gamma'). \end{aligned} \tag{59}$$

Note, that formula (58) is symmetric with respect to any permutation of p_1, p_3, q_1 and p_2, p_4, q_2. The Star – Triangle equation for the Chiral Potts Model is a special case of (58).

To obtain symmetry relations for $r = 3$ and $r = 4$, we use (47), make the Ramanujan summation or T transformation, cancel extra w factors (this gives some constraints) and then, using (47) again, obtain the corresponding ${}_r\Psi_r$ in the right hand side. The formula for $r = 3$ reads

$${}_3\Psi_3 \left(\left. \begin{matrix} (p_1, m_1)(p_3, m_3)(q_1, l_1) \\ (p_2, m_2)(p_4, m_4)(q_2, l_2) \end{matrix} \right| 0 \right) =$$

$$= \left(\frac{\xi_{12}\xi_{43}\xi_{32}\xi_{41}\xi}{\Gamma'^2 \Gamma y(q_1) y(q_2)} \right)^{\frac{N-1}{2}} {}_\omega(l_1 - m_2)(m_1 + m_3 - m_2 - m_4)$$

$$\frac{w(\xi_{12}|m_1 - m_2) w(\xi_{32}|m_3 - m_2)}{w(\xi_{43}|m_4 - m_3) w(\xi_{41}|m_4 - m_1)} \frac{w(\xi|l_2 + m_2 + m_4 - l_1 - m_1 - m_3)}{w(\lambda|l_2 - l_1)}$$

$${}_3\Psi_3 \left(\left. \begin{matrix} (\overline{p}_1, -m_3)(\overline{p}_3, -m_1)(\overline{q}_1, l_1 - m_2 - m_4) \\ (\overline{p}_2, -m_4)(\overline{p}_4, -m_2)(\overline{q}_2, l_2 - m_1 - m_3) \end{matrix} \right| 0 \right), \tag{60}$$

where the connection between arguments in the left hand side is

$$\frac{y_1 y_3 y(q_1)}{y_2 y_4 y(q_2)} = \omega \frac{z_1 z_3 z(q_1)}{z_2 z_4 z(q_2)}, \qquad \frac{\Gamma}{\lambda} = \frac{z_2 z_4 y_1 y_3}{z(q_1) y(q_2)} \tag{61}$$

and new arguments in the right hand side of (60) are

$$\begin{aligned} (\xi) &= (\omega x_2 x_4 x(q_2) y_1 y_3 y(q_1), \xi, \omega x_1 x_3 x(q_1) y_2 y_4 y(q_2)), \\ (\lambda) &= (z(q_1) y(q_2), \lambda, x(q_1) z(q_2)), \\ \overline{q}_1 &= (x(q_1) y(q_2) \Gamma', \xi, \omega \lambda x_2 x_4 y_1 y_3), \\ \overline{q}_2 &= (y(q_1) x(q_2) \Gamma', \xi, \omega \lambda x_1 x_3 y_2 y_4). \end{aligned} \tag{62}$$

Note that this formula is a symmetry transformation for something. Denote (60) as ρ_3. Let τ_3 be the permutation transformation, reordering the columns in ${}_3\Psi_3$ as $\tau_3(1, 2, 3) = (2, 3, 1)$. Then $(\tau_3 \rho_3)^3 = 1$.

The last formula is the symmetry transformation for ${}_4\Psi_4$. A derivation of it is described before formula (60). Let the structure of a set $q_i, \overline{q}_i, \chi_{ij}, \Delta, \Delta'$ be defined identically to that of $p_i, \overline{p}_i, \xi_{ij}, \Gamma, \Gamma'$. Then

$$
\begin{aligned}
&{}_4\Psi_4\left(\left.\begin{matrix}(p_1,m_1)(p_3,m_3)(q_1,l_1)(q_3,l_3)\\(p_2,m_2)(p_4,m_4)(q_2,l_2)(q_4,l_4)\end{matrix}\right|0\right) = \\
&= \left(\frac{\xi_{12}\xi_{43}\xi_{32}\xi_{41}\chi_{12}\chi_{43}\chi_{32}\chi_{41}}{\Gamma'\Gamma\Delta'\Delta}\right)^{\frac{N-1}{2}} \frac{\omega^{(l_2-m_2)(m_1+m_3-m_2-m_4)}}{\widetilde{\Phi}(m_1+m_3-m_2-m_4)} \\
&\frac{w(\xi_{12}|m_1-m_2)w(\xi_{32}|m_3-m_2)}{w(\xi_{43}|m_4-m_3)w(\xi_{41}|m_4-m_1)}\frac{w(\chi_{12}|l_1-l_2)w(\chi_{32}|l_3-l_2)}{w(\chi_{43}|l_4-l_3)w(\chi_{41}|l_4-l_1)} \\
&{}_4\Psi_4\left(\left.\begin{matrix}(\overline{p}_1,-m_3)(\overline{p}_3,-m_1)(\overline{q}_1,l_1-m_2-m_4)(\overline{q}_3,l_3-m_2-m_4)\\(\overline{p}_2,-m_4)(\overline{p}_4,-m_2)(\overline{q}_2,l_2-m_1-m_3)(\overline{q}_4,l_4-m_1-m_3)\end{matrix}\right|0\right),
\end{aligned}
\tag{63}
$$

where the connections are

$$
\begin{aligned}
\frac{y(p_1)y(p_3)y(q_1)y(q_3)}{y(p_2)y(p_4)y(q_2)y(q_4)} &= \omega\frac{z(p_1)z(p_3)z(q_1)z(q_3)}{z(p_2)z(p_4)z(q_2)z(q_4)} \\
&= \omega^{-1}\frac{x(p_1)x(p_3)x(q_1)x(q_3)}{x(p_2)x(p_4)x(q_2)x(q_4)}
\end{aligned}
\tag{64}
$$

and

$$
\begin{aligned}
\frac{\Gamma}{\Delta} &= \omega^{1/2}\frac{y(p_1)y(p_3)z(p_2)z(p_4)}{z(q_1)z(q_3)y(q_2)y(q_4)}, \\
\frac{\Gamma'}{\Delta'} &= \omega^{1/2}\frac{y(p_1)y(p_3)x(p_2)x(p_4)}{x(q_1)x(q_3)y(q_2)y(q_4)},
\end{aligned}
\tag{65}
$$

and the spins in (63) are not independent:

$$
m_1+m_3+l_1+l_3 = m_2+m_4+l_2+l_4. \tag{66}
$$

Note that if we tried to obtain any ${}_{r-1}\Psi_{r-1}$ identity from a ${}_r\Psi_r$ one as a partial case, then we would get some more restricted variant of the lower identity.

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