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LONGITUDINAL DIFFUSION OF A PROTON BUNCH UNDER EXTERNAL NOISE (Wide Band Noise)

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Abstract

Ivanov S.V. Longitudinal Diffusion of a Proton Bunch under External Noise (Wide-Band Noise): IHEP Preprint 93-14. – Protvino, 1993. – p. 31, refs.: 6.

The paper is an extension to the former Preprint IHEP 92–43 with the same title. It presents the diffusion coefficient for a bunched beam subjected to a *wide-band* noise of accelerating voltage in a proton synchrotron. The noise *wide-band* performance infers its spectrum to stretch over many revolution frequency harmonics, i.e., a possibility of an arbitrary ratio of the noise correlation time to the beam revolution period, which turns out essential to treat the noise-induced diffusion in large rings — UNK, LHC, SSC.

Аннотация

Иванов С.В. Продольная диффузия сгустка протонов под действием внешнего шума (Широкополосный шум): Препринт ИФВЭ 93-14. – Протвино, 1993. – 31 с., библиогр.: 6.

Работа является продолжением препринта ИФВЭ 92–43 с тем же названием. В ней приведен коэффициент диффузии для сгруппированного пучка в условиях *широкополосного* шума ускоряющего напряжения в протонном синхротроне. *Широкополосность* шума означает, что его спектр охватывает много гармоник частоты обращения, т.е. возможность произвольного соотношения между временем корреляции шума и периодом обращения пучка. Последнее важно для анализа шумовой диффузии в больших кольцах — УНК, LHC, SSC.

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Иванов С.В.
Продольная диффузия сгустка протонов под действием внешнего шума (Широкополосный шум).

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This paper is, essentially, an extension to Preprint [1] with the same title. Occurring in quadrature, amplitude and phase noises of accelerating voltage build up a convenient model for treating beam dynamics. The approach of ref.[1] allows one to continue these studies — this time under arbitrary relationship between noise correlation time and beam revolution period. Such an analysis shows the results of refs.[1,2,3,4] to be valid for a relatively low-frequency and narrow-band noise whose correlation time exceeds substantially the revolution period. Hence, these results may not be applicable to proton synchrotrons with large orbit perimeters (UNK, LHC, SSC). On the contrary, the present paper offers expressions which hold in the latter case, as well.

1. Status of the Problem

Longitudinal dynamics of a proton bunch subjected to external RF-noise is governed by a diffusion equation. Write down the latter in the form obtained in ref.[1]

$$\frac{\partial\langle F_0\rangle(\mathcal{J},t)}{\partial t} = \frac{\partial}{\partial\mathcal{J}}\left(D(\mathcal{J},t)\frac{\partial\langle F_0\rangle(\mathcal{J},t)}{\partial\mathcal{J}}\right). \tag{1}$$

Here t is time; $\mathcal{J}$ is action variable whose value at separatrix is $\mathcal{J}_{\rm sep}$; F is particle distribution function. The angular brackets $\langle\ldots\rangle$ are the equation symbol for statistical average over noise ensemble. The subscript '0' denotes mathematical average over phase ψ, the canonical conjugate of $\mathcal{J}$. The angle-action variables $(\psi,\mathcal{J})$ are introduced in the longitudinal phase-plane $(\vartheta,\vartheta' \equiv d\vartheta/dt)$, where $\vartheta \sim \Theta - \omega_s t$ is azimuth in a co-rotating frame; Θ is a generalized azimuth around the ring as measured in the laboratory frame in the direction of the beam motion; ω_s is the angular velocity of a reference particle synchronous to the

nominal value of the acceleration frequency. The origin $\vartheta = 0$ of the coordinate ϑ is put on the unperturbed reference particle of the bunch in question.

The diffusion coefficient, say, inside separatrix ($\mathcal{J} \leq \mathcal{J}_{\rm sep}$) reads [1]

$$D(\mathcal{J},t) = \frac{1}{2}\left(\frac{\Omega_0^2}{qV_{\rm ext}\sin\varphi_s}\right)^2 \sum_{m=-\infty}^{\infty} m^2 \sum_{k,k_1=-\infty}^{\infty} \frac{I^*_{mk}(\mathcal{J})}{k}\frac{I_{mk_1}(\mathcal{J})}{k_1} \times \tag{2}$$
$$\times\ 2\int_0^{\infty} \langle \Delta V_k(t)\Delta V^*_{k_1}(t-\tau)\rangle \exp\left(im\Omega_s(\mathcal{J})\tau\right) d\tau.$$

Here, the quantity Ω_0 has the meaning of a circular frequency of small-amplitude oscillations in a potential well $U(\vartheta)$ with $U(\vartheta) \to \vartheta^2/2$ as $\vartheta \to 0$; q is the main RF harmonic number; $V_{\rm ext}$ is the nominal value of accelerating voltage amplitude; φ_s is the stable phase angle ($\varphi_s > 0$ below transition with the synchronous energy gain being $eV_{\rm ext}\cos\varphi_s$); $\Omega_s(\mathcal{J}) = d\psi/dt$ is the frequency of non-linear synchrotron oscillations.

Functions $I^*_{mk}(\mathcal{J})$ are the expansion coefficients of a plane wave into series over multipole oscillations:

$$I^*_{mk}(\mathcal{J}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\psi\ \exp(ik\vartheta(\mathcal{J},\psi) - im\psi), \tag{3}$$
$$\vartheta(\mathcal{J},\psi+2\pi) = \vartheta(\mathcal{J},\psi), \qquad \mathcal{J} \leq \mathcal{J}_{\rm sep}.$$

The beam is subjected to a random voltage $\Delta V(\vartheta,t)$, $|\Delta V(\vartheta,t)| \ll V_{\rm ext}$ treated in the co-rotating frame. Due to periodicity $\Delta V(\vartheta,t) = \Delta V(\vartheta+2\pi,t)$ it can be Fourier series decomposed:

$$\Delta V(\vartheta,t) = \sum_{k=-\infty}^{\infty} \Delta V_k(t)\ \exp(ik\vartheta), \tag{4}$$

$$\Delta V_k(t) = \frac{1}{2\pi}\int_0^{2\pi} \Delta V(\vartheta,t)\ \exp(-ik\vartheta)\, d\vartheta. \tag{5}$$

Diffusion coefficient (2) depends on time correlations of random amplitudes $\Delta V_k(t)$.

Eqs.(1), (2) are quite general in nature. To get them, one only employs a perturbation theory and an assumption on the bunch being both stationary and matched at the initial instant of time $t = 0$, $\Delta V(\vartheta, t \leq 0) = 0$. The latter realizes the so-called *random phase type of approximation* for initial phases of the synchrotron oscillations. Beam-induced self-fields are neglected.

In addition, the following agreements on representative time scales of the problem at issue are involved:

Hereof, a structural peculiarity of $D(\mathcal{J})$, as expressed in terms of modulating currents $i_n^{(\zeta)}(t)$, $q' \neq 0$, is revealed. Function $T_n(\omega)$ always remains inside the weight and amplitude factors. It cannot be taken out of the brackets and thus treated as a factor which merely filters the noise spectrum. The exception is that (infrequent) case when $T_n(\omega)$ features a complex-conjugate symmetry w.r.t. frequency $q'\omega_s$:

$$T_n(q'\omega_s + \omega) = T_n^*(q'\omega_s - \omega). \tag{114}$$

As far as stationary noise $j_n(t)$ is concerned, this condition is met identically due to $q' = 0$, and such a peculiarity of $D(\mathcal{J})$ does not occur.

employ eqs.(105), (106). Equate factors before $2\pi\delta(\omega-\omega_1)$. Thereby get the expressions for eq.(104) in terms of the random currents:

$$P_{nn_1}^{(\zeta\zeta_1)}(\omega)\exp\left(\pm i(\varphi_n^{(\zeta)}-\varphi_{n_1}^{(\zeta_1)})\right)= \\ =T_n(\omega\pm q'\omega_s)\,T_{n_1}^*(\omega\pm q'\omega_s)\;Q_{nn_1}^{(\zeta\zeta_1)}(\omega)\exp\left(\pm i(\phi_n^{(\zeta)}-\phi_{n_1}^{(\zeta_1)})\right); \tag{107}$$

$$P_{nn_1}^{(\zeta\zeta_1)}(\omega)\exp\left(\pm i(\varphi_n^{(\zeta)}+\varphi_{n_1}^{(\zeta_1)})\right)= \\ =T_n(\omega\pm q'\omega_s)\,T_{n_1}^*(\omega\mp q'\omega_s)\;Q_{nn_1}^{(\zeta\zeta_1)}(\omega)\exp\left(\pm i(\phi_n^{(\zeta)}+\phi_{n_1}^{(\zeta_1)})\right). \tag{108}$$

(For instance, it becomes clear hereof that the noise voltage spectrum cannot be wider than the bandwidth of accelerating devices. Consequently, correlation time τ_v of modulation noises $v_n^{(\zeta)}(t)$ is sure not to be less than a filling time of the gaps.)

Insertion of eqs.(107), (108) into the formulae of Sections 3,4 provides a complete solution to the problem in question. Despite unwieldy appearance of eqs.(107), (108), the formal changes of $D(\mathcal{J})$ are not so intricate. We just list the modifications which emerge on passing to the 'current-wise' representation.

First of all, the spectral matrices of noises and the carrier phases, if there are any, are replaced everywhere:

$$P_{nn_1}^{(\zeta\zeta_1)}(\omega)\to Q_{nn_1}^{(\zeta\zeta_1)}(\omega);\qquad \varphi_n^{(\zeta)}\to\phi_n^{(\zeta)}. \tag{109}$$

A new factor emerges in eq.(44):

$$G_{n;k\pm q'}\to G_{n;k\pm q'}\,T_n\left((k\pm q')\omega_s+m\Omega_s(\mathcal{J})\right). \tag{110}$$

Eq.(46), being rewritten for $Q_{nn_1}^{(\zeta\zeta_1)}(\omega)$, is complemented by a requirement of wide-band performance, in the scale of synchrotron frequency Ω_s, of function $T_n(\omega)$:

$$T_n\left(k\omega_s+m\Omega_s(\mathcal{J})\right)\simeq T_n\left(k\omega_s\right). \tag{111}$$

Eq.(48) is entered by factor (110) which is, naturally, rewritten so as to obey approximation (111):

$$G_{n;k\pm q'}\to G_{n;k\pm q'}\,T_n\left((k\pm q')\omega_s\right). \tag{112}$$

Finally, in the unbunched-beam limit, eq.(54) is altered by

$$G_{n;k\pm q'}\to G_{n;k\pm q'}\,T_n\left(k\omega(\mathcal{J})\pm q'\vartheta'(\mathcal{J})\right), \tag{113}$$

while eqs.(56), (57) suffer the same but for $q'=0$.

a) fluctuations of accelerating voltage are assumed to be fast,

$$\tau_{\Delta V}\ll\tau_{dif}; \tag{6}$$

b) bunch evolution is pursued during the intervals of time t which are

$$\tau_{\Delta V}\ll t\lesssim\tau_{dif}. \tag{7}$$

In both of these inequalities time constant τ_{dif} is a rate measure of a slow, systematic dilution of $\langle F_0\rangle(\mathcal{J},t)$ under the noise (diffusion life-time of the bunch), while $\tau_{\Delta V}$ is (auto)correlation time of random voltage $\Delta V(\vartheta,t)$. (The calculations to follow would indicate that τ_{dif} is reciprocal of the noise spectral power intensity. Therefore eq.(6) requires the voltage fluctuations to be, rigorously speaking, both fast and weak.)

A particular expression for the diffusion coefficient depends upon the way of involving the noise source $\Delta V(\vartheta,t)$ into the consideration. Two approaches to this question can be followed:

1. The further treatment in ref.[1], as well as the one in refs.[2,3,4], proceeds from a conjecture that voltage $\Delta V(\vartheta,t)$ is given in the co-moving frame directly, in which it reads

$$\Delta V(\vartheta,t)=v(t)\,g(\vartheta),\qquad g(\vartheta)=g\left(\vartheta+\frac{2\pi}{q}\right). \tag{8}$$

Here $g(\vartheta)$ is a determinate function; $v(t)$ is a stochastic process with a zero average, $\langle v(t)\rangle=0$.

The assumption on noise $v(t)$ to be a 'weakly' stationary one cancels the explicit t-time dependence in $D(\mathcal{J},t)$ at once. As ref.[1] shows, further choice of functions $g(\vartheta)$, $v(t)$ in the form of

$$g^{(a)}(\vartheta)=\cos(q\vartheta+\varphi_s),\qquad v^{(a)}(t)=\Delta V_{\rm ext}(t); \tag{9}$$

$$g^{(\varphi)}(\vartheta)=\sin(q\vartheta+\varphi_s),\qquad v^{(\varphi)}(t)=V_{\rm ext}\Delta\varphi(t); \tag{10}$$

and their substitution into eq.(2) yields the well-known results of refs.[2,3,4] for amplitude (a) and phase (φ) noises of accelerating voltage, respectively. Nevertheless, the acceptability of representation (8) still requires a more thorough investigation.

2. Ref.[1] hints at a more general route to calculate diffusion coefficient (2), as well. It takes into account that the noise voltage is, in fact, driven in the laboratory frame in which it acquires the form of a periodically-unstationary — or, possibly, of a stationary — stochastic process $\Delta V(\Theta,t)$. Due to this

reason $D(\mathcal{J},t)$ from eq.(2) turns out to be a periodic function of time, $2\pi/\omega_s$ being a period, which is extremely unsuitable for the calculations. It is obvious, however, that nothing crucial should be encountered in course of a single turn. What is necessary (and justifiable physically) is to average eq.(1) over a turn. This procedure would allow one to retain the slow motion only. The latter is caused by the non-averaging to zero term of $D(\mathcal{J},t)$ which brings about a systematic effect and is denoted by $D(\mathcal{J})$ later on. An additive correction proportional to the time average of a product between the fast varying parts of $D(\mathcal{J},t)$ and $\langle F_0\rangle(\mathcal{J},t)$ is neglected because of its small value expected (the second order of magnitude in the fast motion). The bunch distribution smoothed over a turn period is denoted by the former equation symbol $\langle F_0\rangle(\mathcal{J},t)$. This function is now supposed to be a slowly varying one in the time scale of $2\pi/\omega_s$:

$$\frac{2\pi}{\omega_s} \ll \tau_{dif}. \tag{11}$$

On these conditions being met the formal modification of eq.(1) after its averaging over t would result in a mere substitution of $D(\mathcal{J},t)$ by $D(\mathcal{J})$.

Quantitative verification of such a procedure is not a simple matter. Rather, we apply to indirect evidences, such as rederivation of the results known by now. These are the diffusion equation of a bunch subjected to low-frequency narrow-band amplitude or phase noises of accelerating voltage (refer to Section 7), and the noise-induced diffusion over momenta in an unbunched beam (refer to the end of Section 4).

It is the second of the afore-said approaches which is followed throughout the present paper. In particular, it is shown thereby that representation of the random voltage by eqs.(8), (9), (10), as studied in refs.[1,2,3,4], is acceptable only for a relatively low-frequency and narrow-band noise $v(t)$. Its correlation time τ_v must comply with limitations

$$\frac{2\pi}{\omega_s} \ll \tau_v \ll \tau_{dif}, \tag{12}$$

which may grow into a crucial constraint. Indeed, the natural noise performances, with τ_v included, are imposed solely by technical features of the RF-system. These cannot be affected strongly by, say, the ring size. However, as the orbit perimeters grow (UNK, LHC, SSC) the revolution frequencies ω_s do decrease. Relationships like $\tau_v \sim 2\pi/\omega_s$ or even $\tau_v \ll 2\pi/\omega_s$ become feasible. Thereby the l.h.s. of inequality (12) can well be violated, results of refs.[1,2,3,4] thus becoming insufficient.

In what follows, more general formulae for the diffusion coefficient under the accelerating voltage noises are arrived at. These hold true not only within

while the quadrature one — that of phase:

$$i_n^{(s')}(t) = I_n \Delta\phi_n(t), \tag{101}$$

$$\Delta\phi(t) = \int^t \Delta\omega'_{rf}(t_1)\,dt_1, \quad \langle\Delta\phi^2\rangle^{1/2} \ll 2\pi,$$

of the feeder current.

Suppression of the carrier by adopting $q' = 0$ results in a particular case of noise current $j_n(t) = \sum_\zeta i_n^{(\zeta)}(t)\cos\phi_n^{(\zeta)}$ which is stationary w.r.t. the laboratory frame.

Transfer-function (97) converts current (98) into the noise voltage

$$u_n(t) = \sum_\zeta v_n^{(\zeta)}(t)\cos(q'\omega_s t - \varphi_n^{(\zeta)}) \tag{102}$$

across the n-th gap. It is the form in which $u_n(t)$ enters the framework of eq.(24) used to represent the random electric field on the orbit.

Perform the Fourier transformation in time of eqs.(98), (102). On applying to definition (97), one establishes the relation between the Fourier transforms of random envelopes $v, i_n^{(\zeta)}(t)$:

$$v_n^{(\zeta)}(\omega)\,\exp(\pm i\varphi_n^{(\zeta)}) = T_n(\omega \pm q'\omega_s)\,i_n^{(\zeta)}(\omega)\,\exp(\pm i\phi_n^{(\zeta)}). \tag{103}$$

Now, let us refer to Sections 3,4. Notice that each and every equation for $D(\mathcal{J})$ given there involves items of four types. These are proportional to

$$P_{nn_1}^{(\zeta\zeta_1)}(\omega)\,\exp\left(\pm i(\varphi_n^{(\zeta)} - \varphi_{n_1}^{(\zeta_1)})\right), \quad P_{nn_1}^{(\zeta\zeta_1)}(\omega)\,\exp\left(\pm i(\varphi_n^{(\zeta)} + \varphi_{n_1}^{(\zeta_1)})\right), \tag{104}$$

and all should be expressed in terms of the spectrum of the random currents. This procedure can easily be carried out.

Really, a 'weak' stationarity of random envelopes $v, i_n^{(\zeta)}(t)$ whose averages are $\langle v_n^{(\zeta)}(t)\rangle = \langle i_n^{(\zeta)}(t)\rangle = 0$ implies their Fourier transforms to be delta-correlated:

$$\left\langle v_n^{(\zeta)}(\omega)\,v_{n_1}^{(\zeta_1)*}(\omega_1)\right\rangle = 2\pi\,P_{nn_1}^{(\zeta\zeta_1)}(\omega)\,\delta(\omega-\omega_1); \tag{105}$$

$$\left\langle i_n^{(\zeta)}(\omega)\,i_{n_1}^{(\zeta_1)*}(\omega_1)\right\rangle = 2\pi\,Q_{nn_1}^{(\zeta\zeta_1)}(\omega)\,\delta(\omega-\omega_1). \tag{106}$$

Here $P_{nn_1}^{(\zeta\zeta_1)}(\omega)$ denote the spectral matrices of modulation noises of voltage (28), while $Q_{nn_1}^{(\zeta\zeta_1)}(\omega)$ are those for the noises of current.

Rewrite eq.(103) as a pair of identities with an arbitrary choice of signs in each of them. Take the relevant indices to be either n,ζ or n_1,ζ_1, respectively. Perform complex conjugation in the latter case, and inter-multiply the expressions thus obtained. Carry out a statistical averaging of their product where

as flowing through, say, a coupling device between the feeder and the n-th gap. (Reflected-wave current does not enter into definition (96).)

Phase ϕ_n of the current is not obliged to coincide with phase φ_n of the (overall) voltage (15). It occurs this way due to two factors:

a) A complex transfer-function $T_n(q'\omega_s) = T_n^*(-q'\omega_s)$ from the feeder current to a component of the gap voltage thereby excited.

b) A phasor diagram of voltages as maintained to compensate beam loading of the accelerating gap.

On taking both these factors into account, one can define amplitude I_n and phase ϕ_n of current (96). It is a standard RF-engineering calculation. Commonly, it is carried out while setting up an operating point for the accelerating system. To save the paper length, its discussion is skipped here. All the more that it would have fallen beyond the context of the diffusion problem under study.

Noises in RF-feeding circuitry originate random additive perturbation $j_n(t)$ to current (96). It drives noise voltage $u_n(t)$ across the n-th gap (19). A feeder, a coupler and a gap constitute a linear and stationary circuit. Hence, the relation between $j_n(t)$ and $u_n(t)$ is linear and time-invariant:

$$u_n(\omega) = T_n(\omega)\, j_n(\omega). \tag{97}$$

Here $u, j_n(\omega) = u, j_n^*(-\omega)$ are the Fourier transforms of $u, j_n(t)$. $T_n(\omega) = T_n^*(-\omega)$ denotes the relevant transfer-function dimensioned as an electric resistance, with eq.(97) being its formal definition.

Due to apparent reasons, consider a random-modulated current which has a general form of

$$j_n(t) = \sum_{\zeta} i_n^{(\zeta)}(t)\, \cos(q'\omega_s t - \phi_n^{(\zeta)}). \tag{98}$$

Here ζ denotes some index again. Carrier phase $\phi_n^{(\zeta)}$ is, rigorously speaking, other than ϕ_n. Currents $i_n^{(\zeta)}(t)$, $\langle i_n^{(\zeta)}(t)\rangle = 0$ are stochastic processes which are (mutually) stationary w.r.t. the laboratory frame.

The particular option of indices $\zeta = c', s'$ and phases

$$\phi_n^{(c')} = \phi_n, \quad \phi_n^{(s')} = \phi_n - \frac{\pi}{2}, \tag{99}$$

allows one to interpret eq.(98) as a decomposition of $j_n(t)$ into sum of its inphase ($\zeta = c'$) and quadrature ($\zeta = s'$) components w.r.t. the reference signal (96). As usual, the inphase component represents the noise of amplitude:

$$i_n^{(c')}(t) = \Delta I_n(t), \tag{100}$$

the range given by eq.(12), but well beyond it:

$$\frac{2\pi}{\omega_s} \sim \tau_v \ll \tau_{dif}, \qquad \tau_v \ll \frac{2\pi}{\omega_s} \ll \tau_{dif}. \tag{13}$$

Note, that up to an order of magnitude $\tau_v \Delta\omega_v \sim 2\pi$, where $\Delta\omega_v$ is the noise spectrum width. Hence, eq.(13) entails that either $\Delta\omega_v \sim \omega_s$ or $\Delta\omega_v \gg \omega_s$. It is in this very sense that the noise is referred to as 'wide-band' in the paper.

2. Accelerating Field and Its Noises

Features peculiar to a random electric field present on the orbit of a synchrotron are, to a large extent, imposed by its accelerating system. We proceed from its brief description.

Main Field $E(\Theta, t)$

Let the accelerating system be specified by three parameters: $q', V'_{\text{ext}}, \varphi'_s$. That is, it should maintain the synchronous-to-beam propagating wave of voltage,

$$V(\vartheta, t) = V'_{\text{ext}} \cos(q'\vartheta + \varphi'_s) \tag{14}$$

which is stationary in the co-rotating frame. The prime, as attached to $q', V'_{\text{ext}}, \varphi'_s$, shows that we are not restricting ourselves to the main accelerating system whose parameters are $q, V_{\text{ext}}, \varphi_s$. To the same extent one may bear in mind an (auxiliary) RF-system driven at a radio-frequency of $q'\omega_s$ – a higher harmonic of the (main) frequency $q\omega_s$ of acceleration. (Assumption $q'/q = 1, 2, \ldots$ might be withdrawn but this would result in unnecessary complications.)

Technically, wave (14) is imposed via excitation of $N \geq 1$ accelerating devices (gaps). Label them with numbers $n = 1, 2, \ldots, N$ in an arbitrary order. Put down the longitudinal electric field according to ref.[5]

$$E(\Theta, t) = \frac{1}{L} \sum_{n=1}^{N} G_n(\Theta) V_n \cos(q'\omega_s t - \varphi_n). \tag{15}$$

Here L is the ring length; V_n and φ_n are the amplitude and phase of the voltage across the n-th gap; $q'\omega_s = \omega'_{\text{rf}}$ is the nominal value of the radio-frequency.

Functions $G_n(\Theta) = G_n(\Theta + 2\pi)$ depend upon the way of field localization around the orbit. They can be Fourier series decomposed:

$$G_n(\Theta) = \sum_{k=-\infty}^{\infty} G_{n;k} \exp(ik\Theta), \tag{16}$$

$$G_{n;k} = \frac{1}{2\pi}\int_0^{2\pi} G_n(\Theta)\exp(-ik\Theta)\,d\Theta. \tag{17}$$

Expansion coefficients $G_{n;k} = G^*_{n;-k}$ (or, to be more exact, their moduli) can be interpreted as transit-time factors at frequencies $k\omega_s$. Suffice it to require

$$\frac{1}{2\pi}\int_0^{2\pi} |G_n(\Theta)|\,d\Theta = 1. \tag{18}$$

Whereof $|G_{n;k}| \leq 1$, the equality sign standing for an infinitely-narrow gap. No other restrictions on $G_n(\Theta)$ are imposed.

Required shape (14) of the synchronous wave can always be attained by a due adjustment of amplitudes V_n and phases φ_n of the gap voltages. For the time being, their practical choice is not important. A few alternatives are possible. One of them — it minimizes the RF-power consumption and, hence, is a preferable one — is introduced later on, when a concrete model of the accelerating system is put forward (refer to Section 5, eqs.(58), (59)).

Reduction of the lumped field given by eq.(15) to the distributed one of synchronous waves (14) yields standard ('smoothed') equations of longitudinal dynamics, ref.[5]. Next step to take may well be the use of the perturbed equations which are deduced from the same model (14) of resonant interaction. Really, noise given by eqs.(9), (10) emerges as a random amplitude or phase variation suffered by wave (14). (Or, possibly, as a sum over such perturbations sufficient to represent function $g(\vartheta)$ from eq.(8) through its Fourier series decomposition.) In this case the required statistical features of the noise (namely, its stationarity and existence of the spectral power density) have to be taken for granted in the *co-moving* coordinate frame directly.

A more straightforward approach which is pursued later on proceeds from the noise as driven in the *laboratory* frame.

Random Field $\Delta E(\Theta, t)$

Due to the RF-system noises a random electric field $\Delta E(\Theta, t)$ appears on the orbit. By analogy with eq.(15) put it as

$$\Delta E(\Theta, t) = \frac{1}{L}\sum_{n=1}^{N} G_n(\Theta)\,u_n(t) \tag{19}$$

where $u_n(t)$ is the noise voltage across the n-th gap. The same gap also bears the main oscillation $\sim \cos(q'\omega_s t - \varphi_n)$. Take it as the reference one. Then,

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Appendix 1

Noises of Current in RF-Feeders

Noises of accelerating voltage are excited by those of RF-drive current in the gap feeders. The latter noises are more logical to deal with while specifying the noise tolerances on a practical, RF-engineering level. Expressions for $D(\mathcal{J})$ in terms of such random currents can be readily established. Suffice it to insert into the above formulae the particular expressions for the spectral matrices of noise voltages — their representation in terms of current noises and relevant transfer-functions.

To excite voltage (15), the external RF-generator drives a forward wave of current in the gap feeder. Let this wave be presented by a current

$$I_n \cos(q'\omega_s t - \phi_n), \tag{96}$$

The unbunched-beam limit corresponds to

$$D^{(\zeta')}(\mathcal{J} \gg \mathcal{J}_{\rm sep}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\rm ext}\sin\varphi_s}\right)^2 \cdot \mathcal{P}^{(\zeta')}(q'\vartheta'(\mathcal{J})) \times 4. \qquad (92)$$

One can readily verify that weight (79), (89) and amplitude (80), (91) functions for the two narrow-band approximations just studied are inter-related:

$$\frac{1}{2}\mathcal{W}_m^{(\zeta')}(\mathcal{J};q') + \frac{1}{2}\mathcal{W}_m^{(\zeta')}(\mathcal{J};-q') = \mathcal{W}_m^{(a)}(\mathcal{J}) + \mathcal{W}_m^{(\varphi)}(\mathcal{J}), \qquad (93)$$

$$\mathcal{A}^{(\zeta')}(\mathcal{J}) = \mathcal{A}^{(a)}(\mathcal{J}) + \mathcal{A}^{(\varphi)}(\mathcal{J}). \qquad (94)$$

The same relation holds for the rightmost numerical factors of asymptotic eqs.(85), (92): $4^{(\zeta')} = 2^{(a)} + 2^{(\varphi)}$.

Though seeming formal at a first glance, these equalities rely on a firm physical basis. Really, let spectrum $P^{(\zeta')}(\omega)$ of noise $u(t)$ be symmetrical w.r.t. frequencies $\omega = \pm q'\omega_s$:

$$P^{(\zeta')}(\pm q'\omega_s + \Omega) = P^{(\zeta')}(\pm q'\omega_s - \Omega), \qquad (95)$$

whereof it follows that $\mathcal{P}, p^{(\zeta')}(\Omega) = \mathcal{P}, p^{(\zeta')}(-\Omega)$. (By its definition, the *band-pass white* noise is precisely of this type.) Due to eqs.(32), (33), (36), (37), the effect exerted by stationary noise $u(t)$ whose spectrum is of eq.(95)-type is equivalent to a superposition of amplitude and phase noises $v^{(a,\varphi)}(t)$ of accelerating voltage, these being uncorrelated and having the same spectrum — $\mathcal{P}^{(a)}(\Omega) = \mathcal{P}^{(\varphi)}(\Omega) = \mathcal{P}^{(\zeta')}(\Omega)$. This statement immediately entails equalities (93), (94). These are but a particular manifestation of the general features of $D(\mathcal{J})$ discussed at the end of Section 2.

Both, the applicability range and the practical significance of eqs.(87)–(92) is by no means less than that of the extensively-used expressions (84)–(85). Nevertheless, the problem of bunch diffusion under effect of stationary noise $u_n(t)$ — even in a simplified, narrow-band approximation (86) — has not yet drawn a proper attention. (On the contrary, the noise-induced diffusion of an unbunched beam was studied for such a problem statement from the very beginning, ref.[5].)

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imposing no restrictions to generality, voltage $u_n(t)$ can be expanded into a sum of its inphase ($\sim$ cos) and quadrature ($\sim$ sin) components:

$$\begin{aligned} u_n(t) &= v_n^{(c)}(t)\cos(q'\omega_s t - \varphi_n) - v_n^{(s)}(t)\sin(q'\omega_s t - \varphi_n) = \\ &= \sum_{\zeta=c,s} v_n^{(\zeta)}(t)\cos(q'\omega_s t - \varphi_n^{(\zeta)}); \end{aligned} \qquad (20)$$

$$\varphi_n^{(c)} = \varphi_n, \quad \varphi_n^{(s)} = \varphi_n - \frac{\pi}{2}. \qquad (21)$$

For brevity, use is made of notation Σ_ζ later on.

Formally, decomposition by eq.(20) is of an absolutely general nature. It involves no extra suggestions on the features of $u_n(t)$, ref.[6]. Commonly, one goes over to it for convenience. For instance, under certain limitations imposed on the spectrum of $u_n(t)$, amplitudes $v_n^{(\zeta)}(t)$ may turn out to be slowly varying functions of time.

But let us have another look at this decomposition now. Here we study stochastic processes. Their further analysis is impossible without an assumption on the noise stationarity w.r.t. the laboratory frame. Which of the random voltages, $u_n(t)$ or $v_n^{(\zeta)}(t)$ exhibits this behavior? (This question is not a senseless one at all. The stationarity of $u_n(t)$ entails that of $v_n^{(\zeta)}(t)$. Vice versa is not always the case.) Technical realization of RF-systems allows both alternatives:

On the one hand, the RF-systems incorporate a low-level driving and shaping circuitry: a master generator of reference oscillations, an amplitude modulator and a phase-shifter. Physical concept of their operation, including the effect of noises, relies on mixing the carrying and modulating signals, a small phase modulation of the reference signal being reducible to an amplitude modulation of the quadrature oscillation in a first approximation. For this reason the hardware maintains not the voltage $u_n(t)$ as such, but rather the random-modulated (in amplitude) oscillations: the inphase and quadrature ones as in the r.h.s. of eq.(20). Either may be driven in a relatively independent way. Given a sustained operation mode under permanent external conditions, envelopes $v_n^{(\zeta)}(t)$ are stationary w.r.t. the laboratory frame, and their spectral intensity exists. The statistical features of voltages $u_n(t)$ across the gaps are derivable from those of $v_n^{(\zeta)}(t)$.

On the other hand, there exist intrinsic noises of the RF-power amplifier: a shot noise of the anode current direct component in the tube, a stray ripple in the tube power supply, etc. These drive stationary gap voltages $u_n(t)$ directly. Decomposition by eqs.(20), (21) is still allowable. But the statistical features of $v_n^{(\zeta)}(t)$ are now secondary, and derivable from those of $u_n(t)$. Any amplifier cascade is a frequency-selective circuit. Hence, it makes sense to anticipate the spectrum of noise $u_n(t)$ to tend to localize in the vicinity of $\omega = \pm q'\omega_s$, and to

have a relatively narrow bandwidth there (in the scale of radio-frequency $q'\omega_s$). The narrower is the noise bandwidth, the slower varying functions of time are the realizations of random amplitudes $v_n^{(\zeta)}(t)$.

Thus due to the technical reasons solely we would draw a distinction between the two types of external *stationary* noises of accelerating voltage. Namely, these are: (i) amplitude-modulating noises $v_n^{(\zeta)}(t)$ carried by oscillations $\cos(q'\omega_s t - \varphi_n^{(\zeta)})$; and (ii) noises $u_n(t)$ without explicit carrier as maintained by the hardware. (In what follows the latter noises are revealed to be a subset of the former ones.) Therefore, envelopes $v_n^{(\zeta)}(t)$ in eq.(20) are allowed to originate from different sources. But this keeps intact the physical content hidden in eqs.(20), (21) which is a rather simple one.

Indeed, let $\Delta E(\Theta, t)$ emerge from a random variation of the main field (15). Let restrict ourselves to a linear approximation in noises $\Delta V_n(t)$, $\Delta\varphi_n(t)$. Then, the inphase ($\zeta = c$) component (20) corresponds to the noises of amplitude

$$v_n^{(c)}(t) = \Delta V_n(t), \tag{22}$$

while the quadrature one ($\zeta = s$) — to that of phase (provided a small deviation of the latter)

$$v_n^{(s)}(t) = V_n \Delta\varphi_n(t), \tag{23}$$

$$\Delta\varphi(t) = \int^t \Delta\omega'_{\rm rf}(t_1)\, dt_1, \quad \langle\Delta\varphi^2\rangle^{1/2} \ll 2\pi.$$

It follows hereof that indices $\zeta = c, s$ of the inphase and quadrature components of perturbation may well be substituted by indices $\zeta = a, \varphi$ of the amplitude- and phase noises of accelerating voltage, respectively. Anyway, it would have adequately reflected the physical nature of the effect inflicted by these perturbations.

Random Field Model

In a full accord with the afore-said let us study the diffusion of particles subjected to a random field

$$\Delta E(\Theta, t) = \frac{1}{L}\sum_{n=1}^{N}\sum_{\zeta} G_n(\Theta)\, v_n^{(\zeta)}(t)\cos(q'\omega_s t - \varphi_n^{(\zeta)}). \tag{24}$$

This model's generality resides in the following.

First of all, a particular meaning of Σ_ζ, features of random function $v_n^{(\zeta)}(t)$ and the value of (determinate) carrier phase $\varphi_n^{(\zeta)}$ would not be specified without

(21) is possible but is not necessary. However, in the latter case envelopes $v^{(\zeta)}(t)$ would be slowly varying functions in the time scale of $2\pi/\omega_s$.

Substitute eq.(64) into the formulae of Section 6. The use of model (86) reduces the sum over k to two items with $k = \pm q'$. Two series over m, j are thus retained whose members have the definite symmetry properties allowing one to resort to a single summation. By acting along the same lines as in deriving eq.(82) one gets

$$D^{(\zeta')}(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\rm ext}\sin\varphi_s}\right)^2 \sum_{j=-\infty}^{\infty} \mathcal{P}^{(\zeta')}(m\Omega_s(\mathcal{J}))\, \mathcal{W}_m^{(\zeta')}(\mathcal{J}; q'); \tag{87}$$

$$m = j, \qquad (\mathcal{J} \le \mathcal{J}_{\rm sep});$$

$$m = 2\frac{q'}{q}{\rm sign}\vartheta' + 2j, \quad \left(\mathcal{J} \ge \mathcal{J}_{\rm sep},\ M = q,\ |\varphi_s| = \frac{\pi}{2}\right).$$

Spectral power density $\mathcal{P}^{(\zeta')}(\Omega)$ of noise 'seen' by the beam is given here by

$$\mathcal{P}^{(\zeta')}(\Omega) = \binom{N^2|f_{q'}|^2}{N} p^{(\zeta')}(\Omega)\, |\widetilde{G}_{q'}|^2. \tag{88}$$

Both, the origin and the meaning of all the co-factors are obvious from the above consideration. What is essential, spectrum $\mathcal{P}^{(\zeta')}(\Omega)$ is no longer obliged to be an even function of frequency Ω. Weight functions $\mathcal{W}_m^{(\zeta')}(\mathcal{J}; q')$ acquire the simple form of

$$\mathcal{W}_m^{(\zeta')}(\mathcal{J}; q') = \frac{m^2 q^2}{q'^2}\left|I^*_{mq'}(\mathcal{J})\right|^2. \tag{89}$$

Introduction of a model of *band-pass white* noise $u(t)$ is possible. Its spectral power density $P^{(\zeta')}(\omega)$ is constant within a range of a few harmonics of the synchrotron frequency near $\omega = \pm q'\omega_s$, while having a bandwidth which does not exceed the revolution frequency $\omega_s \gg \Omega_s$. Hence, functions $\mathcal{P}^{(\zeta')}(\Omega)$ should now comply with eq.(83). Envelopes $v^{(\zeta)}(t)$ of such a noise, if these are introduced, would be fast functions in the time scale of $2\pi/\Omega_s$, and slowly varying ones in the scale of $2\pi/\omega_s$. The diffusion coefficient for the *band-pass white* noise $u(t)$ reads

$$D^{(\zeta')}(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\rm ext}\sin\varphi_s}\right)^2 \mathcal{P}^{(\zeta')}(0)\, \mathcal{A}^{(\zeta')}(\mathcal{J}), \tag{90}$$

$$\mathcal{A}^{(\zeta')}(\mathcal{J}) = \sum_{j=-\infty}^{\infty} \mathcal{W}_m^{(\zeta')}(\mathcal{J}; q') = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\psi \left(\frac{\partial q\vartheta(\mathcal{J}, \psi)}{\partial\psi}\right)^2. \tag{91}$$

Given $q' = q$ and $\varphi'_s = \varphi_s$, eqs.(79)–(84) grow into results of ref.[1]. Two in the role of an amplitude function for $q' \neq 0$ in eq.(85), as well, corresponds to the unbunched-beam limit which was discussed in ref.[1]. (In a particular case of a storage mode when $|\varphi_s| = \pi/2$, eqs.(79), (80) along with the diffusion coefficient are amenable to analytical calculations in terms of the elliptic functions.)

Hereof, the role of inequality (77) is revealed completely. It is the applicability range for the results of ref.[1] and, hence, for those of earlier refs.[2,3,4].

Quite obvious are the following two qualitative implications of eq.(76) which simplify the physical picture involved.

Firstly, there appears a possibility to treat the accelerating system as a single infinitely-narrow gap, spectral power density across which being given by eq.(81).

Secondly, one may come back to a model of a noise that is stationary w.r.t. the co-moving frame, in which model the beam is affected by perturbations of synchronous wave (14) whose pattern is $v^{(\zeta)}(t)\cos(q'\vartheta + \varphi^{(\zeta)})$. It follows from the fact that the results of the present Section were essentially obtained in ref.[1] in frames of that very approach. (Here $v^{(\zeta)}(t)$ is a random function of time given by its spectral power density (81). Phase $\varphi^{(\zeta)}$ is defined through eq.(63). On taking $q' = q$ and $\varphi'_s = \varphi_s$, one gets the former noise representation by eqs.(9), (10).)

As is clear from the afore-said, the withdrawal of conjecture (77) in favor of $\tau_v \sim 2\pi/\omega_s$ or $\tau_v \ll 2\pi/\omega_s$ — being essential for proton synchrotrons with the large orbit perimeters (UNK, LHC, SSC) — is not a trivial one at all. It is not only a quite expectable sum over the noises at revolution frequency harmonics that appears. Under certain conditions (of eq.(66)-type) an account should be taken of locations of the accelerating gaps around the ring.

High-Frequency Noise $u(t)$

Studies of stationary noise $u(t)$ with a low-frequency spectrum like that given by eq.(76) are not of a major interest. More instructive is a case of high-frequency noise $u(t)$ whose spectrum is localized near frequencies $\omega = \pm q'\omega_s$:

$$P^{(\zeta')}(k\omega_s + \Omega) = \frac{1}{2}\delta_{k,-q'}\, p^{(\zeta')}(-\Omega) + \frac{1}{2}\delta_{kq'}\, p^{(\zeta')}(\Omega). \tag{86}$$

Here, function $p^{(\zeta')}(\Omega)$ occupies a lower-frequency domain inside a bandwidth of $\Delta\omega_v \ll \omega_s$. Generally, $p^{(\zeta')}(\Omega) \neq p^{(\zeta')}(-\Omega)$. For definiteness, this noise is labeled by index $\zeta = \zeta'$. To a certain extent, it is a complementary counterpart of the case just studied. The transfer to representation of $u(t)$ given by eqs.(20),

necessity. Thereby we involve into the scope of eq.(24) any random-modulated oscillations with their carrier phases shifted w.r.t. the reference one:

$$u_n(t) = \sum_\zeta v_n^{(\zeta)}(t)\cos(q'\omega_s t - \varphi_n^{(\zeta)}), \quad \varphi_n^{(\zeta)} \neq \varphi_n. \tag{25}$$

Such signals may intentionally be driven by an external noise generator. These can readily be studied without a recourse to their decompositions given by eqs.(20), (21).

Secondly, the framework of eq.(24) embeds a particular case of stationary noise $u_n(t)$ as well. It can be approached not only through a trivial substitution $q' = 0$, $\sum_\zeta v_n^{(\zeta)}(t)\cos\varphi_n^{(\zeta)} = u_n(t)$. Outlined at the end of this Section is another route which is not so much suitable for the practical calculations as the former one though being essential in principle.

Let modulating voltages $v_n^{(\zeta)}(t)$, $\langle v_n^{(\zeta)}(t)\rangle = 0$ in eq.(24) be ('weakly') stationary and stationarily-correlated real-valued stochastic processes given in the laboratory frame. By definition, their cross-correlation functions do not depend upon the current instant of time t

$$\langle v_n^{(\zeta)}(t)\, v_{n_1}^{(\zeta_1)}(t-\tau)\rangle \equiv \langle v\, v\rangle_{nn_1}^{(\zeta\zeta_1)}(\tau) \tag{26}$$

and exhibit an obvious symmetry

$$\langle v\, v\rangle_{nn_1}^{(\zeta\zeta_1)}(-\tau) = \langle v\, v\rangle_{n_1 n}^{(\zeta_1\zeta)}(\tau) \tag{27}$$

with $n, n_1 = 1, 2, \ldots, N$. These define functional matrices of cross-correlations between the gap noises. (Each is dimensioned as $N \times N$. According to eq.(27), not all of the matrices are independent.) From the viewpoint of a correlative theory, matrices (26) entirely describe the statistical features of the external noise in the time domain.

The noise properties in the frequency domain are given by (complex) spectral matrices. Their elements, the (mutual) spectral power densities are the Fourier transforms of cross-correlations (26):

$$P_{nn_1}^{(\zeta\zeta_1)}(\omega) = \int_{-\infty}^{\infty} \langle v\, v\rangle_{nn_1}^{(\zeta\zeta_1)}(\tau)\exp(i\omega\tau)\, d\tau \tag{28}$$

with $n, n_1 = 1, 2, \ldots, N$. Due to real-valueness and symmetry (27) of the cross-correlations, matrix elements $P_{nn_1}^{(\zeta\zeta_1)}(\omega)$ acquire the property of

$$P_{nn_1}^{(\zeta\zeta_1)}(-\omega) = P_{nn_1}^{(\zeta\zeta_1)}(\omega)^* = P_{n_1 n}^{(\zeta_1\zeta)}(\omega). \tag{29}$$

Main-diagonal elements $P_{nn}^{(\zeta\zeta)}(\omega)$ are real-valued and symmetrical functions of frequency ω. These can be interpreted physically as (double-sided) spectral power densities of stationary noises $v_n^{(\zeta)}(t)$. Due to evenness of auto-correlation function $\langle v\,v\rangle_{nn}^{(\zeta\zeta)}(\tau) = \langle v\,v\rangle_{nn}^{(\zeta\zeta)}(-\tau)$, the double-sided Fourier cosine transformation, rather than that of eq.(28), is commonly applied to calculate $P_{nn}^{(\zeta\zeta)}(\omega)$.

Quite transparent are the physical reasons that force us to resort to eqs.(19), (20), (24) which are put down in the *laboratory* frame. Particular samples of random voltages $v_n^{(\zeta)}(t)$ are, by no means, obliged to be slowly varying functions in the time scale of $2\pi/\omega_s$. That is, the frequency spectrum of random-modulated oscillations $v_n^{(\zeta)}(t)\cos(q'\omega_s t - \varphi_n^{(\zeta)})$ may well contain harmonics of the revolution frequency other than q'. It is due to these harmonics that the new synchronous-to-beam propagating waves of voltage appear on the orbit. They affect the beam systematically at harmonic numbers other than q'. Their phases being uncorrelated, the new waves involved sum up incoherently, and do not build up any determinate spatial pattern (like the one given by function $g(\vartheta)$ from eq.(8)).

Of course, these effects can hardly be studied whenever the noises are introduced by eqs.(8), (9), (10) in the *co-moving* frame directly, in terms of synchronous wave perturbations (14). Such an approach allows one to treat stochastic processes that are slowly varying in the time scale of $2\pi/\omega_s$. The latter is apparent from general physical reasons, and is confirmed by the calculations which follow.

Stationary Noises $u_n(t)$ and $\Delta E(\Theta, t)$

Due to multiplication of stationary signals $v_n^{(\zeta)}(t)$ by harmonic carrier oscillations at frequency $q'\omega_s$ in eq.(24), stochastic processes $u_n(t)$ and $\Delta E(\Theta, t)$ are almost sure to be periodically-unstationary, their moments being periodic functions of t.

The only exception is the case when inphase and quadrature amplitudes $v_n^{(c,s)}(t)$ show quite definite statistical features, ref.[6]:

$$\langle v\,v\rangle_{nn_1}^{(cc)}(\tau) = \langle v\,v\rangle_{nn_1}^{(ss)}(\tau) = C_{nn_1}(\tau), \tag{30}$$

$$\langle v\,v\rangle_{nn_1}^{(cs)}(\tau) = -\langle v\,v\rangle_{nn_1}^{(sc)}(\tau) = -S_{nn_1}(\tau). \tag{31}$$

Or otherwise, in terms of the spectral matrices,

$$P_{nn_1}^{(cc)}(\omega) = P_{nn_1}^{(ss)}(\omega) = C_{nn_1}(\omega), \tag{32}$$

$$P_{nn_1}^{(cs)}(\omega) = -P_{nn_1}^{(sc)}(\omega) = -S_{nn_1}(\omega) \tag{33}$$

Let us put coefficients (N^2, N), along with $|\widetilde{G}_{q'}|^2$ from eq.(78), into the definition of effective spectral power density $\mathcal{P}^{(\zeta)}(\Omega)$ of noise $v^{(\zeta)}(t)$ as 'seen' by the beam:

$$\mathcal{P}^{(\zeta)}(\Omega) = \binom{N^2}{N} p^{(\zeta)}(\Omega)\,|\widetilde{G}_{q'}|^2, \qquad \mathcal{P}^{(\zeta)}(\Omega) = \mathcal{P}^{(\zeta)}(-\Omega). \tag{81}$$

Factor N^2 or N should again be taken according to the context as established by eqs.(66) or (67), respectively. Factor $|\widetilde{G}_{q'}|^2 \le 1$ accounts for the transit-time effects at frequency $q'\omega_s$.

As a result, eqs.(68), (43) and (50) convert into

$$D^{(\zeta)}(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\text{ext}}\sin\varphi_s}\right)^2 \sum_{j=-\infty}^{\infty} \mathcal{P}^{(\zeta)}(m\Omega_s(\mathcal{J}))\,\mathcal{W}_m^{(\zeta)}(\mathcal{J}); \tag{82}$$

$$m = j, \qquad (\mathcal{J} \le \mathcal{J}_{\text{sep}});$$

$$m = 2j, \quad \left(\mathcal{J} \ge \mathcal{J}_{\text{sep}},\ M = q,\ |\varphi_s| = \frac{\pi}{2}\right).$$

By adhering to constraints (76) and (72) simultaneously, one arrives at a model of the so-called *(low-pass) white* noise $v^{(\zeta)}(t)$. It has the spectral power density which is constant throughout a range of a few synchrotron frequency harmonics, its bandwidth, however, not exceeding the revolution frequency $\omega_s \gg \Omega_s$:

$$\mathcal{P}^{(\zeta)}(\Omega) \simeq \begin{cases} \mathcal{P}^{(\zeta)}(0), & |\Omega| \lesssim |m|\Omega_s; \\ 0, & |\Omega| \gtrsim \omega_s. \end{cases} \tag{83}$$

Realizations $v^{(\zeta)}(t)$ of such a noise are both, fast functions in the time scale of $2\pi/\Omega_s$ and slowly varying ones in the scale of $2\pi/\omega_s$. On inserting eqs.(76) and (83) into eq.(73), one gets the diffusion coefficient for the *(low-pass) white* noise:

$$D^{(\zeta)}(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\text{ext}}\sin\varphi_s}\right)^2 \mathcal{P}^{(\zeta)}(0)\,\mathcal{A}^{(\zeta)}(\mathcal{J}). \tag{84}$$

Eq.(75) yields for the unbunched-beam limit

$$D^{(\zeta)}(\mathcal{J} \gg \mathcal{J}_{\text{sep}}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\text{ext}}\sin\varphi_s}\right)^2 \mathcal{P}^{(\zeta)}(q'\vartheta'(\mathcal{J})) \times 2(1 + \delta_{q'0}\cos 2\varphi^{(\zeta)}). \tag{85}$$

(Notice, due to evenness of $\mathcal{P}^{(\zeta)}(\Omega)$, there is no dependence on the sign of ϑ' here.)

Possibility of further simplifications to yield the results of refs.[1,2,3,4] for the amplitude and phase noises of accelerating voltage emerges from a rather stringent conjecture on the spectrum shape of noise $v^{(\zeta)}(t)$.

7. Narrow-Band Noises

Low-Frequency Noise $v^{(\zeta)}(t)$

Let the spectrum of (modulation) noise $v^{(\zeta)}(t)$ be localized in a lower-frequency domain within a narrow bandwidth:

$$P^{(\zeta)}(k\omega_s + \Omega) = \delta_{k0}\, p^{(\zeta)}(\Omega), \tag{76}$$

where $p^{(\zeta)}(\Omega) = p^{(\zeta)}(-\Omega)$ is a low-frequency function whose bandwidth at base is subject to constraint $\Delta\omega_v \ll \omega_s$. Therefore, correlation time τ_v of noise $v^{(\zeta)}(t)$ should exceed the revolution period, $\tau_v \gg 2\pi/\omega_s$, which implies realizations $v^{(\zeta)}(t)$ to be slowly varying in the time scale of $2\pi/\omega_s$. Combine the latter inequality with the initial conjecture (6) where $\tau_{\Delta V} = \tau_v$ is taken. The result is eq.(12) which was discussed formerly in Section 1:

$$\frac{2\pi}{\omega_s} \ll \tau_v \ll \tau_{dif}. \tag{77}$$

Due to assumption (76), the only term with $k = 0$ should be retained in all the formulae of the preceding Section, which has the following consequences:

Firstly, $|f_0|^2 = 1$, and the diffusion coefficient under noise (66) fails to depend on gap coordinates Θ_n.

Secondly, it can be easily established that functions $\mathcal{W}^{(\zeta)}_{mk}(\mathcal{J})$ and $\mathcal{A}^{(\zeta)}_k(\mathcal{J})$ from eqs.(71), (74) factorize when $k = 0$,

$$\mathcal{W}^{(\zeta)}_{m0}(\mathcal{J}) = |\widetilde{G}_{q'}|^2 \cdot \mathcal{W}^{(\zeta)}_m(\mathcal{J}); \qquad \mathcal{A}^{(\zeta)}_0(\mathcal{J}) = |\widetilde{G}_{q'}|^2 \cdot \mathcal{A}^{(\zeta)}(\mathcal{J}). \tag{78}$$

Here, the following notations are used

$$\mathcal{W}^{(\zeta)}_m(\mathcal{J}) = \frac{m^2 q^2}{4q'^2}\left|I^*_{mq'}(\mathcal{J})\exp(+i\varphi^{(\zeta)}) - I^*_{m,-q'}(\mathcal{J})\exp(-i\varphi^{(\zeta)})\right|^2; \tag{79}$$

$$\begin{aligned}\mathcal{A}^{(\zeta)}(\mathcal{J}) &= \sum_{j=-\infty}^{\infty} \mathcal{W}^{(\zeta)}_m(\mathcal{J}) = \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\psi \left(\frac{\partial q\vartheta(\mathcal{J},\psi)}{\partial\psi}\right)^2 \cos^2(q'\vartheta(\mathcal{J},\psi) + \varphi^{(\zeta)}).\end{aligned} \tag{80}$$

where $C, S_{nn_1}(\omega) = C, S_{nn_1}(-\omega)^*$ are the Fourier transforms of the r.h.s. of eqs.(30), (31).

Under these conditions one gets noises $u_n(t)$ and $\Delta E(\Theta, t)$ which are *stationary* w.r.t. the laboratory frame. Really, multiply $u_n(t)$ by $u_{n_1}(t-\tau)$ both taken in the form of their expansions (20). Average the result statistically. Due to eqs.(30), (31), all the oscillatory (at frequency $2q'\omega_s$) functions of current time t would cancel each other, which would exclusively retain the dependence on τ. Cross-correlation function of gap voltages $u_n(t)$ converts into

$$\begin{aligned}\langle u_n(t)\, u_{n_1}(t-\tau)\rangle &= C_{nn_1}(\tau)\cos(q'\omega_s\tau - \Delta\varphi_{nn_1}) - \\ &- S_{nn_1}(\tau)\sin(q'\omega_s\tau - \Delta\varphi_{nn_1}), \\ \Delta\varphi_{nn_1} &= \varphi_n - \varphi_{n_1}.\end{aligned} \tag{34}$$

Which, in itself, is an evidence for the stationarity of $u_n(t)$ and $\Delta E(\Theta, t)$.

Like any other stationary noise, $u_n(t)$ can be described by its spectral matrix $P_{nn_1}(\omega)$. By applying the Fourier transformation to identity (34) one gets

$$\begin{aligned}P_{nn_1}(\omega) &= \frac{1}{2}\Big(C_{nn_1}(\omega + q'\omega_s)e^{-i\Delta\varphi_{nn_1}} + C_{nn_1}(\omega - q'\omega_s)e^{+i\Delta\varphi_{nn_1}}\Big) + \\ &+ \frac{i}{2}\Big(S_{nn_1}(\omega + q'\omega_s)e^{-i\Delta\varphi_{nn_1}} - S_{nn_1}(\omega - q'\omega_s)e^{+i\Delta\varphi_{nn_1}}\Big).\end{aligned} \tag{35}$$

Functions $C, S_{nn_1}(\tau)$ may be referred to as the inphase and quadrature amplitudes of expansion (34) of the cross-correlation. Thus, they are not independent at all. Indeed, by a mere inspection one concludes that eq.(35) is solvable for $C, S_{nn_1}(\omega)$:

$$C_{nn_1}(\omega) = \frac{1}{2}\Big(P_{nn_1}(\omega + q'\omega_s)e^{-i\Delta\varphi_{nn_1}} + P_{nn_1}(\omega - q'\omega_s)e^{+i\Delta\varphi_{nn_1}}\Big), \tag{36}$$

$$S_{nn_1}(\omega) = \frac{i}{2}\Big(P_{nn_1}(\omega + q'\omega_s)e^{-i\Delta\varphi_{nn_1}} - P_{nn_1}(\omega - q'\omega_s)e^{+i\Delta\varphi_{nn_1}}\Big). \tag{37}$$

In this route we express spectral matrices (32), (33) of random amplitudes $v^{(c,s)}_n(t)$ through spectrum $P_{nn_1}(\omega)$ of stationary noises $u_n(t)$ across the gaps. Now, they are voltages $u_n(t)$ which can be treated as the primary noise sources, while $v^{(\zeta)}_n(t)$ and their features by eqs.(30)–(33) turn out to be the secondary ones. The possibility itself of such an inversion is closely related to the aforesaid generality of decomposition by eqs.(20), (21). The latter is applicable to a stationary noise $u_n(t)$ as well.

However, in this case an admission of oscillation $\cos(q'\omega_s t - \varphi_n)$ to be the reference one is not predetermined physically. (Provided, of course, we are not

going to treat eqs.(20)–(23) along with eqs.(32), (33) as a way of expanding stationary noise $u_n(t)$ into sum over the amplitude and phase noises of accelerating voltage.) Any frequency from the spectrum of $u_n(t)$ can be taken as a reference one. Moreover, even such a choice is not obligatory if we are not insisting on envelopes $v_n^{(\zeta)}(t)$ to be slowly varying functions of time. Thus, the presence of parameters q' and φ_n seems to be redundant as only the final results are concerned. And it happens precisely this way ultimately.

Indeed, it was noticed earlier that there existed a more direct way to attack stationary noise $u_n(t)$. Eqs.(20), (21) show that this approximation can be approached, e.g., by putting

$$\zeta = c, s; \quad q' = \varphi_n = 0; \quad v_n^{(c)}(t) = u_n(t); \quad P_{nn}^{(cc)}(\omega) = P_{nn}(\omega) \tag{38}$$

into the ultimate formulae. From this viewpoint, the stationarity of noise $u_n(t)$ is but a particular manifestation of its periodical non-stationarity — with the relevant period tending to infinity.

From this it follows that the diffusion coefficient of particles in random field (24) would show quite an unexpected feature. Namely, the insertion of particular spectral matrices (32), (33), (36), (37), with q' and φ_n embraced explicitly, must yield the same result as the use of eq.(38) with $q' = \varphi_n = 0$. At a first glance, such a structure of the diffusion coefficient is not self-evident. Still, the expressions for $D(\mathcal{J})$ obtained later do exhibit such a behavior.

All this offers a key to a physical interpretation of dynamical effect inflicted by stationary noise $u_n(t)$ of accelerating voltage. It acts as a superposition of the inphase and quadrature — otherwise, of the amplitude and phase — noises $v_n^{(c,s)}(t)$ which have quite definite statistical features as given by eqs.(32), (33), (36), (37).

3. Diffusion inside Buckets

Random electric field (24) originates azimuthal (along ϑ) voltage harmonics whose amplitudes are

$$\Delta V_k(t) = \sum_{n=1}^{N} \sum_{\zeta} G_{n;k}\, v_n^{(\zeta)}(t)\, \exp(ik\omega_s t)\, \cos(q'\omega_s t - \varphi_n^{(\zeta)}). \tag{39}$$

Therefore, the cross-correlation functions entering diffusion coefficient (2) acquire the form of

This factor has an out-of-phase interference nature. It takes into account a possible incoherence of an effect exerted on beam by the spectral components of noise at frequencies $(k \pm q')\omega_s$, the reason of the incoherence residing in a non-optimal (for the latter frequencies) phase-lock of fields in the accelerating gaps.

Emergence of N^2 as of an (upper) factor in eq.(68) is quite understandable. Given eq.(66), a near-to-coherent summation $\sim N$ of the random voltages across the gaps occurs. Hence, spectral power density of the noise 'seen' by the beam may increase $\lesssim N^2$, which is contrasted to an increase $\sim N$ in case of eq.(67) for the uncorrelated random voltages.

Eq.(68) also embraces weight factors $\mathcal{W}_{mk}^{(\zeta)}(\mathcal{J})$ which equal

$$\mathcal{W}_{mk}^{(\zeta)}(\mathcal{J}) = \frac{m^2 q^2}{4} \left| \frac{I^*_{m,k+q'}(\mathcal{J})}{k+q'} \widetilde{G}_{k+q'} \exp(+i\varphi^{(\zeta)}) + \right. \tag{71}$$
$$\left. + \frac{I^*_{m,k-q'}(\mathcal{J})}{k-q'} \widetilde{G}_{k-q'} \exp(-i\varphi^{(\zeta)}) \right|^2.$$

Essentially, it is the absolute value squared of $\mathcal{V}_{n;mk}^{(\zeta)}(\mathcal{J})$ from eq.(44), the latter function being expressed in terms of gap form-function $\widetilde{G}(\Theta)$, eq.(60).

Given a noise with a slowly varying (in the frequency scale of $\Omega_s \ll \omega_s$) spectral power density

$$P^{(\zeta)}(k\omega_s + m\Omega_s(\mathcal{J})) \simeq P^{(\zeta)}(k\omega_s), \tag{72}$$

eqs.(47), (48) convert into

$$D^{(\zeta)}(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\rm ext} \sin\varphi_s}\right)^2 \sum_{k=-\infty}^{\infty} \binom{N^2|f_k|^2}{N} P^{(\zeta)}(k\omega_s)\, \mathcal{A}_k^{(\zeta)}(\mathcal{J}); \tag{73}$$

$$\mathcal{A}_k^{(\zeta)}(\mathcal{J}) = \sum_{j=-\infty}^{\infty} \mathcal{W}_{mk}^{(\zeta)}(\mathcal{J}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\psi \left(\frac{\partial q\vartheta(\mathcal{J},\psi)}{\partial\psi}\right)^2 \times \tag{74}$$
$$\times \frac{1}{4}\left|\widetilde{G}_{k+q'} \exp(+iq'\vartheta(\mathcal{J},\psi) + i\varphi^{(\zeta)}) + \widetilde{G}_{k-q'} \exp(-iq'\vartheta(\mathcal{J},\psi) - i\varphi^{(\zeta)})\right|^2.$$

Finally, in the unbunched-beam limit it follows from eq.(54) that

$$D^{(\zeta)}(\mathcal{J} \gg \mathcal{J}_{\rm sep}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\rm ext} \sin\varphi_s}\right)^2 \sum_{k=-\infty}^{\infty} \binom{N^2|f_k|^2}{N} \times \tag{75}$$
$$\times \Big[P^{(\zeta)}(k\omega(\mathcal{J}) + q'\vartheta'(\mathcal{J}))\, |\widetilde{G}_{k+q'}|^2 + P^{(\zeta)}(k\omega(\mathcal{J}) - q'\vartheta'(\mathcal{J}))\, |\widetilde{G}_{k-q'}|^2 +$$
$$+ P^{(\zeta)}(k\omega(\mathcal{J}))\, |\widetilde{G}_k|^2\, 2\delta_{q'0} \cos 2\varphi^{(\zeta)} \Big].$$

Statistical Interplay between Gaps

To simplify the matters, let the noise spectral power density be the same for each of N gaps. It is the level of cross-correlation between the random voltages across the gaps which is taken to be the only subject to distinction in extent. Two qualitatively opposite conjectures are possible which yield quite simple results. Namely,

1. Absolute gap-to-gap cross-correlation of noises:

$$P^{(\zeta)}_{nn_1}(\omega) = P^{(\zeta)}(\omega). \tag{66}$$

Such a noise source may be introduced by low-level driving and shaping circuits of the RF-system (a master oscillator, amplitude and phase modulators) which are shared by all accelerating gaps.

2. Complete statistical gap-to-gap independence (uncorrelation) of noises:

$$P^{(\zeta)}_{nn_1}(\omega) = P^{(\zeta)}(\omega)\,\delta_{nn_1}. \tag{67}$$

Such features may be appropriate to noises driven by the RF-power amplifiers which are coupled each to a certain accelerating gap.
Both these assumptions are studied concurrently.

6. Wide-Band Noise

Insert eq.(65) followed by eqs.(66) or (67) into eqs.(42), (44). Sum up over n to get

$$\begin{aligned} D^{(\zeta)}(\mathcal{J}) &= \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\rm ext}\sin\varphi_s}\right)^2 \sum_{k,j=-\infty}^{\infty}\binom{N^2|f_k|^2}{N} \times \\ &\times\ P^{(\zeta)}(k\omega_s + m\Omega_s(\mathcal{J}))\,\mathcal{W}^{(\zeta)}_{mk}(\mathcal{J}). \end{aligned} \tag{68}$$

Either of factors, $N^2|f_k|^2$ or N should be taken according to the assumption studied, the one given by eqs.(66) or (67), respectively.

Function $|f_k|^2$

$$0 \le |f_k|^2 = \left|\frac{1}{N}\sum_{n=1}^{N}\exp(-ik\Theta_n)\right|^2 \le 1, \tag{69}$$

depends upon locations of the gaps along the ring. For example, their regular layout $\Theta_n \sim \Delta\Theta\cdot n$ gives

$$|f_k|^2 = \frac{\sin^2(k\Delta\Theta N/2)}{N^2\sin^2(k\Delta\Theta/2)}. \tag{70}$$

$$\begin{aligned} \langle \Delta V_k(t)\,\Delta V^*_{k_1}(t-\tau)\rangle &= \sum_{n,n_1=1}^{N}\sum_{\zeta,\zeta_1} G_{n;k}G^*_{n_1;k_1}\langle v\,v\rangle^{(\zeta\zeta_1)}_{nn_1}(\tau) \times \\ &\times\ \cos(q'\omega_s t - \varphi^{(\zeta)}_n)\ \cos(q'\omega_s(t-\tau) - \varphi^{(\zeta_1)}_{n_1}) \times \\ &\times\ \exp(i(k-k_1)\omega_s t)\ \exp(ik_1\omega_s\tau). \end{aligned} \tag{40}$$

Thereby, a periodically-unstationary stochastic process is encountered in the co-moving frame, as well. Fortunately, being time-averaged over a period of $2\pi/\omega_s$, its correlations show a property of the moments pertaining to a stationary stochastic process. Perform such an averaging in eq.(40) to get

$$\begin{aligned} \frac{\omega_s}{2\pi}\int_t^{t+\frac{2\pi}{\omega_s}} dt_1\,\langle \Delta V_k(t_1)\,\Delta V^*_{k_1}(t_1-\tau)\rangle &= \sum_{n,n_1=1}^{N}\sum_{\zeta,\zeta_1}\frac{1}{4}G_{n;k}G^*_{n_1;k_1} \times \\ \times\ \langle v\,v\rangle^{(\zeta\zeta_1)}_{nn_1}(\tau)\ \Big[\delta_{k_1k}\ &\exp(i(k+q')\omega_s\tau\ -\ i(\varphi^{(\zeta)}_n - \varphi^{(\zeta_1)}_{n_1})) + \\ +\delta_{k_1,k+2q'}\ &\exp(i(k+q')\omega_s\tau\ -\ i(\varphi^{(\zeta)}_n + \varphi^{(\zeta_1)}_{n_1})) + \\ &+\ldots\ (q'\to -q',\ \varphi\to-\varphi)\Big] \end{aligned} \tag{41}$$

with δ_{k_1k} denoting the Kronecker's delta-symbol. Non-vanishing contribution to sum (41) is borne by those of its items which comply with the conditions of frequency resonance: $(k-k_1)\omega_s = 0, \pm 2q'\omega_s$.

Insert eq.(41) into eq.(2). Straightforward calculations, unwieldy though they are, provide the diffusion coefficient (or, to be more precise — its systematic part $D(\mathcal{J})$) in the form of

$$\begin{aligned} D(\mathcal{J}) &= \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\rm ext}\sin\varphi_s}\right)^2 \times \\ &\times\ \sum_{n,n_1=1}^{N}\sum_{\zeta,\zeta_1}\sum_{k,j=-\infty}^{\infty} P^{(\zeta\zeta_1)}_{nn_1}(k\omega_s + m\Omega_s(\mathcal{J}))\ \mathcal{V}^{(\zeta)}_{n;mk}(\mathcal{J})\,\mathcal{V}^{(\zeta_1)*}_{n_1;mk}(\mathcal{J}). \end{aligned} \tag{42}$$

Inside separatrix, the label m of multipole coincides with the summation index j:

$$m = j, \qquad (\mathcal{J} \le \mathcal{J}_{\rm sep}). \tag{43}$$

Such a notation is employed intentionally. Its convenience will be confirmed later, refer to eq.(50).

Factors $\mathcal{V}^{(\zeta)}_{n;mk}(\mathcal{J})$ specify the efficiency of the bunch perturbation excitation at the m-th multipole as caused by spectral components of noise $v^{(\zeta)}_n(t)$ at

frequency $\omega \simeq k\omega_s$:

$$\mathcal{V}^{(\zeta)}_{n;mk}(\mathcal{J}) = \frac{mq}{2}\left(\frac{I^*_{m,k+q'}(\mathcal{J})}{k+q'}G_{n;k+q'}\exp(+i\varphi^{(\zeta)}_n) + \right. \tag{44}$$
$$\left. + \frac{I^*_{m,k-q'}(\mathcal{J})}{k-q'}G_{n;k-q'}\exp(-i\varphi^{(\zeta)}_n)\right).$$

These functions depend on both, the carrier frequency $q'\omega_s$ and its phase $\varphi^{(\zeta)}_n$ w.r.t. the bunched beam. The presence of expansion coefficients $I^*_{m,k\pm q'}$ which decompose plane waves $\sim \exp(i(k\pm q')\vartheta)$ into series over multipoles $\sim \exp(im\psi)$ is quite explainable. Multiplication of signal $v^{(\zeta)}_n(t)$ by high-frequency harmonic oscillation $\cos(q'\omega_s t - \varphi^{(\zeta)}_n)$ translates spectral components of voltage $v^{(\zeta)}_n(t)$ from frequency $\omega \simeq k\omega_s$ into a region of (higher) frequencies $\omega \simeq (k \pm q')\omega_s$. From which these as such affect the beam by driving the multipole oscillations of the bunches. Quantities $G_{n;k\pm q'}$ are the transit-time factors at the relevant frequencies (of interaction).

Apart from noise correlation time $\tau_v \sim \tau_{\Delta V}$, beam revolution period $2\pi/\omega_s$ and time constant τ_{dif} of the bunch diffusion (refer to Section 1), the problem in question has one more representative time scale which is imposed by unperturbed motion of particles in the external accelerating field. It is the period of synchrotron oscillations $2\pi/\Omega_s$ with $\Omega_s \sim \Omega_0$ denoting some typical value of synchrotron frequency. So far no restrictions were imposed on a relationship between τ_v and $2\pi/\Omega_s$. Nevertheless, a quite universal way to simplify general eq.(42) may follow from this.

Let realizations of noise $v^{(\zeta)}_n(t)$ be fast functions in the time scale of the synchrotron oscillation period:

$$\tau_v \ll \frac{2\pi}{\Omega_s}. \tag{45}$$

(Commonly $\Omega_s \ll \omega_s$, and the above assumption may turn actual for wide-band noise (13).)

In which case $P^{(\zeta\zeta_1)}_{nn_1}(\omega)$ should vary insignificantly throughout any interval of frequency $\omega \sim \Omega_s \ll \omega_s$. Thus, take

$$P^{(\zeta\zeta_1)}_{nn_1}(k\omega_s + m\Omega_s(\mathcal{J})) \simeq P^{(\zeta\zeta_1)}_{nn_1}(k\omega_s) \tag{46}$$

and factor this quantity out of the sum over m, j in eq.(42). The series of the weight functions thus retained is summable (reducible to quadratures). The result is the diffusion coefficient:

$$D(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\text{ext}}\sin\varphi_s}\right)^2 \sum_{n,n_1=1}^{N}\sum_{\zeta,\zeta_1}\sum_{k=-\infty}^{\infty} P^{(\zeta\zeta_1)}_{nn_1}(k\omega_s)\,\mathcal{A}^{(\zeta\zeta_1)}_{nn_1;k}(\mathcal{J}), \tag{47}$$

To study noise $u_n(t)$ which is stationary w.r.t. the laboratory frame take

$$q' = 0, \qquad \varphi^{(\zeta)}_n = \varphi^{(\zeta)}\,(=0). \tag{64}$$

(Identity $\varphi^{(\zeta)} = 0$ is not an essential one. Its dismissal would only alter instantaneous values of noise $u_n(t) \sim \cos\varphi^{(\zeta)}$, and its spectral intensity $\sim \cos^2\varphi^{(\zeta)}$.)

Uncorrelated Noises $\zeta = a, \varphi$

Let us follow the interpretation of the inphase and quadrature components of the random perturbation as offered by eqs.(21), (22), (23). Replace the noise type indices $\zeta = c, s$ by those $\zeta = a, \varphi$ attributed to the amplitude and phase noises of accelerating voltage. Consider noises $\zeta = a, \varphi$ to be statistically independent (or, to be more correct, uncorrelated):

$$P^{(\zeta\zeta_1)}_{nn_1}(\omega) = P^{(\zeta)}_{nn_1}(\omega)\,\delta_{\zeta\zeta_1}. \tag{65}$$

Thus, components $\zeta = a, \varphi$ would enter $D(\mathcal{J})$ additively: $D = \sum_{\zeta=a,\varphi} D^{(\zeta)}$. The objective of calculations is a component $D^{(\zeta)}(\mathcal{J})$. It stands for a noise of type $\zeta = a, \varphi$ depending on a particular phase option in eq.(63). (Subsequent summation over $\zeta = a, \varphi$ is implied, but is not commonly carried out.) The expression for $D^{(\zeta)}(\mathcal{J})$ copes with a transition to stationary noise $u_n(t)$ as well. It can be performed via substitution (64).

Rigorously speaking, assumption (65) is a concession to a customary approach of refs.[2,3,4]. To justify it, one might recall that the beam dynamics still remains the subject matter under study. Noises $\zeta = a, \varphi$ of the accelerating voltage do occur in quadrature, and build up a basic ('mutually-orthogonal') set of the external perturbations. To a certain extent, beam responses to these two types of perturbation diverge to the utmost in their features. It is for this very reason that these are themselves of interest.

As from a practical standpoint, the situation seems to be somewhat different. A question arises regarding technical feasibility of eq.(65). A more natural and technically imposed system of basic perturbations appears to be that of the amplitude and phase noises of the forward wave of current in the feeder of accelerating gap. Their cross-uncorrelated performance, being quite plausible practically, does not necessarily results in eq.(65). The latter is an exclusion rather than a rule. Expressions for $D(\mathcal{J})$ in terms of these random currents are obtained in the Appendix.

Accelerating System Model

For definiteness, let the co-rotating frame be introduced as $\vartheta = \Theta - \omega_s t$ (i.e. the reference particle $\vartheta = 0$ of the bunch in question traverses the origin $\Theta = 0$ of the generalized azimuth at the moment of time $t = 0$, and after each revolution period of $2\pi/\omega_s$ further on). Substitute decomposition (16) into eq.(15) and transfer to the co-moving frame. The required form (14) of the synchronous wave can be obtained by, say, the following choice of amplitudes V_n and phases φ_n of the gap voltages:

$$\sum_{n=1}^{N} V_n |G_{n;q'}| = V'_{\text{ext}}, \tag{58}$$

$$\varphi_n = \varphi'_s - \arg G_{n;q'} \pmod{2\pi}. \tag{59}$$

The last of these expressions specifies a way of phase-locking of fields in the gaps, which depends on their locations along the ring. While the first one entails that the lowest level of RF-power is consumed to drive the required amplitude V'_{ext} of the accelerating voltage: the RF-power to feed the n-th gap is $\sim V_n^2$. It is the last argument that makes conditions (58), (59) optimal, and these are commonly realized in practice.

Consider the field geometry to be identical in all the accelerating gaps. (In many cases it does not conflict with the reality.) Then, the entire distinction between functions $G_n(\Theta)$, eq.(16) for the different gaps is reduced to an azimuthal shift of their common form-function $\widetilde{G}(\Theta)$:

$$G_n(\Theta) = \widetilde{G}(\Theta - \Theta_n), \tag{60}$$

where Θ_n is the coordinate of, say, the n-th gap center. Hereof, harmonics $G_{n;k}$ (17) can be presented in the form of

$$G_{n;k} = \widetilde{G}_k \exp(-ik\Theta_n). \tag{61}$$

Without a loss of generality, from now on we require coefficients $\widetilde{G}_{\pm q'}$ to be real-valued, i.e., having $\arg \widetilde{G}_{\pm q'} = 0$.

Insert eq.(61) into phase-lock identity (59). Then, by applying to definitions (21), one determines carrier phases $\varphi_n^{(\zeta)}$ for both, the inphase and quadrature noises $v_n^{(\zeta)}(t)$:

$$q' \neq 0, \qquad \varphi_n^{(\zeta)} = \varphi^{(\zeta)} + q'\Theta_n, \tag{62}$$

$$\varphi^{(c)} = \varphi'_s, \qquad \varphi^{(s)} = \varphi'_s - \frac{\pi}{2}. \tag{63}$$

where $\mathcal{A}_{nn_1;k}^{(\zeta\zeta_1)}(\mathcal{J})$ is the equation symbol for amplitude function:

$$\mathcal{A}_{nn_1;k}^{(\zeta\zeta_1)}(\mathcal{J}) = \sum_{j=-\infty}^{\infty} \mathcal{V}_{n;mk}^{(\zeta)}(\mathcal{J})\, \mathcal{V}_{n_1;mk}^{(\zeta_1)*}(\mathcal{J}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\psi \left(\frac{\partial q\vartheta(\mathcal{J},\psi)}{\partial\psi}\right)^2 \times \tag{48}$$
$$\times \frac{1}{4}\Big[G_{n;k+q'} \exp(+iq'\vartheta(\mathcal{J},\psi) + i\varphi_n^{(\zeta)}) + G_{n;k-q'} \exp(-iq'\vartheta(\mathcal{J},\psi) - i\varphi_n^{(\zeta)})\Big] \times$$
$$\times \Big[\ \ldots \quad (n \to n_1, \zeta \to \zeta_1)\Big]^*.$$

4. Diffusion beyond Stationary Buckets

Let us remind the principal guide-lines to extend the diffusion equation to beam halo region, ref.[1]. Only a storage operation mode is involved, i.e., $|\varphi_s| = \pi/2$. Angle-action variables $(\tilde{\psi}, \widetilde{\mathcal{J}})$ of finite orbital motion of untrapped particle are introduced. Choice of $q'/q = 1, 2, \ldots$ results in coincidence of the diffusion coefficients inside all the beam buckets. Therefore, on injecting $M = q$ identical bunches, the beam halo would be built up uniformly w.r.t. phase $\tilde{\psi}$. (In other words, the *random phase type of approximation* for the orbital motion is realized.) Consequently, particle dynamics in the halo region may be ultimately reduced to a single closed equation — the diffusion one along coordinate $\widetilde{\mathcal{J}}$.

Further, periodicity (with a period of the main RF) of the orbital motion allows one to transform $(\tilde{\psi}, \widetilde{\mathcal{J}})$ to scaled variables $(\psi, \mathcal{J})$. The latter are defined within the one RF-period, near the separatrix at issue, and manifest the following behavior:

1. At $\mathcal{J} = \mathcal{J}_{\text{sep}}$ go over continuously into ψ, $\mathcal{J}$ and $d\psi/dt = \Omega_s(\mathcal{J})$ of the synchrotron oscillations of trapped particles.

2. In case of $\mathcal{J} \gg \mathcal{J}_{\text{sep}}$ (unbunched-beam approximation) passage is performed to the limit:

$$\psi \to \frac{q\vartheta}{2}\text{sign}\vartheta', \qquad \mathcal{J} \to \frac{2|\vartheta'|}{q}, \qquad \Omega_s(\mathcal{J}) \to \frac{q|\vartheta'|}{2}. \tag{49}$$

As a result, the diffusion coefficient along coordinate $\mathcal{J}$, $\mathcal{J} \geq \mathcal{J}_{\text{sep}}$ is obtained which resembles outwardly eqs.(42), (44) or (47), (48). All the formal difference is reducible to the following three modifications:

1. Instead of eq.(43), use should be made of an analogue for the multipole index m beyond separatrix:

$$m = 2\frac{k}{q}\text{sign}\vartheta' + 2j, \quad \left(\mathcal{J} \geq \mathcal{J}_{\text{sep}},\ M = q,\ |\varphi_s| = \frac{\pi}{2}\right). \tag{50}$$

2. Functions $I^*_{mk}(\mathcal{J})$ inside weight factors (44) grow into a continuation of eq.(3) beyond separatrix:

$$I^*_{mk}(\mathcal{J}) = \frac{1}{\pi}\int_0^{\pi} d\psi \, \exp(ik\vartheta(\mathcal{J},\psi) - im\psi), \tag{51}$$

$$\vartheta(\mathcal{J},\psi+\pi) = \vartheta(\mathcal{J},\psi) + \frac{2\pi}{q}\mathrm{sign}\vartheta', \qquad \mathcal{J} \geq \mathcal{J}_{\mathrm{sep}}.$$

3. Amplitude factors (48) get the new limits of integration over phase:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} d\psi \to \frac{1}{\pi}\int_0^{+\pi} d\psi. \tag{52}$$

Up to these reservations, we are not going to distinguish between cases $\mathcal{J} \leq \mathcal{J}_{\mathrm{sep}}$ and $\mathcal{J} \geq \mathcal{J}_{\mathrm{sep}}$, $M = q$, $|\varphi_s| = \pi/2$ altogether. (Naturally, save the unbunched-beam limit to follow.)

Unbunched-Beam Limit

Wide-band performance (in the frequency scale of ω_s) of noise $v_n^{(\zeta)}(t)$ inflicts, though weakly so, an asymmetry to diffusion coefficient $D(\mathcal{J})$ as $\mathcal{J} \geq \mathcal{J}_{\mathrm{sep}}$. Now it starts to depend on where the particles move, in the upper ($\vartheta' > 0$) or in the lower ($\vartheta' < 0$) half-planes of phase-plane (ϑ, ϑ'). (Such a possibility can first be noticed as early as from eq.(50).)

The most apparent illustration to this effect can be gained by passing to the limit (49) of unbunched beam in which case functions (51) do degenerate:

$$I^*_{mk}(\mathcal{J}) \to \delta_{m,2\frac{k}{q}\mathrm{sign}\vartheta'}. \tag{53}$$

Substitution of these values into eqs.(42), (44) yields

$$D(\mathcal{J} \gg \mathcal{J}_{\mathrm{sep}}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\mathrm{ext}}\sin\varphi_s}\right)^2 \sum_{n,n_1=1}^{N}\sum_{\zeta,\zeta_1}\sum_{k=-\infty}^{\infty} \times \tag{54}$$

$$\times \Big[P^{(\zeta\zeta_1)}_{nn_1}(k\omega(\mathcal{J}) + q'\vartheta'(\mathcal{J}))\, G_{n;k+q'}G^*_{n_1;k+q'} \exp(+i(\varphi_n^{(\zeta)} - \varphi_{n_1}^{(\zeta_1)})) +$$

$$+ P^{(\zeta\zeta_1)}_{nn_1}(k\omega(\mathcal{J}) - q'\vartheta'(\mathcal{J}))\, G_{n;k-q'}G^*_{n_1;k-q'} \exp(-i(\varphi_n^{(\zeta)} - \varphi_{n_1}^{(\zeta_1)})) +$$

$$+ P^{(\zeta\zeta_1)}_{nn_1}(k\omega(\mathcal{J}))\, G_{n;k}G^*_{n_1;k}\, 2\delta_{q'0}\cos(\varphi_n^{(\zeta)} + \varphi_{n_1}^{(\zeta_1)}) \Big],$$

where $\omega(\mathcal{J})$ denotes the angular velocity of the off-set particles:

$$\omega(\mathcal{J}) = \omega_s + \vartheta'(\mathcal{J}); \qquad \vartheta'(\mathcal{J}) = \frac{q}{2}\mathcal{J}\mathrm{sign}\vartheta'. \tag{55}$$

Given $q' \neq 0$, eq.(54) retains its dependence on carrier-phase differences $(\varphi_n^{(\zeta)} - \varphi_{n_1}^{(\zeta_1)})$ only, i.e., on the cross-lock in phase of fields across the accelerating gaps. Their phase-lock w.r.t. an unbunched beam is inessential (and, moreover, is not well defined at all).

Given $q' = 0$, one goes over to a particular case of noise $u_n(t)$ that is stationary w.r.t. the laboratory frame, in which case eq.(54) is subject to a noticeable simplification:

$$D(\mathcal{J} \gg \mathcal{J}_{\mathrm{sep}}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\mathrm{ext}}\sin\varphi_s}\right)^2 \sum_{n,n_1=1}^{N}\sum_{\zeta,\zeta_1}\sum_{k=-\infty}^{\infty} \times \tag{56}$$

$$\times\ 4\, P^{(\zeta\zeta_1)}_{nn_1}(k\omega(\mathcal{J}))\, G_{n;k}G^*_{n_1;k} \cos\varphi_n^{(\zeta)} \cos\varphi_{n_1}^{(\zeta_1)}.$$

Emergence of factor $\cos\varphi_n^{(\zeta)}\cos\varphi_{n_1}^{(\zeta_1)}$ is quite explainable. It takes into account that voltage $u_n(t) = \sum_\zeta v_n^{(\zeta)}(t)\cos\varphi_n^{(\zeta)}$ is, in fact, applied to the gap as $q' = 0$. While spectral matrix $P^{(\zeta\zeta_1)}_{nn_1}(\omega)$ is related to amplitudes $v_n^{(\zeta)}(t)$ solely.

For example, consider one gap ($n = N = 1$) and a single stationary noise source (say, $\zeta = 1$):

$$P^{(\zeta\zeta_1)}_{nn_1}(\omega) = P^{(\zeta)}_n(\omega)\,\delta_{nn_1}\delta_{\zeta\zeta_1}, \quad \varphi_n^{(\zeta)} = 0.$$

As a result get from eq.(56)

$$D(\mathcal{J} \gg \mathcal{J}_{\mathrm{sep}}) = 2\left(\frac{\Omega_0^2}{q^2 V_{\mathrm{ext}}\sin\varphi_s}\right)^2 \sum_{k=-\infty}^{\infty} P^{(\zeta)}_n(k\omega(\mathcal{J}))\,|G_{n;k}|^2. \tag{57}$$

Convert $\mathcal{J}$ into a new independent variable $\Delta p\, L/2\pi$ with Δp being the deviation of momentum from the reference one. Consequently, find out the above expression to coincide with the diffusion coefficient which was obtained in ref.[5] in treatment of the so-called stochastic acceleration technique.

5. Simplifications

General expressions for $D(\mathcal{J})$ are too complicated. More particular results are of practical interest. Major possibility of getting them is to involve a few physically justifiable assumptions on the structure of cross-correlation matrices (26) or, equivalently, of spectral matrices (28). A simplified model of the accelerating system is required, as well.

Let us introduce these simplifications one-by-one with all the suitable comments accompanying.