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LONGITUDINAL DIFFUSION OF A PROTON BUNCH UNDER EXTERNAL NOISE

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Abstract

S.V. Ivanov. Longitudinal Diffusion of a Proton Bunch under External Noise: IHEP Preprint 92-43. – Protvino, 1992. – p. 38, figs. 11, refs.: 6.

Evolution of longitudinal distribution of a proton bunch subjected to stationary (amplitude or phase) RF-noises is governed by a diffusion equation. Its diffusion coefficient is essentially nonlinear and, possibly, diverges near separatrix. The paper deals with the dynamical foundations of this diffusive approximation. Treated in detail is the motion of particles located either inside, or outside stationary buckets (beam halo). The formal statement of boundary-value problem for the noise-induced diffusion of a bunch (zero boundary conditions at separatrix, or at infinity) is discussed. Both these problems are solved numerically for arbitrarily long bunches. Use is made of the Finite Element Technique (spatial discretization), and of the Crank-Nicolson's scheme (time-domain integration). Computed estimates of the representative bunch life-times for the wide-band (*white*) noise approximation are presented. These emerge from the criteria of either bunch quality degradation, or of its population loss.

Аннотация

С.В. Иванов. Продольная диффузия сгустка протонов под действием внешнего шума: Препринт ИФВЭ 92-43. – Протвино, 1992. – 38 с., 11 рис., библиогр.: 6.

Эволюция продольного распределения сгустка протонов под действием стационарного (амплитудного или фазового) ВЧ-шума описывается уравнением диффузии. Его коэффициент диффузии существенно нелинеен и, возможно, расходится вблизи сепаратрисы. В работе проводится динамическое обоснование этого диффузионного приближения. Подробно исследуется движение частиц как внутри, так и вне стационарных сепаратрис (гало пучка). Обсуждается формальная постановка краевой задачи шумовой диффузии сгустка (нулевые граничные условия на сепаратрисе либо на бесконечности). Обе эти задачи решаются численно для сгустков произвольной длины. Используется метод конечных элементов (пространственная дискретизация) и схема Кранка-Николсона (интегрирование по времени). Приведены численные оценки характерных времён жизни сгустка в приближении широкополосного (*белого*) шума. Они определяются исходя из критериев ухудшения качества сгустка либо потерь его интенсивности.

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Иванов С.В.

Продольная диффузия сгустка протонов под действием внешнего шума.

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INTRODUCTION

Studies of bunched-beam dynamics under RF-noise in a proton synchrotron were initiated long ago. Diffusion equation for the longitudinal distribution of a bunch was obtained in refs.[1,2]. A more rigorous, theoretical-physical derivation of the result was published in ref.[3].

All the papers mentioned arrive at diffusion equation within the frames of a probabilistic approach. The latter proceeds from the Fokker-Planck equation for the probability of a particle, affected by an external perturbation, to transit from its phase-plane trajectory to another one. The subject to calculations are the coefficients of this equation. These are expressed through a linear in Δt component of time dependence of both the first ($\langle\Delta\psi\rangle$; $\langle\Delta\mathcal{J}\rangle$) and the second ($\langle\Delta\psi^2\rangle$; $\langle\Delta\psi\Delta\mathcal{J}\rangle$; $\langle\Delta\mathcal{J}^2\rangle$) moments of a change in phase-plane coordinates of a particle during time intervals $\Delta t \to 0$. (Here, for definiteness, angle-action variables $(\psi, \mathcal{J})$ are used. The angular brackets $\langle\ldots\rangle$ denote a statistical average over the noise ensemble. Mathematical averaging over phase ψ reduces the problem in question to a one-dimensional one.)

Virtually, such an approach 'a priory' incorporates a few qualitative suggestions. Namely, it assumes:

1. Absence of any probabilistic after-effect (the so-called Markovian random process).
2. Certain type of behavior of the transition probability during vanishingly small time intervals: the third and higher-order moments of phase-plane displacement of a particle decrease faster than the first power of Δt.

This approach is not free of its intrinsic limitations. It circumvents the analysis of a particular relation of the dynamical properties of the system in

question to statistical features of an external perturbation. Whereof, an uncertainty arises regarding dynamical foundation and physical prerequisites for applicability of the diffusion approximation. Appreciable efforts are as well required to prove an exact compensation of the systematic-transfer velocity of particles along $\mathcal{J}$-coordinate due to the diffusion coefficient gradient (identity $A_1 = \frac{1}{2}\partial A_2/\partial\mathcal{J}$ in refs.[1]–[3]). It is this very condition which reduces the Fokker-Planck equation to a standard diffusion one.

This paper derives the longitudinal diffusion equation for a bunch under RF-noise in a quite different way — in frames of a consistent dynamical approach. Similar techniques is employed, say, in ref.[4] to study a Brownian motion. To put it briefly, the main idea of this approach is as follows. First, the dynamics of particles in a bunch under the sole effect of determinate, i.e. not random, perturbations is studied with the Vlasov equation. It is at a concluding step only that stochastic processes themselves are incorporated, and proper statistical averages are taken.

This approach allows one to apply the formal techniques already developed in the theory of bunched-beam instabilities. The general results of ref.[3] are rederived in a more transparent way, these being valid for an arbitrary azimuthal pattern of a perturbing-voltage wave synchronous to the beam. Hence, the results of [1,2] for a beam stored inside stationary buckets (amplitude and phase noises) are also obtained. Methodologically, this alternative derivation is itself of interest, and facilitates a deeper physical insight into the problem at issue. The approach involved can be readily extended to treat the motion of particles beyond separatrices.

However, refs.[1]–[3] seem to skip over a detailed solution of the diffusion problem. Indeed, the approximation of short bunches turns out to be the only one studied practically. Therefore, a highly nonlinear dependence of the diffusion coefficient on oscillation amplitudes is not taken into full account, which might be crucial for some near-separatrix region. The motion of particles which have escaped from (stationary) buckets is not considered. These off-set particles build up a beam halo whose presence must be taken into account in putting boundary conditions for the noise-induced diffusion problem in a storage regime. No quantitative data on life-times of arbitrarily long bunches are presented. The paper attempts to fill in the blanks mentioned.

1. INTRODUCTIVE REMARKS

Let $\vartheta = \Theta - \omega_s t$ be an azimuth in the co-moving coordinate frame. Here Θ is a generalized azimuth along the ring in the laboratory frame; ω_s is the angular

References

[1] Boussard D., Dôme G., Graziani C. *The Influence of RF Noise on the Lifetime of Bunched Proton Beams.* — Proceed. of 11-th Internat. Conf. on High-Energy Accel., CERN, Geneva, 1980, pp. 620–626.

[2] Dôme G. *Theory of RF Acceleration and RF Noise.* — CAS Proceed.: "Antiprotons for Colliding Beam Facilities", CERN/84–15, 1984, pp. 215-260; Preprint CERN/SPS/84–14 (ARF), Geneva, 1984.

[3] Krinsky S., Wang J.M. *Bunch Diffusion due to RF Noise.* — Part. Accel., 1982, Vol. 12, pp. 107-117.

[4] Klimontovich Yu.L. *Statistical Physics.* — Moscow, Nauka, 1982 (in Russian).

[5] Kolomensky A.A., Lebedev A.N. *Theory of Cyclic Accelerators.* — Moscow, Physmathgiz, 1962 (in Russian).

[6] Zienkiewicz O.C., Morgan K. *Finite Elements and Approximation* — New York, Chichester, etc., A Wiley-Interscience Publication, John Wiley & Sons, Inc., 1983.

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revolution frequency of a reference particle which is synchronous to the nominal value of the RF-frequency; t is the time. Put the origin $\vartheta = 0$ of longitudinal ϑ-coordinate on the unperturbed reference particle of the bunch in question. Neglect beam-induced self-fields.

Consider the pair $(\vartheta; \vartheta' \equiv d\vartheta/dt)$ as the canonical coordinate and momentum. First, let us study dynamics of particles under effect of determinate, i.e. not random, fields exclusively. The equations of longitudinal motion can be put down in their canonical form:

$$\frac{d\vartheta}{dt} = \frac{\partial \mathcal{H}}{\partial \vartheta'}; \qquad \frac{d\vartheta'}{dt} = -\frac{\partial \mathcal{H}}{\partial \vartheta}. \tag{1}$$

Hamiltonian $\mathcal{H}(\vartheta, \vartheta', t)$ is a sum of two items:

$$\mathcal{H}(\vartheta, \vartheta', t) = H(\vartheta, \vartheta') + \Delta H(\vartheta, t). \tag{2}$$

Here $H(\vartheta, \vartheta')$ is Hamiltonian of the unperturbed motion

$$H(\vartheta, \vartheta') = \Omega_0^2 U(\vartheta) + \frac{\vartheta'^2}{2}, \tag{3}$$

$$U(\vartheta) = \frac{1}{q^2}((1 - \cos q\vartheta) + (q\vartheta - \sin q\vartheta)\cot\varphi_s) \overset{\vartheta \to 0}{\to} \frac{\vartheta^2}{2}, \tag{4}$$

where Ω_0 is the small-amplitude angular frequency of synchrotron oscillations; $U(\vartheta)$ is the scaled potential energy; q is the RF harmonic number; φ_s is the nominal value of stable-phase angle ($\varphi_s > 0$ below transition; synchronous energy gain varies as $\cos\varphi_s$). However, the results of Sections 1, 2 are quite general and do not depend on the particular shape of $U(\vartheta)$.

Time-dependent additive term ΔH reads

$$\Delta H(\vartheta, t) = -\frac{\Omega_0^2}{qV_{\text{ext}} \sin\varphi_s} \int^{\vartheta} \Delta V(\vartheta_1, t)\, d\vartheta_1, \tag{5}$$

where V_{ext} is the nominal value of accelerating voltage amplitude; $\Delta V(\vartheta, t) = \Delta V(\vartheta + 2\pi, t)$ is a perturbing voltage reduced to the co-rotating frame. The explicit form of $\Delta V(\vartheta, t)$-term is specified later on. For the time being, $\Delta V(\vartheta, t)$ is considered as a given, or particular, realization of a random (noise) process. Therefore, it should be treated as a determinate value. Introduction of stochastic processes themselves is delayed until the end of calculations, i.e. until the moment when statistical averaging over a noise ensemble is carried out.

The bunch is given by its single-particle distribution function $\mathcal{F}(\vartheta, \vartheta', t)$ which is normalized at the initial instant of time $t = 0$:

$$\frac{1}{2\pi} \iint_{(\text{sep})} d\vartheta d\vartheta'\, \mathcal{F}(\vartheta, \vartheta', 0) = 1. \tag{6}$$

The integration goes over the region bounded by separatrix. Specification of the instant of time $t = 0$ is essential. Indeed, in course of evolution of the bunch its particles eventually abandon the bucket, which changes the normalization. But extension of the integration limits over a full beam phase-plane would keep the normalization unaltered (naturally, in the absence of a physical aperture). This situation would indicate conservation of the total number of particles in both the beam and its halo.

Time-evolution of the distribution $\mathcal{F}$ obeys the Vlasov equation

$$\frac{\partial \mathcal{F}(\vartheta, \vartheta', t)}{\partial t} + \{\mathcal{H}; \mathcal{F}\}_{\vartheta\vartheta'} = 0, \tag{7}$$

where

$$\{\mathcal{A}; \mathcal{B}\}_{xy} = \frac{\partial \mathcal{A}}{\partial y}\frac{\partial \mathcal{B}}{\partial x} - \frac{\partial \mathcal{A}}{\partial x}\frac{\partial \mathcal{B}}{\partial y} \tag{8}$$

is the Poisson Bracket.

Consider $\Delta V(\vartheta, t)$ as a small perturbation which allows to split $\mathcal{F}(\vartheta, \vartheta', t)$ into a sum

$$\mathcal{F}(\vartheta, \vartheta', t) = F(\vartheta, \vartheta', t) + \Delta F(\vartheta, \vartheta', t), \tag{9}$$

where $\Delta F \to 0$ in a direct proportion with ΔV, and the following normalizations are imposed

$$\frac{1}{2\pi} \iint_{(\mathrm{sep})} d\vartheta d\vartheta' \, F(\vartheta, \vartheta', 0) = 1, \tag{10}$$

$$\frac{1}{2\pi} \iint_{(\mathrm{sep})} d\vartheta d\vartheta' \, \Delta F(\vartheta, \vartheta', 0) = 0. \tag{11}$$

In accord with this assumption, eq.(7) can as well be split into two. This is the equation which governs slowly varying (with a rate of $\sim \Delta V^2$) function F:

$$\frac{\partial F(\vartheta, \vartheta', t)}{\partial t} + \{H; F\}_{\vartheta\vartheta'} = -\{\Delta H; \Delta F\}_{\vartheta\vartheta'}, \tag{12}$$

and the equation for additive disturbance ΔF:

$$\frac{\partial \Delta F(\vartheta, \vartheta', t)}{\partial t} + \{H; \Delta F\}_{\vartheta\vartheta'} = -\{\Delta H; F\}_{\vartheta\vartheta'}. \tag{13}$$

To begin with, let us study inside-separatrix motion only. Transform the pair (ϑ, ϑ') into angle-action variables $(\psi, \mathcal{J})$ of the unperturbed longitudinal motion. Equations of motion (without noise) in terms of these variables are canonical as well

$$\frac{d\psi}{dt} = \frac{\partial \mathcal{E}}{\partial \mathcal{J}} = \Omega_s(\mathcal{E}); \qquad \frac{d\mathcal{J}}{dt} = -\frac{\partial \mathcal{E}}{\partial \psi} = 0, \tag{14}$$

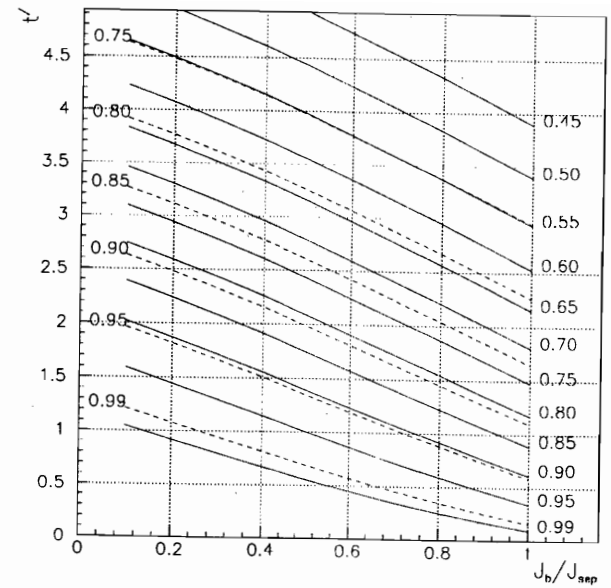

Figure 10. Population-loss life-times (phase noise).

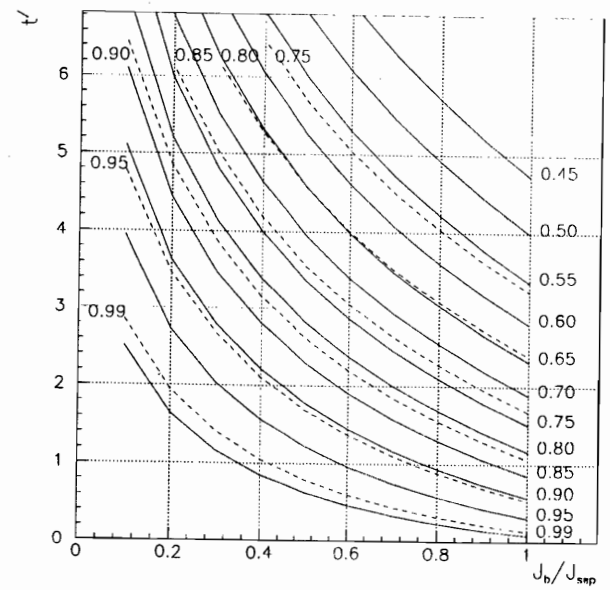

Figure 11. Population-loss life-times (amplitude noise).

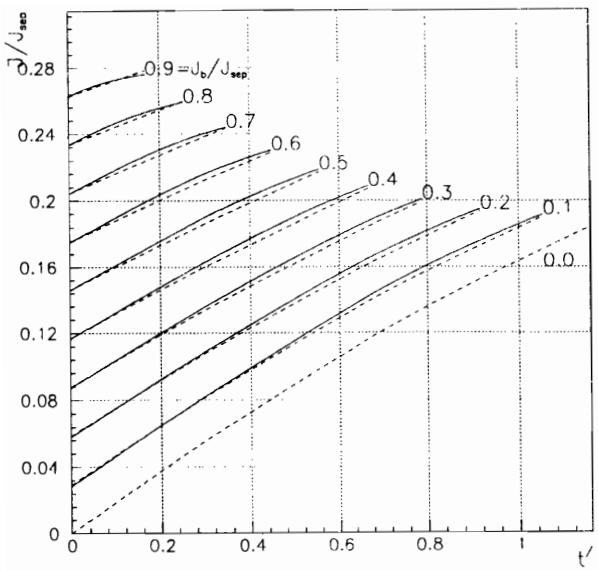

Figure 8. Increase of bunch emittance (phase noise).

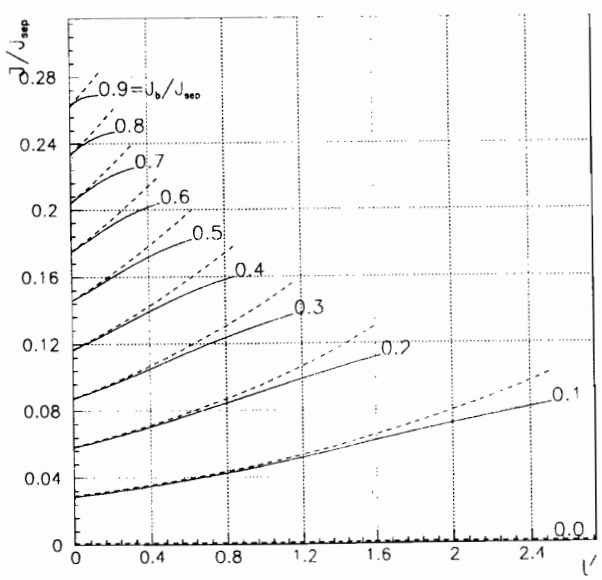

Figure 9. Increase of bunch emittance (amplitude noise).

where $\Omega_s(\mathcal{E})$ denotes the frequency of synchrotron oscillations along the phase-plane trajectory $H(\vartheta,\vartheta') = \mathcal{E}$. Energy $\mathcal{E}(\mathcal{J})$ coincides with the value of new (unperturbed) Hamiltonian, and is defined formally as a function inverse to $\mathcal{J}(\mathcal{E})$:

$$\mathcal{J}(\mathcal{E}) = \frac{1}{\pi}\int_{\vartheta_1(\mathcal{E})}^{\vartheta_2(\mathcal{E})} d\vartheta\,\sqrt{2\left(\mathcal{E} - \Omega_0^2 U(\vartheta)\right)}, \tag{15}$$

where $\vartheta_1(\mathcal{E}) < \vartheta_2(\mathcal{E})$ are coordinates of the turning points of phase-plane trajectory $H(\vartheta,\vartheta') = \mathcal{E}$.

For the coordinate-transformation Jacobian $\partial(\vartheta,\vartheta')/\partial(\psi,\mathcal{J}) = 1$, normalizations (10), (11) become

$$\frac{1}{2\pi}\iint_{(\mathrm{sep})} d\mathcal{J}\,d\psi\; F(\mathcal{J},\psi,0) \;=\; 1, \tag{16}$$

$$\frac{1}{2\pi}\iint_{(\mathrm{sep})} d\mathcal{J}\,d\psi\; \Delta F(\mathcal{J},\psi,0) \;=\; 0. \tag{17}$$

In what follows, periodic function $\Delta V(\vartheta,t)$ is expanded in azimuthal harmonics via the Fourier series representation:

$$\Delta V(\vartheta,t) \;=\; \sum_{k=-\infty}^{\infty} \Delta V_k(t)\exp(ik\vartheta), \tag{18}$$

$$\Delta V_k(t) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \Delta V(\vartheta,t)\exp(-ik\vartheta)\,d\vartheta. \tag{19}$$

Functions $F, \Delta F(\mathcal{J},\psi,t)$, as all the other periodic functions of phase ψ inside bucket, are Fourier series decomposed in multipole harmonics

$$F, \Delta F(\mathcal{J},\psi,t) \;=\; \sum_{m=-\infty}^{\infty} F, \Delta F_m(\mathcal{J},t)\exp(im\psi), \tag{20}$$

$$F, \Delta F_m(\mathcal{J},t) \;=\; \frac{1}{2\pi}\int_{-\pi}^{+\pi} F, \Delta F(\mathcal{J},\psi,t)\exp(-im\psi)\,d\psi. \tag{21}$$

2. EQUATION OF NOISE-INDUCED DIFFUSION

General Case

Eq.(12) in terms of $(\psi,\mathcal{J})$-variables reads

$$\frac{\partial F(\mathcal{J},\psi,t)}{\partial t} + \Omega_s(\mathcal{J})\frac{\partial F(\mathcal{J},\psi,t)}{\partial \psi} = -\left\{\Delta H(\mathcal{J},\psi,t), \Delta F^*(\mathcal{J},\psi,t)\right\}_{\psi\mathcal{J}}. \tag{22}$$

Here, the advantage is taken of the Poisson Bracket kept intact by the canonical transformation $(\vartheta, \vartheta') \to (\psi, \mathcal{J})$. Convenience of complex conjugation applied to real-valued function ΔF is confirmed later on.

Let us study the evolution of the particle distribution in $\mathcal{J}$-variable. Define it formally as the ψ-average of function $F(\mathcal{J}, \psi, t)$:

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} F(\mathcal{J}, \psi, t)\, d\psi \equiv F_0(\mathcal{J}, t). \tag{23}$$

Replace functions ΔH, ΔF^* in the r.h.s. of eq.(22) by their Fourier series expansions in multipole harmonics (20). Integrate both sides of eq.(22) over ψ between limits (23). In the r.h.s., this integration would retain only the diagonal items of the double sum over multipoles. As a result, one gets

$$\frac{\partial F_0(\mathcal{J}, t)}{\partial t} = i \frac{\partial}{\partial \mathcal{J}} \sum_{m=-\infty}^{\infty} m\, \Delta H_m(\mathcal{J}, t)\, \Delta F_m^*(\mathcal{J}, t). \tag{24}$$

Definition (5) allows one to relate the multipole harmonic of ΔH to azimuthal harmonics of ΔV

$$\Delta H_m(\mathcal{J}, t) = i \frac{\Omega_0^2}{q V_{\text{ext}} \sin \varphi_s} \sum_{k=-\infty}^{\infty} \frac{I_{mk}^*(\mathcal{J})}{k} \Delta V_k(t), \tag{25}$$

where functions

$$I_{mk}^*(\mathcal{J}) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\psi\, \exp(ik\vartheta(\mathcal{J}, \psi) - im\psi) \tag{26}$$

are coefficients of the Fourier series expansion of a plane wave in the multipole oscillations. These coefficients are extensively used in the theory of longitudinal instabilities of bunched beams. Thus, eq.(24) turns into

$$\begin{aligned} \frac{\partial F_0(\mathcal{J}, t)}{\partial t} &= -\frac{\Omega_0^2}{q V_{\text{ext}} \sin \varphi_s} \times \\ &\times \frac{\partial}{\partial \mathcal{J}} \sum_{k,m=-\infty}^{\infty} \text{Re} \left(m \frac{I_{mk}^*(\mathcal{J})}{k} \Delta V_k(t)\, \Delta F_m^*(\mathcal{J}, t) \right). \end{aligned} \tag{27}$$

Explicit extraction of the real part in the r.h.s. emphasizes that the series in eq.(27) involves coefficients of the Fourier series decomposition with respect to ϑ, ψ of a real-valued function. It simplifies subsequent passing to spectral power densities.

Fig.7 also marks an interval of time $t' \lesssim 2$ during which more than 99%-fraction of particles is left confined inside the physical aperture. This interval may well be treated as a time domain in which numerical solutions obtained (say, times $t'_{0.99}$ and $t'_{0.96}$) diverge insignificantly from those for the boundary-value problem inside the half-infinite straight-line segment. Of course, the time domain involved is subject to convention. It depends entirely on the accuracy level adopted. Thus, in the following, while applying a similar approach to solving the degenerate boundary-value problem, it is required that the finite physical aperture should contain at least 99.9%-fraction of particles injected.

The very onset ($t' \lesssim t'_{0.99}$) of the noise-induced diffusion features a beam quality degradation, its quantitative measure being an increase of a bunch longitudinal emittance $\overline{\mathcal{J}}$, refer to eq.(79).

Figs.8,9 plot curves of $\overline{\mathcal{J}}(t')$ for the phase and amplitude noises, respectively. Solid lines are numerical results, while dashed lines plot analytical solutions of eqs.(83) in the short-bunch approximation. Notice a rather good fitting together of the curves for the phase noise. The parameter of the curves is bunch size at base $\mathcal{J}_b(t' = 0)$. As for distribution (136), it is related to the initial emittance via the identity:

$$\overline{\mathcal{J}}(t' = 0) = \frac{1}{4} \left(\frac{1 - a^3}{1 - a^2} \right) \mathcal{J}_b(t' = 0)|_{a=\frac{1}{2}} \simeq 0.29\, \mathcal{J}_b(t' = 0).$$

All these curves are cut off at time $t'_{0.99}$ coming from the boundary-value problem with zero condition at separatrix.

A course of further (as $t' \gtrsim t'_{0.99}$) evolution of a bunch depends upon the boundary condition imposed. While a loss of population trapped in buckets turns out to be a more adequate quantitative measure of the diffusion at issue.

Figs.10,11 plot the life-time of the bunch against its initial size $\mathcal{J}_b(t' = 0)$ at different levels of the population loss. Again, the phase (Fig.10) and amplitude (Fig.11) noises are considered. Two families of curves are shown, their parameter being the fraction of particles left inside bucket. Solid lines correspond to zero boundary condition at separatrix (their parameter value is plotted on the left-hand side), and dashed lines stand for zero boundary condition at infinity (parameter is on the right). Practically, the latter case incorporates the aperture put at a distance of $\mathcal{J}_{\max} = (4\text{–}5)\mathcal{J}_{\text{sep}}$. Until time $t'_{0.75}$ is exceeded, that aperture is crossed by a less than 0.1%-fraction of the initial population. As it should be expected, with $t' \to \infty$ the effect of boundary conditions on the life-time estimates increases.

Author thanks Drs. V.I. Balbekov and G.G. Gurov for many instructive discussions and critical notes on the subject matter of the paper.

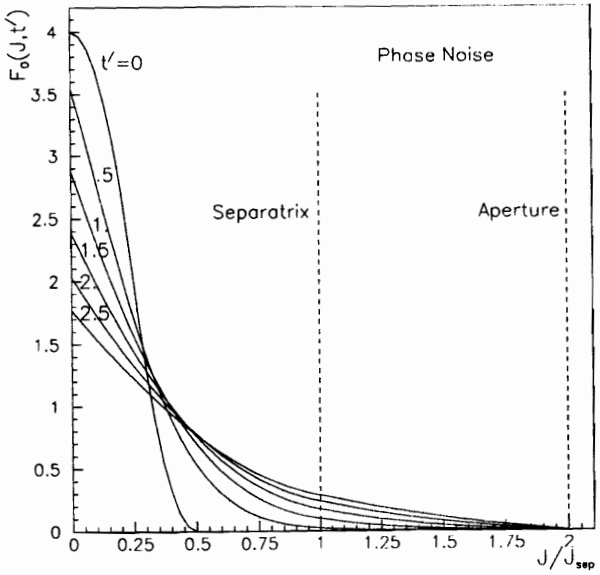

Figure 6. Bunch evolution.

Similarly, bunch life-times extracted from other levels of population losses are introduced (e.g., time $t'_{0.96} \simeq 1.2$ in Fig.7).

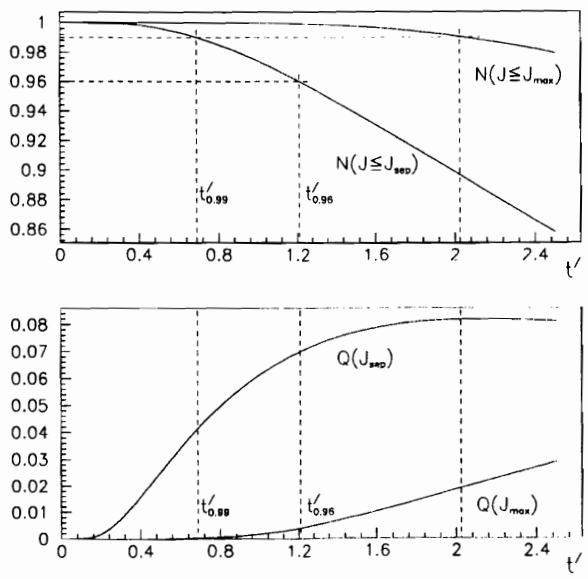

Figure 7. Bunch evolution (continued)

Function $\Delta F_m(\mathcal{J},t)$ embedded in the r.h.s. of eq.(27) is a solution of eq.(13). Coordinate transformation to $(\psi,\mathcal{J})$ converts this equation into

$$\frac{\partial \Delta F(\mathcal{J},\psi,t)}{\partial t} + \Omega_s(\mathcal{J})\frac{\partial \Delta F(\mathcal{J},\psi,t)}{\partial \psi} = -\left\{\Delta H(\mathcal{J},\psi,t); F(\mathcal{J},\psi,t)\right\}_{\psi\mathcal{J}}. \quad (28)$$

Let us assume that for all $t<0$ voltage $\Delta V \equiv 0$, while a bunch is both matched to the phase-plane trajectories and stationary in time. In which case $F(\mathcal{J},\psi,t<0) = F_0(\mathcal{J})$. Hence, it is solely the voltage ΔV, acting at $t \geq 0$, which is responsible for both the bunch non-stationarity ($\partial F/\partial t \neq 0$) and its multipole modulation ($\partial F/\partial \psi \neq 0$). As only the slow processes are studied, attribute small values to the r.h.s. of eqs.(12), (27). Therefore, in solving eq.(28) one may well restrict himself to the first-order approximation in ΔV-perturbation, and ignore a negligible induced multipole modulation of F: $\partial F/\partial\psi \sim \Delta V^2$. Thus, up to the accuracy adopted, substitute $F_0(\mathcal{J},t)$ for $F(\mathcal{J},\psi,t)$ in eq.(28) (ref.[3] incorporates the similar assumption directly, which is referred to as *a random phase type of approximation*). Then, derive term-wise multipole harmonics (21), and take account of eq.(25). As a result, get

$$\frac{\partial \Delta F_m(\mathcal{J},t)}{\partial t} + im\Omega_s(\mathcal{J})\Delta F_m(\mathcal{J},t) = -\frac{\Omega_0^2}{qV_{\text{ext}}\sin\varphi_s}\frac{\partial F_0(\mathcal{J},t)}{\partial \mathcal{J}} m \sum_{k=-\infty}^{\infty}\frac{I^*_{mk}(\mathcal{J})}{k}\Delta V_k(t). \quad (29)$$

It is easy to get the forced solution of this equation which satisfies the initial condition of $\Delta F(\mathcal{J},\psi,t=0)=0$:

$$\Delta F_m(\mathcal{J},t) = -\frac{\Omega_0^2}{qV_{\text{ext}}\sin\varphi_s} m \sum_{k=-\infty}^{\infty}\frac{I^*_{mk}(\mathcal{J})}{k} \times \int_0^t \frac{\partial F_0(\mathcal{J},t-\tau)}{\partial \mathcal{J}}\Delta V_k(t-\tau)\exp\left(-im\Omega_s(\mathcal{J})\tau\right)d\tau. \quad (30)$$

Put this result into the r.h.s. of eq.(27) to yield

$$\frac{\partial F_0^{(\xi)}(\mathcal{J},t)}{\partial t} = \left(\frac{\Omega_0^2}{qV_{\text{ext}}\sin\varphi_s}\right)^2 \frac{\partial}{\partial \mathcal{J}}\Bigg(\text{Re}\sum_{m=-\infty}^{\infty} m^2 \times \sum_{k,k_1=-\infty}^{\infty}\frac{I^*_{mk}(\mathcal{J})}{k}\frac{I_{mk_1}(\mathcal{J})}{k_1}\int_0^t \frac{\partial F_0^{(\xi)}(\mathcal{J},t-\tau)}{\partial \mathcal{J}} \times \Delta V_k^{(\xi)}(t)\Delta V_{k_1}^{(\xi)*}(t-\tau)\exp\left(im\Omega_s(\mathcal{J})\tau\right)d\tau\Bigg). \quad (31)$$

Eq.(31) describes the evolution of the bunch distribution in $\mathcal{J}$-coordinate under the particular (ξ-th) realization of perturbing (noise) voltage. This fact is denoted by (ξ)-superscript attributed to functions F_0 and ΔV. So far, they were treated as determinate (not random).

As a matter of fact, such a detailed information about perturbing voltage $\Delta V(\vartheta, t_1)$ covering an arbitrary time interval of $0 \le t_1 \le t$, which could have allowed a direct employment of eq.(31), is unavailable. Moreover, it is not required practically.

A theory of stochastic (random) processes implies the concept of a statistical ensemble of noise realizations [4]. (In practice, such an ensemble can be introduced as a series of experimental observations of the bunch in different accelerator cycles. Macroscopic conditions for these experiments are assumed to be sufficiently unaltered, so as to reproduce exactly the statistical features of noise at issue: its mean value, correlation function and correlation time, spectral power density.) Within the frames of this ensemble of realizations, a procedure of statistical averaging is well defined, and is denoted later on via angular brackets $\langle \ldots \rangle$. Now one can define the (ensemble) average over realizations, alongside with the relevant fluctuations:

$$\langle F_0^{(\xi)}(\mathcal{J}, t)\rangle = \langle F_0\rangle(\mathcal{J}, t), \tag{32}$$

$$\delta F_0^{(\xi)}(\mathcal{J}, t) = F_0^{(\xi)}(\mathcal{J}, t) - \langle F_0\rangle(\mathcal{J}, t); \qquad \langle \delta F_0^{(\xi)}(\mathcal{J}, t)\rangle = 0. \tag{33}$$

Perform the statistical averaging of eq.(31). Consider fluctuations δF_0 (33) to be small, and neglect on this footing the term $\sim \langle \delta F_0 \Delta V_k \Delta V_{k_1}^*\rangle$. As a result, get the equation for the averaged (over realizations) bunch distribution in $\mathcal{J}$-variable:

$$\begin{aligned} \frac{\partial \langle F_0\rangle(\mathcal{J}, t)}{\partial t} &= \left(\frac{\Omega_0^2}{qV_{\text{ext}} \sin\varphi_s}\right)^2 \frac{\partial}{\partial \mathcal{J}}\Bigg(\text{Re} \sum_{m=-\infty}^{\infty} m^2 \times \\ &\times \sum_{k,k_1=-\infty}^{\infty} \frac{I_{mk}^*(\mathcal{J})}{k} \frac{I_{mk_1}(\mathcal{J})}{k_1} \int_0^t \frac{\partial \langle F_0\rangle(\mathcal{J}, t-\tau)}{\partial \mathcal{J}} \times \\ &\times \langle \Delta V_k(t) \Delta V_{k_1}^*(t-\tau)\rangle \exp\left(im\Omega_s(\mathcal{J})\tau\right) d\tau\Bigg). \end{aligned} \tag{34}$$

Practically, correlation function $\langle \Delta V \Delta V^*\rangle$ is essentially non-zero only during the time interval of $|\tau| \lesssim \tau_{\Delta V}$, where $\tau_{\Delta V}$ is a representative correlation time of the noise voltages. Let another time constant, τ_{dif}, be a rate measure of a slow, systematic dilution of $\langle F_0\rangle$ under the RF-noise effect (diffusion life-time of the bunch). Consider the voltage fluctuations to be fast

$$\tau_{\Delta V} \ll \tau_{dif}, \tag{35}$$

The integration of eq.(118) has resulted in a quadrature formula of trapezia.

Numerical computation necessarily employs a finite value of $\mathcal{J}_{\max}$. A numerical solution on a finite straight-line segment $0 \le \mathcal{J} \le \mathcal{J}_{\max}$ would insignificantly differ from that for the half-infinite straight-line $0 \le \mathcal{J} \le \infty$, provided either $Q(\mathcal{J}_{\max}, t) \simeq 0$ or integral (134) remain (almost) unaltered. To treat a more prolonged process, boundary $\mathcal{J}_{\max}$ is removed farther, etc.

The approach discussed here is realized in computer codes for numerical solution of the diffusion equation in question.

8. NUMERICAL RESULTS

Approximation (46) of the wide-band (*white*) noise is studied. The amplitude and phase noises in question are treated as independent ones. To present results, use is made of a scaled time

$$t' = t\,\frac{\Omega_0^2\,\mathcal{P}^{(a,\varphi)}}{V_{\text{ext}}^2}. \tag{135}$$

The initial distribution of particles in a bunch is combined of two conjugated parabolas:

$$\begin{aligned} \langle F_0\rangle(\mathcal{J}, t'=0) &= \frac{3}{(1-a^2)\mathcal{J}_b^3} \times \\ &\times \begin{cases} (\mathcal{J}_b - \mathcal{J})^2 - (a\mathcal{J}_b - \mathcal{J})^2/a, & 0 \le \mathcal{J} < a\mathcal{J}_b; \\ (\mathcal{J}_b - \mathcal{J})^2, & a\mathcal{J}_b \le \mathcal{J} \le \mathcal{J}_b; \end{cases} \end{aligned} \tag{136}$$

where $\mathcal{J}_b = \mathcal{J}_b(t'=0)$, $\mathcal{J}_b \le \mathcal{J}_{\text{sep}}$ is a boundary of injected bunch, $0 < a < 1$ is a parameter. In computation the value of $a = \frac{1}{2}$ is taken. Final results depend weakly on the shape of $\langle F_0\rangle(\mathcal{J}, t'=0)$.

Figs.6,7 clarify the meaning of the time characteristics introduced, and explain the method of solving the degenerate boundary-value problem.

Fig.6 shows evolution in time of the distribution function (area bounded under the curve $t' = 0$ equals unit). The phase noise is treated. The initial bunch occupies half of the bucket: $\mathcal{J}_b(t'=0) = \frac{1}{2}\mathcal{J}_{\text{sep}}$. Physical aperture is put at a distance of $\mathcal{J}_{\max} = 2\mathcal{J}_{\text{sep}}$.

The top part of Fig.7 plots time dependence of fractional number of particles trapped either inside the separatrix $N(\mathcal{J} \le \mathcal{J}_{\text{sep}}, t')$ or within the physical aperture $N(\mathcal{J} \le \mathcal{J}_{\max}, t')$. While the bottom part of Fig.7 shows diffusive flux $Q(\mathcal{J}, t')$ of particles across the same sections. Parameter $t'_{0.99}$ denotes the time until which more than 99%-fraction of particles is left trapped in the bucket. By convention, it may well be referred to as a *running-into-separatrix* life-time.

On substituting these expressions into eq.(120) one gets the following two-layer time-domain integration procedure (Crank-Nicolson's numerical scheme, see ref.[6]):

$$(\widehat{A}+\frac{1}{2}\Delta t\,\widehat{B})\,\vec{F}^{(n+1)}=(\widehat{A}-\frac{1}{2}\Delta t\,\widehat{B})\,\vec{F}^{(n)}. \tag{130}$$

This iteration process is unconditionally stable for an arbitrary Δt. Moreover, a numerical solution will be free of any unwanted oscillations with respect to the exact one, given the following inequality is satisfied

$$\lambda_{\max}\Delta t \quad < \quad 2. \tag{131}$$

Here $\lambda_{\max}>0$ denotes the maximum eigenvalue of a generalized problem:

$$\lambda\,\widehat{A}\,\vec{F} \quad = \quad \widehat{B}\,\vec{F}. \tag{132}$$

By the order of its magnitude, $\lambda_{\max}\simeq D_*/\Delta\mathcal{J}_*^2$ where D_* and $\Delta\mathcal{J}_*$ are some representative values of the diffusion coefficient and of the discretization step. Exact evaluation of $\lambda_{\max}$ is carried out numerically whenever either grid $\mathcal{J}_\ell$ or function $D(\mathcal{J})$ are altered. Inequality (131) itself is treated as the practical limitation on the choice of a time-domain integration step.

Degenerate Boundary-Value Problem

To get a numerical solution of the degenerate boundary-value problem, it is important to specify a quantitative criterion which allows passing to the limit of $\mathcal{J}_{\max}\to\infty$.

Insert approximations given in eqs.(128), (129) alongside with boundary conditions $F_L(t)=0$, $dF_L(t)/dt=0$ into eq.(125). Thus, the approximation of the diffusive flux through the outer boundary $\mathcal{J}_L=\mathcal{J}_{\max}$ is obtained:

$$Q(\mathcal{J}_{\max})^{(n+1/2)}\simeq -A_{L,L-1}\frac{F_{L-1}^{(n+1)}-F_{L-1}^{(n)}}{\Delta t}-B_{L,L-1}\frac{F_{L-1}^{(n+1)}+F_{L-1}^{(n)}}{2}. \tag{133}$$

Naturally, it should coincide with the negative time derivative of normalization integral (i.e. with loss rate of the fractional number of particles trapped inside the physical aperture). The integral just mentioned is

$$\begin{aligned} N(\mathcal{J}\le\mathcal{J}_{\max},t^{(n)}) \quad &= \quad \int_0^{\mathcal{J}_{\max}}\langle F_0\rangle(\mathcal{J},t^{(n)})\,d\mathcal{J}\simeq \\ &\simeq \quad F_{\ell=0}^{(n)}\frac{\Delta\mathcal{J}_1}{2}+\sum_{\ell=1}^{L-1}F_\ell^{(n)}\frac{\Delta\mathcal{J}_\ell+\mathcal{J}_{\ell+1}}{2}+F_L^{(n)}\frac{\Delta\mathcal{J}_L}{2}. \end{aligned} \tag{134}$$

and study the bunch evolution during intervals of time t:

$$\tau_{\Delta V}\ll t\lesssim\tau_{dif}. \tag{36}$$

Inequality (36), alongside with a fast fall-off expected of the correlation function, allows one to replace the finite upper limit of integration in eq.(34) by $+\infty$. Formally, it is the same as to put the forced and sustained (at $t\to\infty$) solution of eq.(29) directly into the r.h.s. of eq.(31).

On the other hand, on satisfying inequality (35) and due to the cut-off effect of the correlation function in the integrand, derivative $\partial\langle F_0\rangle(\mathcal{J},t-\tau)/\partial\mathcal{J}$ may be factored out of $d\tau$-integration in eq.(34) and treated at the instant of time $\tau=0$.

As a result, eq.(34) is reduced to a desired diffusion equation in $\mathcal{J}$-coordinate:

$$\frac{\partial\langle F_0\rangle(\mathcal{J},t)}{\partial t}=\frac{\partial}{\partial\mathcal{J}}\left(D(\mathcal{J})\frac{\partial\langle F_0\rangle(\mathcal{J},t)}{\partial\mathcal{J}}\right) \tag{37}$$

with the diffusion coefficient

$$\begin{aligned} D(\mathcal{J}) \quad = \quad &\frac{1}{2}\left(\frac{\Omega_0^2}{qV_{\rm ext}\sin\varphi_s}\right)^2\mathrm{Re}\sum_{m=-\infty}^{\infty}m^2\sum_{k,k_1=-\infty}^{\infty}\frac{I_{mk}^*(\mathcal{J})}{k}\frac{I_{mk_1}(\mathcal{J})}{k_1}\times \\ &\times\,2\int_0^\infty\langle\Delta V_k(t)\Delta V_{k_1}^*(t-\tau)\rangle\,\exp\left(im\Omega_s(\mathcal{J})\tau\right)d\tau. \end{aligned} \tag{38}$$

Stationary Noise in Co-Rotating Frame

Coefficient D does not, in fact, depend on the current time t, provided correlation $\langle\Delta V\Delta V^*\rangle$ is a function of difference time τ only. It occurs so, say, in the following two cases.

1. The noise voltage is stationary or periodically-unstationary with respect to the laboratory frame, while D takes account only of its systematic effect which does not average to zero during a revolution period of a beam round the orbit.
2. The noise is a ('weakly') stationary stochastic process as defined directly in the co-moving coordinate frame.

This paper treats only the last, less general case, which implies that the noise correlations exceed the revolution period: $\tau_{\Delta V}\gg 2\pi/\omega_s$. It is also assumed that independent variables ϑ and t in perturbing voltage $\Delta V(\vartheta,t)$ are separated

$$\Delta V(\vartheta,t)=v(t)\,g(\vartheta),\qquad g(\vartheta)=g(\vartheta+\frac{2\pi}{q}). \tag{39}$$

Here $g(\vartheta)$ specifies determinate (possibly, unharmonic) wave-shape of propagating wave of the perturbing voltage synchronous to the beam. Its periodicity entails that the diffusion coefficient would be the same for all buckets of the beam. (This assumption can be withdrawn by taking the ring orbit to be a period of $g(\vartheta)$: $g(\vartheta) = g(\vartheta + 2\pi)$. But this would result in unnecessary complications, especially in treating the beyond-separatrix motion in Section 5.)

A real-valued function $v(t)$ denotes a ('weakly') stationary stochastic process with a zero average, $\langle v(t) \rangle = 0$. Its (auto)correlation function $\langle v(t)\, v(t-\tau) \rangle$ does not depend on the current instant of time t. It is an even function of difference time τ only:

$$\langle v(t)\, v(t-\tau) \rangle = \langle v\, v \rangle(\tau); \;\; \langle v\, v \rangle(-\tau) = \langle v\, v \rangle(\tau).$$

Put eq.(39) into eq.(38) to yield

$$D(\mathcal{J}) = \frac{1}{2} \left(\frac{\Omega_0^2}{q^2 V_{\rm ext} \sin \varphi_{\rm s}} \right)^2 \sum_{m=-\infty}^{\infty} \mathcal{W}_m(\mathcal{J})\, P_v\left(m\Omega_{\rm s}(\mathcal{J})\right) \geq 0. \tag{40}$$

The value $P_v(\Omega)$ denotes a (double-sided) spectral power density of the stationary random process $v(t)$:

$$P_v(\Omega) = \int_{-\infty}^{\infty} \langle v\, v \rangle(\tau)\, \cos(\Omega\tau)\, d\tau, \qquad P_v(\Omega) = P_v(-\Omega). \tag{41}$$

Weight functions

$$\begin{aligned} \mathcal{W}_m(\mathcal{J}) &= m^2 q^2 \left| \sum_{k=-\infty}^{\infty} \frac{I^*_{mk}(\mathcal{J})}{k} g_k \right|^2 = \\ &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial q\vartheta(\mathcal{J},\psi)}{\partial \psi} g(\vartheta(\mathcal{J},\psi))\, \exp(-im\psi)\, d\psi \right|^2 \end{aligned} \tag{42}$$

specify the contribution of the noise at the m-th harmonic of synchrotron frequency, while $\mathcal{W}_{-m} = \mathcal{W}_m$. (To allow for a joint effect of both harmonics $\pm m$, take $2\mathcal{W}_m$.) The values g_k are coefficients of the Fourier series decomposition of $g(\vartheta)$ in azimuthal harmonics (19). Due to periodicity imposed by eq.(39) their numbers are multiples of q.

White Noise

Calculations of weight functions $\mathcal{W}_m(\mathcal{J})$ would allow to estimate the indices of multipoles $|m| \lesssim m'(\mathcal{J})$ which bear a dominant contribution to series (40). On varying insignificantly throughout a band of lower frequencies

Diffusive flux of particles which enters an arbitrary section $\mathcal{J}_\ell > 0$ is given by

$$Q(\mathcal{J}_\ell - 0, t) \simeq -A_{\ell,\ell-1} \frac{d}{dt}(F_{\ell-1}(t) + 2F_\ell(t)) - B_{\ell,\ell-1}(F_{\ell-1}(t) - F_\ell(t)). \tag{125}$$

What is essential, the above formula does not apply to definition (112) at all. (The latter can hardly be used practically, e.g., at separatrix.) The flux which leaves the same section $\mathcal{J}_\ell < \mathcal{J}_L$ is

$$Q(\mathcal{J}_\ell + 0, t) \simeq +A_{\ell,\ell+1} \frac{d}{dt}(F_{\ell+1}(t) + 2F_\ell(t)) + B_{\ell,\ell+1}(F_{\ell+1}(t) - F_\ell(t)). \tag{126}$$

In either section the diffusive flux should be continuous, $Q(\mathcal{J}_\ell - 0, t) = Q(\mathcal{J}_\ell + 0, t)$. Subtract the second of above equations from the first one. On accounting for properties

$$A_{\ell\ell} = 2(A_{\ell,\ell-1} + A_{\ell,\ell+1}), \qquad B_{\ell\ell} = -(B_{\ell,\ell-1} + B_{\ell,\ell+1}), \tag{127}$$

of the matrix elements, one immediately arrives at the ℓ-th equation of set (120).

Whereof, a physical content hidden in this set becomes perfectly clear. The one expresses continuity against $\mathcal{J}$-variable of both the distribution of particles and of their diffusive flux, which is shown in discretized form. It is by this reason that an integrable singularity of diffusion coefficient (under a conjugation of solutions via eqs.(115), (116)) would never reveal itself in a formal structure of set (120).

Time-Domain Component

Introduce an equidistant time-domain grid $t^{(n)} = n\Delta t; n = 0, 1, 2, \ldots$ Denote the nodal values of vector $\vec{F}(t = t^{(n)})$ by $\vec{F}^{(n)}$.

Consider vector equation (120) at the intermediate (central) instant of time $t^{(n+1/2)} = \frac{1}{2}(t^{(n)} + t^{(n+1)})$. Replace derivative $d\vec{F}^{(n+1/2)}/dt$ by its central finite difference approximation (whose error is $\sim \Delta t^2$):

$$\frac{d\vec{F}^{(n+1/2)}}{dt} \simeq \frac{1}{\Delta t} \left(\vec{F}^{(n+1)} - \vec{F}^{(n)} \right). \tag{128}$$

Express vector $\vec{F}^{(n+1/2)}$ by linear interpolation through the adjacent nodal values, which entails an approximation error of the same order of Δt^2:

$$\vec{F}^{(n+1/2)} \simeq \frac{1}{2} \left(\vec{F}^{(n+1)} + \vec{F}^{(n)} \right). \tag{129}$$

this point boundary conditions (113), (116), imposed in terms of diffusive fluxes $Q(\mathcal{J},t)$, are naturally incorporated. Any practical computation of the solution gradients at boundaries is excluded altogether.

As a result, diffusion eqs.(37), (93) entail the following set of linear homogeneous equations

$$\widehat{A}\,\frac{d}{dt}\vec{F}(t)+\widehat{B}\,\vec{F}(t)=\vec{0} \tag{120}$$

with symmetrical three-diagonal matrices $\widehat{A}$ and $\widehat{B}$. Their non-zero elements read:

$$\begin{pmatrix} A_{00}=\frac{1}{3}\Delta\mathcal{J}_1 & A_{01}=\frac{1}{6}\Delta\mathcal{J}_1 & \dots \\ \dots & \dots & \dots \\ A_{\ell,\ell-1}=\frac{1}{6}\Delta\mathcal{J}_\ell & A_{\ell\ell}=\frac{1}{3}(\Delta\mathcal{J}_\ell+\Delta\mathcal{J}_{\ell+1}) & A_{\ell,\ell+1}=\frac{1}{6}\Delta\mathcal{J}_{\ell+1} \\ \dots & \dots & \dots \\ \dots & A_{L,L-1}=\frac{1}{6}\Delta\mathcal{J}_L & A_{LL}=\frac{1}{3}\Delta\mathcal{J}_L \end{pmatrix}; \tag{121}$$

$$\begin{pmatrix} B_{00}=\frac{\overline{D}_1}{\Delta\mathcal{J}_1} & B_{01}=-\frac{\overline{D}_1}{\Delta\mathcal{J}_1} & \dots \\ \dots & \dots & \dots \\ B_{\ell,\ell-1}=-\frac{\overline{D}_\ell}{\Delta\mathcal{J}_\ell} & B_{\ell\ell}=\frac{\overline{D}_\ell}{\Delta\mathcal{J}_\ell}+\frac{\overline{D}_{\ell+1}}{\Delta\mathcal{J}_{\ell+1}} & B_{\ell,\ell+1}=-\frac{\overline{D}_{\ell+1}}{\Delta\mathcal{J}_{\ell+1}} \\ \dots & \dots & \dots \\ \dots & B_{L,L-1}=-\frac{\overline{D}_L}{\Delta\mathcal{J}_L} & B_{LL}=\frac{\overline{D}_L}{\Delta\mathcal{J}_L} \end{pmatrix}. \tag{122}$$

Generally speaking, set (120) involves only sub-matrices of dimension $L\times L$ whose row-column indices ℓ,ℓ_1 run in the range of $\ell,\ell_1=0,1,\dots,L-1$. Both the L-th row and column are excluded due to boundary conditions $F_L(t)=0$, $dF_L(t)/dt=0$. Nevertheless, elements of the L-th row are reserved for later use in calculation of boundary flux through the section $\mathcal{J}_L=\mathcal{J}_{\max}$.

Symbol $\overline{D}_\ell$ denotes the diffusion coefficient averaged over the ℓ-th element of discretization

$$\overline{D}_\ell=\frac{1}{\Delta\mathcal{J}_\ell}\int\limits_{\mathcal{J}_{\ell-1}}^{\mathcal{J}_\ell} D(\mathcal{J})\,d\mathcal{J}. \tag{123}$$

Column-vector is made of unknown nodal values:

$$\vec{F}=\left(F_{\ell=0},F_1,\dots,F_{\ell_{\rm sep}},\dots,F_{L-1}\right)^T, \tag{124}$$

with $F_L=0$ due to boundary condition.

$|\Omega|\lesssim m'(\mathcal{J})\Omega_{\rm s}(\mathcal{J})$, the noise spectral density $P_v(m\Omega_{\rm s}(\mathcal{J}))\simeq P_v$ can well be factored out of the sum over m. Thus retained series of weight functions is reducible to quadratures. (Suffice it to take an integral presentation of eq.(42).) The result is the diffusion coefficient which reads

$$D(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2V_{\rm ext}\sin\varphi_{\rm s}}\right)^2 P_v\,\mathcal{A}(\mathcal{J}), \tag{43}$$

$$\mathcal{A}(\mathcal{J}) = \sum_{m=-\infty}^{\infty}\mathcal{W}_m(\mathcal{J}) = \tag{44}$$

$$= \frac{1}{2\pi}\int\limits_{-\pi}^{+\pi} d\psi\left(\frac{\partial q\vartheta(\mathcal{J},\psi)}{\partial\psi}\right)^2 g^2(\vartheta(\mathcal{J},\psi)).$$

(An integral presentation of eq.(44)-type valid for arbitrary functions $g(\vartheta)$ was first obtained in ref.[3].)

It is the so called approximation of a wide-band noise. From the time-domain viewpoint, it implies the noise correlation time τ_v to be short as compared to a period of synchrotron oscillations

$$\tau_v \ll \frac{2\pi}{m'(\mathcal{J})\Omega_{\rm s}(\mathcal{J})}. \tag{45}$$

Of course, this condition should be satisfied for all $\mathcal{J}$ belonging to the segment $0\le\mathcal{J}\le\mathcal{J}_{\max}$ where the solution is looked for.

Due to an appreciable distinction of representative time-scales, as expressed via inequalities (35), (36) and (45), such a noise can be referred to as a delta-correlated, or *white* one. Formally, the latter entails

$$\langle v\,v\rangle(\tau)\simeq P_v\,\delta(\tau);\qquad P_v(\Omega)\simeq P_v=\text{const}. \tag{46}$$

Here $\delta(\tau)$ is delta function; $P_v\simeq 2\tau_v\langle v^2\rangle$ with $\langle v^2\rangle$ denoting a (finite) dispersion of the actual noise voltage $v(t)$.

Ergodicity

Function $\langle F_0\rangle(\mathcal{J},t)$ from eq.(37) (the ensemble average) has a quite formal origin. It can hardly be measured experimentally. Of a greater practical interest is the time-average of a particular bunch observation (under a given noise realization). With the stochastic process $F_0^{(\xi)}(\mathcal{J},t)$ being ergodic, these two averages directly correspond one for one:

$$\langle F_0\rangle(\mathcal{J},t)\approx\frac{1}{T}\int\limits_t^{t+T} F_0^{(\xi)}(\mathcal{J},t_1)\,dt_1. \tag{47}$$

Period T is taken such as to satisfy inequalities

$$\tau_{\delta F_0} \ll T \ll \tau_{dif}, \tag{48}$$

where $\tau_{\delta F_0}$ denotes the correlation time for δF_0-fluctuations.

By virtue of the r.h.s. of eq.(48) this time averaging does not result in a loss of data on a systematic dilution of the bunch.

On the other hand, the l.h.s. of eq.(48) provides an effective smoothing of fluctuations δF_0. Indeed, on averaging (over realizations) in a quasi-stationary approximation, a square of difference between the r.h.s. and l.h.s. of eq.(47) is just [4]

$$\frac{1}{T^2}\int_t^{t+T} dt_1 \int_t^{t+T} dt_2 \langle \delta F_0(\mathcal{J},t_1)\delta F_0(\mathcal{J},t_2)\rangle \lesssim \frac{2\tau_{\delta F_0}}{T}\langle \delta F_0^2(\mathcal{J},t)\rangle \ll 1. \tag{49}$$

Its small value is, in fact, provided by the l.h.s. of eq.(48) together with the finite value of the dispersion of fluctuations δF_0.

In estimating the value of the double integral an account is taken of its integrand — correlation of (quasi-stationary) fluctuations δF_0 — to be essentially a function of time difference $t_1 - t_2$, this function being non-zero for $|t_1 - t_2| \lesssim \tau_{\delta F_0} \ll T$.

Usually, these fluctuations as such are omitted from consideration. Hence, the question of a qualitative verification of the ergodic hypothesis for stochastic process $F_0^{(\xi)}(\mathcal{J},t)$ (i.e. the proof of inequality (49)) remains unanswered.

3. AMPLITUDE AND PHASE NOISES INSIDE SEPARATRICES

Many of the formal results presented in Sections 3–4 are already known from refs.[1]–[3]. These Sections should rather be treated as a review on the topic.

An Acceleration Mode

Perturbations, precisely of eq.(39)-type, are imposed by the accelerating voltage noises. Namely,

– either by an amplitude noise:

$$\Delta V^{(a)}(\vartheta,t) = \Delta V_{\text{ext}}(t)\,\cos(q\vartheta + \varphi_s), \tag{50}$$

– or by a (small) phase noise:

$$\Delta V^{(\varphi)}(\vartheta,t) = V_{\text{ext}}\Delta\varphi(t)\,\sin(q\vartheta + \varphi_s), \tag{51}$$

$$\Delta\varphi(t) = \int^t \Delta\omega_{\text{rf}}(t_1)\,dt_1, \qquad \langle\Delta\varphi^2\rangle^{1/2} \ll 2\pi.$$

Of course, as far as the onset of short-bunch evolution is only concerned — until the essential fraction of particles crosses separatrix — the particular formulation of boundary-value problem, by eqs.(114) or (117), does not matter at all. The advantage was taken of this fact earlier, in Section 4, to study the diffusion equation via the method of moments.

7. METHOD OF NUMERICAL SOLUTION

Spatial and time-domain discretizations of the diffusion equation are performed in different ways.

Spatial Component

Finite difference schemes do not suit: troubles arise in approximating solution gradients at boundaries (113), (116). Instead, use is made of the Finite Element Method [6] applied, possibly, to a non-equidistant grid.

A straight-line segment 0–$\mathcal{J}_{\max}$, $\mathcal{J}_{\max} \geq \mathcal{J}_{\text{sep}}$ is covered by $(L+1)$ nodes $\mathcal{J}_\ell$, where $\ell = 0, 1, \ldots, L$. The boundaries correspond to $\mathcal{J}_0 = 0$, $\mathcal{J}_L = \mathcal{J}_{\max}$. An intermediate node $0 < \ell_{\text{sep}} \leq L$ is placed precisely on the separatrix. The latter is subject to continuity conditions (115), (116). Thereby, segment 0–$\mathcal{J}_{\max}$ is broken into L (non-identical) elements with their lengths being $\Delta\mathcal{J}_\ell = \mathcal{J}_\ell - \mathcal{J}_{\ell-1}$, $\ell \neq 0$.

A standard basic set of piece-wise linear functions $\Phi_\ell(\mathcal{J})$ is introduced. (Function $\Phi_\ell(\mathcal{J})$ equals unit at the ℓ-th node, and is zero at all other nodes.) By virtue of this set a (piece-wise linear) approximation of the spatial component of solution $\langle F_0\rangle(\mathcal{J},t)$ is obtained:

$$\langle F_0\rangle(\mathcal{J},t) \simeq \sum_{\ell=0}^{L} F_\ell(t)\,\Phi_\ell(\mathcal{J}), \tag{118}$$

with F_ℓ being an approximation of the exact solution $\langle F_0\rangle$ at the ℓ-th node:

$$F_\ell(t) \simeq \langle F_0\rangle(\mathcal{J}_\ell,t). \tag{119}$$

Then, use is made of a weak formulation of Galerkin's weighted residual technique [6], which practically implies the following:

(a) Weight factors' roles in the definition of integral residual for trial solution (118) are played by the basic functions $\Phi_\ell(\mathcal{J})$ themselves.

(b) In the residual, an integration by parts is carried out once, which decreases the order of differentiation with respect to $\mathcal{J}$ down to the first one. At

As far as the storage mode with a beam made of $M = q$ identical bunches is concerned, the second of boundary conditions emerges rather rigorously. An ultimate loss of particles occurs when these reach quite a distant physical aperture, which corresponds to an "absorbing wall" erected at a coordinate $\mathcal{J} = \mathcal{J}_{\max} \gg \mathcal{J}_{\rm sep}$. Therefore,

$$\langle F_0 \rangle (\mathcal{J} = \mathcal{J}_{\max}, t) = 0, \qquad \mathcal{J}_{\max} \to \infty, \tag{114}$$

and one arrives at the degenerate boundary-value problem for the half-infinite straight-line segment $0 \le \mathcal{J} \le \infty$. The diffusion coefficient $D(\mathcal{J})$ possibly suffers a discontinuity at separatrix. At this break-point the continuity is retained of the density of particles, and of their diffusive flux:

$$\langle F_0 \rangle (\mathcal{J} = \mathcal{J}_{\rm sep} - 0, t) = \langle F_0 \rangle (\mathcal{J} = \mathcal{J}_{\rm sep} + 0, t), \tag{115}$$

$$Q(\mathcal{J} = \mathcal{J}_{\rm sep} - 0, t) = Q(\mathcal{J} = \mathcal{J}_{\rm sep} + 0, t). \tag{116}$$

In a more general case (either $M \neq q$, or $|\varphi_{\rm s}| \neq \frac{\pi}{2}$) the formulation of the boundary-value problem in terms of eqs.(114)–(116) is no longer valid: beam dynamics beyond separatrix is not at all reducible to a diffusion equation. In this context, it makes sense to resort to the (approximate) boundary condition of ref.[2]:

$$\langle F_0 \rangle (\mathcal{J} = \mathcal{J}_{\rm sep}, t) = 0. \tag{117}$$

It assumes the ultimate loss of particles to occur immediately on their reaching separatrix. Essentially, this condition is equivalent to that of $D(\mathcal{J}) = \infty$ as $\mathcal{J} \ge \mathcal{J}_{\rm sep}$. Its application is open to no serious objections whenever physical mechanisms are involved which are fast enough to withdraw particles from the bunch vicinity. Namely, these might be the following ones:

– Acceleration. Phase-plane trajectories beyond separatrices are unclosed and unbounded; there is no periodical (orbital) motion of particles near separatrices.

– Storage of single bunches in a ring with a large orbit circumference ($M \ll q$). The untrapped particles waste most of their time near empty buckets, which would not practically affect the diffusion of bunches in a few filled ones.

In all the rest cases incorporation of boundary condition (117) would probably lower estimates of expected bunch life-time.

In studying the noise-induced diffusion in a storage mode the paper treats both the boundary-value problems, of eqs.(114)–(116) and that of eq.(117). If $M \neq q$ it would hopefully provide the upper and lower margins for bunch life-times, respectively.

In which particular cases eqs.(40)–(42) entail

$$D^{(a,\varphi)}(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\rm ext}\sin\varphi_{\rm s}}\right)^2 \sum_{m=-\infty}^{\infty} \mathcal{W}_m^{(a,\varphi)}(\mathcal{J})\, \mathcal{P}^{(a,\varphi)}(m\Omega_{\rm s}(\mathcal{J})), \tag{52}$$

where $\mathcal{P}^{(a,\varphi)}(\Omega)$ is the equation symbol for the spectral density measured in mutually comparable units of $[V]^2 \cdot [t]$:

$$\mathcal{P}^{(a)}(\Omega) = P_{\Delta V_{\rm ext}}(\Omega), \tag{53}$$

$$\mathcal{P}^{(\varphi)}(\Omega) = V_{\rm ext}^2\, P_{\Delta\varphi}(\Omega). \tag{54}$$

Weight functions acquire a simple form of

$$\mathcal{W}_m^{(a,\varphi)}(\mathcal{J}) = \frac{m^2}{4}\left|I_{mq}^*(\mathcal{J})\,\exp(+i\varphi_{\rm s}) \mp I_{m,-q}^*(\mathcal{J})\,\exp(-i\varphi_{\rm s})\right|^2. \tag{55}$$

Here and in the following, the upper sign (or notation) stands for the amplitude noise, while the lower one — for the phase noise.

Under assumption of *white* noise, when $\mathcal{P}(m\Omega_{\rm s}(\mathcal{J})) \simeq \mathcal{P}$, eqs.(43)–(44) yield

$$D^{(a,\varphi)}(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\rm ext}\sin\varphi_{\rm s}}\right)^2 \mathcal{P}^{(a,\varphi)}\mathcal{A}^{(a,\varphi)}(\mathcal{J}), \tag{56}$$

$$\mathcal{A}^{(a,\varphi)}(\mathcal{J}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\psi \left(\frac{\partial q\vartheta(\mathcal{J},\psi)}{\partial\psi}\right)^2 \times \begin{cases} \cos^2(q\vartheta(\mathcal{J},\psi) + \varphi_{\rm s}), & (a) \\ \sin^2(q\vartheta(\mathcal{J},\psi) + \varphi_{\rm s}). & (\varphi) \end{cases} \tag{57}$$

A Storage Mode

As is quite evident, long-time beam-storage operation modes involving stationary separatrices ($|\varphi_{\rm s}| = \frac{\pi}{2}$) are of the most practical interest. They are studied in what follows.

In this case the potential well of longitudinal motion is symmetrical ($U(\vartheta) = U(-\vartheta)$). For phase-plane trajectories trapped inside separatrix it entails that

$$I_{m,-k}(\mathcal{J}) = (-1)^m I_{mk}(\mathcal{J}). \tag{58}$$

Wherefrom,

$$\mathcal{W}_m^{(a,\varphi)}(\mathcal{J}) = m^2\,|I_{mq}(\mathcal{J})|^2 \times \frac{1}{4}(1 \pm (-1)^m)^2. \tag{59}$$

Therefore, the storage mode (as any other regime offering a symmetrical $U(\vartheta)$) features a complete partition of noise effect between synchrotron frequency

harmonics of opposite parity: the amplitude noise excites the even multipoles m, while the phase noise affects the odd m's only. With this fact taken into account, it is convenient to put down eq.(52) as

$$D^{(a,\varphi)}(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\rm ext}}\right)^2 \sum_{j=-\infty}^{\infty} \mathcal{W}_m^{(a,\varphi)}(\mathcal{J})\, \mathcal{P}^{(a,\varphi)}(m\Omega_{\rm s}(\mathcal{J})), \tag{60}$$

$$m^{(a)} = 2j; \qquad m^{(\varphi)} = 2j+1.$$

Section 5 offers a similar presentation for the diffusion coefficient outside separatrices.

Eqs.(56), (57) for wide-band noise convert into

$$D^{(a,\varphi)}(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\rm ext}}\right)^2 \mathcal{P}^{(a,\varphi)} \mathcal{A}^{(a,\varphi)}(\mathcal{J}), \tag{61}$$

$$\mathcal{A}^{(a,\varphi)}(\mathcal{J}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\psi \left(\frac{\partial q\vartheta(\mathcal{J},\psi)}{\partial\psi}\right)^2 \times \begin{cases} \sin^2 q\vartheta(\mathcal{J},\psi), & (a) \\ \cos^2 q\vartheta(\mathcal{J},\psi). & (\varphi) \end{cases} \tag{62}$$

It is important that analysis of single-particle longitudinal dynamics in the storage mode can be carried out analytically in terms of elliptic functions [1]–[3]. Namely, the law of motion $\vartheta(\mathcal{E},t)$ along the phase-plane trajectory $H(\vartheta,\vartheta') = \mathcal{E}$, together with the essential dynamical functions $\Omega_{\rm s}(\mathcal{J})$ and $\mathcal{J}(\mathcal{E})$, can be presented parametrically via the functions of an independent variable $0 \le x \le 1$, where $x = \sin(\frac{1}{2}q\Delta\vartheta)$; $\Delta\vartheta$ is an oscillation amplitude along ϑ:

$$\mathcal{E}(x) = \frac{\Omega_0^2}{q^2} 2x^2, \tag{63}$$

$$\Omega_{\rm s}(x) = \Omega_0 \frac{\pi}{2{\rm K}(x)}, \tag{64}$$

$$\mathcal{J}(x) = \mathcal{J}_{\rm sep}\cdot({\rm E}(x) - (1-x^2){\rm K}(x)), \tag{65}$$

$$\mathcal{J}_{\rm sep} = \mathcal{J}(x=1) = \frac{8\Omega_0}{\pi q^2}, \tag{66}$$

$$\vartheta(\mathcal{E}(x),t) = \frac{2}{q}\arcsin\left(x\,{\rm sn}(\Omega_0 t, x)\right), \quad \vartheta(\mathcal{E}, t=0) = 0. \tag{67}$$

Action $\mathcal{J}_{\rm sep}$ is the properly scaled phase-plane area bounded by the separatrix (longitudinal acceptance).

On putting the law of motion (67) into eqs.(26), (59), one gets expressions for weight functions $\mathcal{W}_m(x)$ which coincide with the result of [1]–[3]:

$$\mathcal{W}_m^{(a,\varphi)}(x) = (1 \pm (-1)^m)^2 \left(\frac{\pi m}{2{\rm K}(x)}\right)^4 \times \tag{68}$$

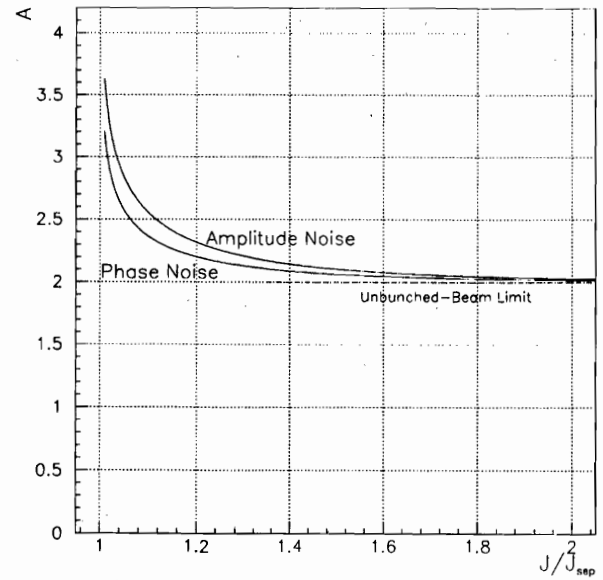

Figure 5. White-Noise amplitude functions (outer).

6. STATEMENT OF BOUNDARY-VALUE PROBLEM

To get a unique solution of the noise-induced diffusion equation

$$\frac{\partial\langle F_0\rangle(\mathcal{J},t)}{\partial t} = \frac{\partial}{\partial\mathcal{J}}\left(D(\mathcal{J})\frac{\partial\langle F_0\rangle(\mathcal{J},t)}{\partial\mathcal{J}}\right) \tag{111}$$

in the domain of $0 \le t < \infty$, $0 \le \mathcal{J} \le \mathcal{J}_{\rm max}$, one should complement it by initial and boundary conditions.

The initial condition is established straightforwardly. It is just a distribution of particles in a bunch $\langle F_0\rangle(\mathcal{J}, t=0)$, $0 \le \mathcal{J} \le \mathcal{J}_{\rm sep}$ taken at the initial instant of time $t = 0$.

To impose boundary conditions, it is convenient to introduce the concept of a diffusive flux

$$Q(\mathcal{J},t) = -D(\mathcal{J})\frac{\partial\langle F_0\rangle}{\partial\mathcal{J}}(\mathcal{J},t). \tag{112}$$

Then the first of boundary conditions reads

$$Q(\mathcal{J}=0,t) = 0, \tag{113}$$

which physically signifies the absence of any source of particles in the center of bucket.

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(As calculations show, limits (109), (110) do describe the noise-
ing-up of distribution in kinetic momentum of an unbunched beam
perturbations (50), (51).)
ction $\mathcal{J}(x)$ given by eq.(104) together with eqs.(106), (108) com-
the problem of calculating functions $\mathcal{W}_m(\mathcal{J})$, $\mathcal{A}(\mathcal{J})$, and diffusion
$(\mathcal{J})$ beyond stationary buckets.
s weight functions $2\mathcal{W}_m(\mathcal{J})$ (106) for some lower "multipoles".
e amplitude and phase noises merge and are practically undistin-
$|m| \geq 4$. Like in the region of $\mathcal{J} < \mathcal{J}_{\rm sep}$, all these functions are
vanish at $\mathcal{J} = \mathcal{J}_{\rm sep}$. As $|m|$ increases, functions $\mathcal{W}_m$ are localized
er layer beyond separatrix. Maximums of $\mathcal{W}_m$ and their coordi-
$| \gtrsim 6$) are defined by formulae which differ from eqs.(71), (72)
n index m for phase noise, and by 'plus' sigh put before figure
(72).

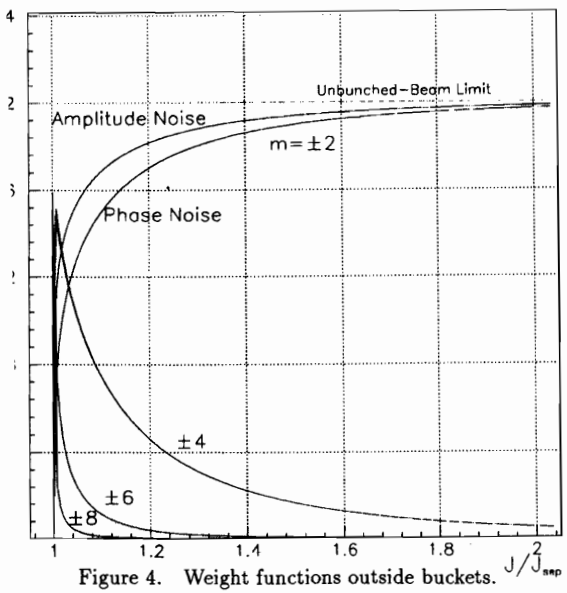

Figure 4. Weight functions outside buckets.

f amplitude functions $\mathcal{A}(\mathcal{J})$ (108) are presented in Fig.5. These
able logarithmic singularities at separatrix.

$$\times \begin{cases} \sinh^{-2}\left(\dfrac{\pi m}{2\mathrm{K}(x)}\mathrm{K}(\sqrt{1-x^2})\right), & (a) \\ \cosh^{-2} & (\varphi) \end{cases}$$

and amplitude functions $\mathcal{A}(x)$

$$\mathcal{A}^{(a,\varphi)}(x) = \frac{16}{\pi^2}x^2\mathrm{K}(x)\int_0^{\mathrm{K}(x)} du\ \mathrm{cn}^2(u,x)\times \tag{69}$$
$$\times \begin{cases} 4x^2\,\mathrm{sn}^2(u,x)\,\mathrm{dn}^2(u,x), & (a) \\ \left(1-2x^2\,\mathrm{sn}^2(u,x)\right)^2. & (\varphi) \end{cases}$$

Here $\mathrm{K}(x)$ and $\mathrm{E}(x)$ are complete elliptic integrals of the first and second kind, x is their modulus; $\mathrm{sn}(u,x)$, $\mathrm{cn}(u,x)$ and $\mathrm{dn}(u,x)$ are the Jacobian elliptic functions. Integration involved can be carried out analytically, which yields [2]

$$\mathcal{A}^{(a,\varphi)}(x) = \frac{16}{\pi^2}\mathrm{K}(x)\left(\frac{15\pm 1}{30}\left(\mathrm{E}(x)-(1-x^2)\,\mathrm{K}(x)\right)\mp \right. \tag{70}$$
$$\left. \mp\frac{4}{15}x^2(1-x^2)\left(2\mathrm{E}(x)-\mathrm{K}(x)\right)\right).$$

Function $\mathcal{J}(x)$ given by eq.(65), together with eqs.(68), (70), finally solves the problem of calculating functions $\mathcal{W}_m(\mathcal{J})$, $\mathcal{A}(\mathcal{J})$ and, hence, diffusion coefficients $D(\mathcal{J})$ for arbitrary amplitudes of oscillations inside a stationary separatrix.

Fig.1 plots weight functions $2\mathcal{W}_m(\mathcal{J})$ (68) of some lower multipoles. Notice, the effect of phase noise at harmonics $|m| = 3$ drastically enhances as $\mathcal{J} \gtrsim \frac{1}{2}\mathcal{J}_{\rm sep}$. All the weight functions are bounded and vanish at $\mathcal{J} = \mathcal{J}_{\rm sep}$. As $|m|$ increases, the $\mathcal{W}_m$-shapes do not differ qualitatively from $\mathcal{W}_{2,3}$. Rather, they are localized in a thinner layer near separatrix, and (for $|m| \gtrsim 6$) exhibit their maximums of

$$\left(2\mathcal{W}_m^{(a,\varphi)}\right)_{\max} \simeq \begin{cases} 1.60 & m = \text{even}, \quad (a) \\ 1.49 & m = \text{odd}, \quad (\varphi) \end{cases} \tag{71}$$

at points

$$\left(\frac{\mathcal{J}}{\mathcal{J}_{\rm sep}}\right)_{\max} \simeq 1 - \begin{cases} 10.3|m|\exp(-2.6|m|) & m = \text{even}, \quad (a) \\ 9.6\,|m|\exp(-2.4|m|) & m = \text{odd}. \quad (\varphi) \end{cases} \tag{72}$$

Plots of amplitude functions $\mathcal{A}(\mathcal{J})$ are shown in Fig.2. By virtue of factor $\mathrm{K}(x)$ in eq.(70) they diverge logarithmically near separatrix. The singularity is integrable.

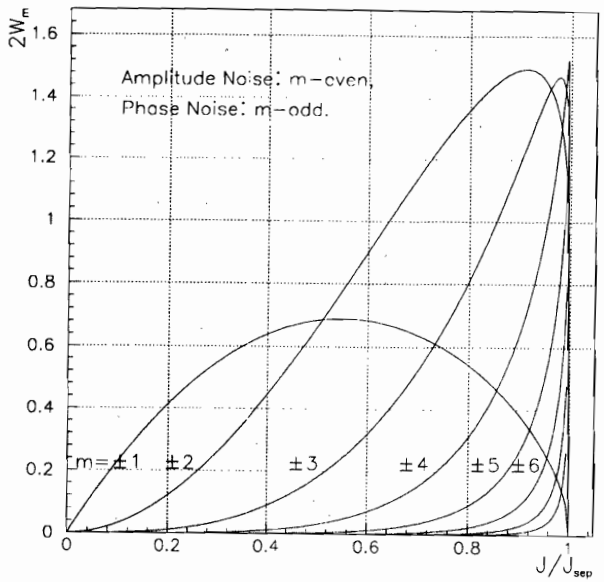

Figure 1. Weight functions inside buckets.

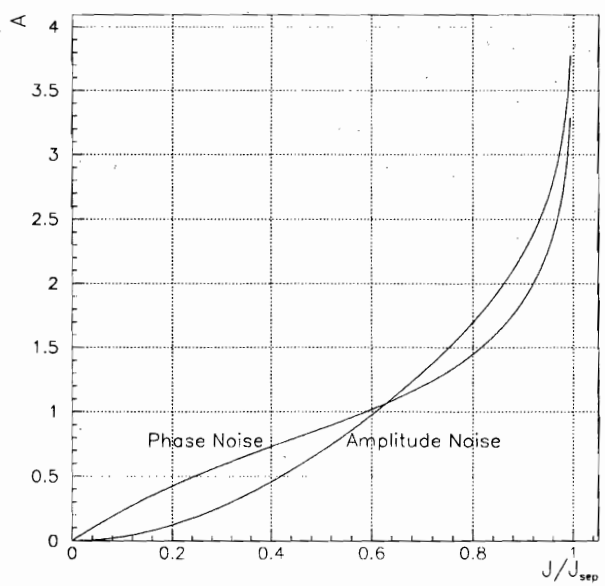

Figure 2. Amplitude functions inside buckets.

where $\mathrm{am}(u, x^{-1})$ is the Jacobian elliptic amplitude function. Parameter x is still proportional to $\sqrt{\mathcal{E}}$, but is no longer expressed through the amplitude of oscillations along ϑ. The modulus of the elliptic functions runs within the standard range: $0 \le x^{-1} \le 1$.

Weight functions $\mathcal{W}_m(x)$ beyond separatrix now read

$$\mathcal{W}_m^{(a,\varphi)}(x) = (1+(-1)^m)^2 \left(\frac{\pi m x}{2\mathrm{K}(x^{-1})}\right)^4 \times \times \begin{cases} \sinh^{-2}\left(\dfrac{\pi m}{2\mathrm{K}(x^{-1})}\mathrm{K}(\sqrt{1-x^{-2}})\right). & (a) \\ \cosh^{-2} & (\varphi) \end{cases} \tag{106}$$

As opposed to eq.(68), here index m runs over the even integers only.

Amplitude functions $\mathcal{A}(x)$ are as well expressed via definite integrals of elliptic functions

$$\mathcal{A}^{(a,\varphi)}(x) = \frac{16}{\pi^2}\mathrm{K}(x^{-1}) \int_0^{\mathrm{K}(x^{-1})} du\, \mathrm{dn}^2(u, x^{-1}) \times \times \begin{cases} 4\,\mathrm{sn}^2(u, x^{-1})\,\mathrm{cn}^2(u, x^{-1}), & (a) \\ \left(1 - 2\,\mathrm{sn}^2(u, x^{-1})\right)^2. & (\varphi) \end{cases} \tag{107}$$

On integrating, they become

$$\mathcal{A}^{(a,\varphi)}(x) = \frac{16}{\pi^2}\mathrm{K}(x^{-1})\left(\frac{15 \pm 1}{30}\mathrm{E}(x^{-1}) \mp \mp \frac{4}{15}(1-x^2)\left(2\,x^2\mathrm{E}(x^{-1}) + (1-2x^2)\,\mathrm{K}(x^{-1})\right)\right). \tag{108}$$

Amplitude functions have the common asymptotics at infinity:

$$\mathcal{A}^{(a,\varphi)}(x) \to 2; \qquad \mathcal{E}, x \to \infty. \tag{109}$$

These limits are entirely determined by weight functions $\mathcal{W}_m$ with $|m| = 2$. Indeed, it can readily be verified that

$$\mathcal{W}_m^{(a,\varphi)}(x) \to \delta_{|m|,2}; \qquad \mathcal{E}, x \to \infty. \tag{110}$$

The merging of asymptotes (a, φ) is quite explainable. As the energy of particles increases, azimuthal modulation in their velocities disappears. The particles move as in an unbunched beam. Naturally, they fail to distinguish phase shift of $\pi/2q$ in ϑ between harmonics (50), (51) of the amplitude and

$$m^{(a,\varphi)} = 2j.$$

Weight functions $\mathcal{W}_m = \mathcal{W}_{-m}$ of the "multipole" components of noise beyond buckets become

$$\mathcal{W}_m^{(a,\varphi)}(\mathcal{J}) = \frac{m^2}{4}\left|I^*_{mq}(\mathcal{J}) \pm I^*_{m,-q}(\mathcal{J})\right|^2. \tag{99}$$

Coefficients (92) are now lacking the property of eq.(58). Hence, weight functions of the even harmonics of the phase noise are not zeros at all, as contrasted to these inside separatrices. Irrespective of the noise type (either amplitude or phase), they are the weight functions of the odd harmonics that are equal to zero identically.

Of course, the very introduction of an (even) "multipole" m beyond separatrix is quite a conventional thing. Rather, it is a way to present wave-number $m_1 = \frac{1}{2}qm$ of the decomposition in phase harmonics $\sim \exp(im_1\tilde{\psi})$. If $\mathcal{E} \to \infty$ (an unbunched-beam approximation) $\tilde{\psi} \to \pm\vartheta$, see eq.(87). By virtue of assumptions (50), (51) on the nature of the noise voltages, they are capable of maintaining the sole azimuthal harmonics $\sim \exp(\pm iq\vartheta)$ of unbunched-beam perturbations. Whereof, at $\mathcal{E} \to \infty$ the effect of phase harmonics $|m_1| = q$ (i.e. of "multipoles" $|m| = 2$) should dominate, which is confirmed by later calculations, eq.(110).

The results for the (*white*) noise acquire the form

$$D^{(a,\varphi)}(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2V_{\text{ext}}}\right)^2 \mathcal{P}^{(a,\varphi)}\mathcal{A}^{(a,\varphi)}(\mathcal{J}), \tag{100}$$

$$\mathcal{A}^{(a,\varphi)}(\mathcal{J}) = \frac{1}{\pi}\int_0^\pi d\psi \left(\frac{\partial q\vartheta(\mathcal{J},\psi)}{\partial\psi}\right)^2 \times \begin{cases} \sin^2 q\vartheta(\mathcal{J},\psi), & (a) \\ \cos^2 q\vartheta(\mathcal{J},\psi), & (\varphi) \end{cases} \tag{101}$$

which coincides formally with eqs.(61) and (62).

Calculation of single-particle motion beyond stationary separatrices (within limits of an RF-imposed period) allows to present counterparts of eqs.(63)–(67) for the beam halo. These are given parametrically via functions of an independent variable $1 \le x < \infty$ ($x = 1$ at separatrix):

$$\mathcal{E}(x) = \frac{\Omega_0^2}{q^2}2x^2, \tag{102}$$

$$\Omega_s(x) = \Omega_0\frac{\pi x}{2\mathrm{K}(x^{-1})}, \tag{103}$$

$$\mathcal{J}(x) = \mathcal{J}_{\text{sep}}\, x\,\mathrm{E}(x^{-1}), \tag{104}$$

$$\vartheta(\mathcal{E}(x),t) = \frac{2}{q}\,\mathrm{am}\left(x\Omega_0 t, x^{-1}\right), \quad \vartheta(\mathcal{E}, t=0) = 0, \tag{105}$$

4. EVOLUTION OF SHORT BUNCHES

Diffusion Coefficient

Simplify the obtained equations for bunches occupying the central region in separatrix. Expand $\mathcal{J}(x)$, $\mathcal{W}_m(x)$ and $\mathcal{A}(x)$ in the Taylor series in powers of x, up to items $\sim x^4$:

$$\mathcal{J}(x) \simeq \frac{2\Omega_0}{q^2}\left(x^2 + \frac{x^4}{8}\right), \tag{73}$$

$$\begin{aligned} \mathcal{A}^{(a)}(x) &\simeq 2\mathcal{W}^{(a)}_{\pm 2}(x) \simeq 2\,x^4, \\ \mathcal{A}^{(\varphi)}(x) &\simeq 2\mathcal{W}^{(\varphi)}_{\pm 1}(x) \simeq 2\left(x^2 - \frac{5x^4}{8}\right). \end{aligned} \tag{74}$$

Invert these sums to get the functions of interest

$$\begin{aligned} \mathcal{A}^{(a)}(\mathcal{J}) &\simeq 2\mathcal{W}^{(a)}_{\pm 2}(\mathcal{J}) \simeq \frac{32}{\pi^2}\mathcal{J}'^2, \\ \mathcal{A}^{(\varphi)}(\mathcal{J}) &\simeq 2\mathcal{W}^{(\varphi)}_{\pm 1}(\mathcal{J}) \simeq \frac{8}{\pi}\left(\mathcal{J}' - \frac{3}{\pi}\mathcal{J}'^2\right), \end{aligned} \tag{75}$$

where to abbreviate the notations we have denoted $\mathcal{J}' = \mathcal{J}/\mathcal{J}_{\text{sep}}$.

Fig.3 compares these decompositions (dashed lines) with the exact curves by eqs.(68), (70) and (65) (solid lines). Notice, approximation (75) is quite acceptable for $\mathcal{J} \lesssim \frac{1}{2}\mathcal{J}_{\text{sep}}$. (As a reference, the value of $\mathcal{J} = \frac{1}{2}\mathcal{J}_{\text{sep}}$ is reached at the phase-plane trajectory with oscillation amplitude of $q\Delta\vartheta \simeq 99.30°$. This estimates the upper bound for the maximum phase half-length (at base) of a bunch which can yet be treated in a short-bunch approximation (75).)

Thus, with the noise being sufficiently wide-band, the motion of particles in the center of separatrix ($\mathcal{J} \lesssim \frac{1}{2}\mathcal{J}_{\text{sep}}$) is entirely governed either by the phase noise at the ± 1-st or by the amplitude noise at the ± 2-nd harmonics of synchrotron frequency. The diffusion coefficient reads

$$\begin{aligned} D^{(a)}(\mathcal{J}) &\simeq \frac{1}{2}\left(\frac{\Omega_0^2}{q^2V_{\text{ext}}}\right)^2 \mathcal{P}^{(a)}(2\Omega_s(\mathcal{J})) \times \frac{32}{\pi^2}\mathcal{J}'^2, \\ D^{(\varphi)}(\mathcal{J}) &\simeq \frac{1}{2}\left(\frac{\Omega_0^2}{q^2V_{\text{ext}}}\right)^2 \mathcal{P}^{(\varphi)}(\,\Omega_s(\mathcal{J})) \times \frac{8}{\pi}\left(\mathcal{J}' - \frac{3}{\pi}\mathcal{J}'^2\right). \end{aligned} \tag{76}$$

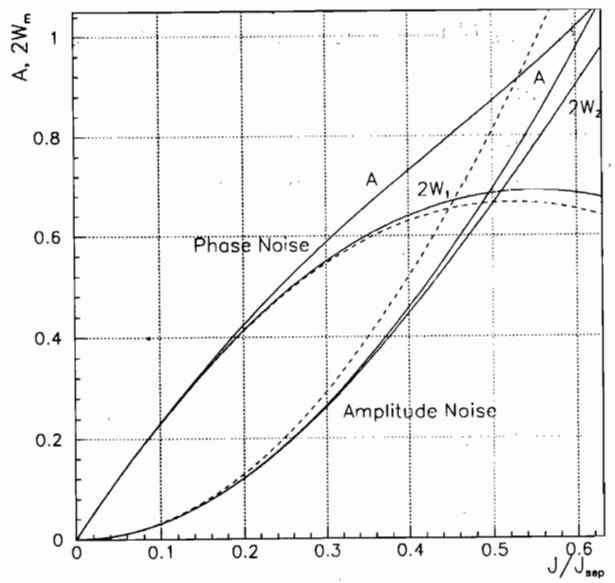

Figure 3. Qudratic approximation.

As the noise is assumed to be wide-band, substitute $\Omega_s(\mathcal{J}) \to \Omega_s(0) \equiv \Omega_0$ in the argument of its spectral density $\mathcal{P}(\Omega)$. Thus, the diffusion coefficient in the center of separatrix ($\mathcal{J} \lesssim \frac{1}{2}\mathcal{J}_{\rm sep}$) reduces to a quadratic function of variable $\mathcal{J}$:

$$D(\mathcal{J}) \simeq D_1\mathcal{J} + D_2\mathcal{J}^2. \tag{77}$$

Evolution of Longitudinal Emittance

It is quite evident, in treating the diffusion equation under this very assumption one cannot take into account the effect boundary conditions which are put at separatrix or beyond it. It is the onset of a short-bunch evolution which is the subject under study. Therefore, this analysis can well be carried out yet before a complete statement of the relevant boundary-value problem. It is suitable to employ the "moment-of-distribution" techniques. Its equation for the first moment is of the most practical significance.

Suppose that the following conditions are met at the boundary $\mathcal{J}_b(t)$ of a short bunch

$$\langle F_0\rangle(\mathcal{J}_b(t),t) = \frac{\partial\langle F_0\rangle}{\partial\mathcal{J}}(\mathcal{J}_b(t),t) = 0. \tag{78}$$

Introduce the natural definition of the distribution-averaged longitudinal emit-

continuation of the inside-bunch distribution $\langle F_0\rangle(\mathcal{J} < \mathcal{J}_{\rm sep}, t)$ over separatrix. (Though, of course, the distribution $\langle F_0\rangle(\mathcal{J} > \mathcal{J}_{\rm sep}, t)$ consists of particles which have never belonged to the bunch at issue, provided $M > 1$.)

Diffusion eq.(88) in terms of $\langle F_0\rangle(\mathcal{J},t)$ reads

$$\frac{\partial\langle F_0\rangle(\mathcal{J},t)}{\partial t} = \frac{\partial}{\partial\mathcal{J}}\left(D(\mathcal{J})\frac{\partial\langle F_0\rangle(\mathcal{J},t)}{\partial\mathcal{J}}\right). \tag{93}$$

On adopting all the above simplifications and substitutions of variables caused by periodicity, the diffusion coefficient

$$D(\mathcal{J} > \mathcal{J}_{\rm sep}) \equiv \frac{4}{q^2}\widetilde{D}\left(\widetilde{\mathcal{J}} = \frac{q\mathcal{J}}{2}\right)$$

beyond separatrices converts into

$$D(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2V_{\rm ext}}\right)^2 \sum_{j=-\infty}^{\infty} \mathcal{W}_m(\mathcal{J})\, P_v(m\Omega_s(\mathcal{J})) \geq 0, \tag{94}$$

$$\mathcal{W}_m(\mathcal{J}) = m^2q^2\left|\sum_{k=-\infty}^{\infty}\frac{I^*_{mk}(\mathcal{J})}{k}g_k\right|^2 = \tag{95}$$

$$= \left|\frac{1}{\pi}\int_0^{\pi}\frac{\partial q\vartheta(\mathcal{J},\psi)}{\partial\psi}g(\vartheta(\mathcal{J},\psi))\,\exp(-im\psi)\,d\psi\right|^2,$$

where $m = 2j$ is introduced to shorten the notations. Approximation (46) of wide-band (*white*) noise now results in

$$D(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2V_{\rm ext}}\right)^2 P_v\,\mathcal{A}(\mathcal{J}), \tag{96}$$

$$\mathcal{A}(\mathcal{J}) = \sum_{j=-\infty}^{\infty}\mathcal{W}_m(\mathcal{J}) = \tag{97}$$

$$= \frac{1}{\pi}\int_0^{\pi}d\psi\left(\frac{\partial q\vartheta(\mathcal{J},\psi)}{\partial\psi}\right)^2 g^2(\vartheta(\mathcal{J},\psi)).$$

Amplitude and Phase Noises

Put here particular perturbing voltages of eqs.(50), (51) to get the outer counterpart of eq.(60):

$$D^{(a,\varphi)}(\mathcal{J}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2V_{\rm ext}}\right)^2 \sum_{j=-\infty}^{\infty}\mathcal{W}_m^{(a,\varphi)}(\mathcal{J})\,\mathcal{P}^{(a,\varphi)}(m\Omega_s(\mathcal{J})), \tag{98}$$

with that of eqs.(40)–(42):

$$\widetilde{D}(\widetilde{\mathcal{J}}) = \frac{1}{2}\left(\frac{\Omega_0^2}{q^2 V_{\rm ext}\sin\varphi_{\rm s}}\right)^2 \sum_{m_1=-\infty}^{\infty} m_1^2 q^2 \times \tag{89}$$
$$\times \left|\sum_{k_1=-\infty}^{\infty} \frac{\tilde{I}^*_{m_1 k_1}(\widetilde{\mathcal{J}})}{k_1} g_{k_1}\right|^2 P_v\left(m_1\widetilde{\Omega}_{\rm s}(\widetilde{\mathcal{J}})\right) \geq 0,$$

$$\tilde{I}^*_{m_1 k_1}(\widetilde{\mathcal{J}}) = \frac{1}{2\pi}\int_0^{2\pi} d\tilde{\psi}\, \exp(ik_1\vartheta(\widetilde{\mathcal{J}},\tilde{\psi}) - im_1\tilde{\psi}). \tag{90}$$

Now it is convenient to introduce scaled phase variables of the orbital motion:

$$\frac{2\widetilde{\mathcal{J}}}{q} = \mathcal{J}, \quad \frac{q\tilde{\psi}}{2} = \psi, \quad \frac{q\widetilde{\Omega}_{\rm s}(\widetilde{\mathcal{J}})}{2} = \Omega_{\rm s}(\mathcal{J}).$$

At $\mathcal{J} = \mathcal{J}_{\rm sep}$ they go over continuously into variables $\mathcal{J}, \psi$ (up to a shift multiple to π) and frequency $\Omega_{\rm s}(\mathcal{J})$ of synchrotron oscillations.

Periodicity of unperturbed motion beyond stationary buckets manifests itself in

$$\vartheta(\mathcal{J},\psi+\pi) = \vartheta(\mathcal{J},\psi) + \frac{2\pi}{q}, \qquad \vartheta'(\mathcal{J},\psi+\pi) = \vartheta'(\mathcal{J},\psi).$$

Make use of this periodicity to pass over in eq.(90) to integration between limits of an RF-imposed period. As a result, get

$$\tilde{I}^*_{m_1 k_1}(\widetilde{\mathcal{J}}) = I^*_{2m_1/q,k_1}(\mathcal{J}) \times \sum_{p=-\infty}^{\infty} \delta_{k_1, m_1+pq}\,. \tag{91}$$

Here $\delta_{kk'}$ denotes the Kronecker's delta symbol, and the continuation of function $I^*_{mk}(\mathcal{J})$, eq.(26), over separatrix is introduced by

$$I^*_{2m_1/q,k_1}(\mathcal{J}) = \frac{1}{\pi}\int_0^{\pi} d\psi\, \exp\left(ik_1\vartheta(\mathcal{J},\psi) - i\frac{2m_1}{q}\psi\right). \tag{92}$$

Azimuthal wave-shape $g(\vartheta)$ of perturbation (39) has, by assumption, a period of $2\pi/q$. Hence, coefficients g_{k_1} are non-zero only for those azimuthal harmonics whose indices are $k_1 = lq$; $l = \pm1, \pm2, \ldots$ Hereof, and by virtue of identity (91), it becomes clear that sum (89) retains the phase harmonics with their indices $m_1 = jq$ only, where $j = \pm1, \pm2, \ldots$

As is evident, the $(2\pi/q)$-peiodicity of the orbital motion allows to introduce distribution $\langle F_0\rangle(\mathcal{J},t) \sim \langle\widetilde{F}_0\rangle(\widetilde{\mathcal{J}} = q\mathcal{J}/2, t)$ averaged over phase ψ within limits of an RF-imposed period. This function may conventionally be treated as a

tance of bunch. It is just the first moment of the distribution involved

$$\overline{\mathcal{J}}(t) = \int_0^{\mathcal{J}_{\rm b}(t)} \mathcal{J}\,\langle F_0\rangle(\mathcal{J},t)\,d\mathcal{J}. \tag{79}$$

Here, the account is taken of the fact that by initial normalization (16) and definition (23)

$$\int_0^{\mathcal{J}_{\rm b}(t)} \langle F_0\rangle(\mathcal{J},t)\,d\mathcal{J} = 1. \tag{80}$$

Differentiate eq.(79) with respect to time and replace derivative $\partial\langle F_0\rangle/\partial t$ by the r.h.s. of diffusion eq.(37). On inserting $D(\mathcal{J})$ from eq.(77), carry out two-fold integration by parts. Impose boundary conditions (78). As a result, get a closed equation in terms of $\overline{\mathcal{J}}$ which describes evolution of longitudinal emittance:

$$\frac{d\overline{\mathcal{J}}(t)}{dt} = D_1 + 2D_2\overline{\mathcal{J}}(t). \tag{81}$$

It can readily be solved to yield

$$\overline{\mathcal{J}}(t) = \left(\overline{\mathcal{J}}(0) + \frac{D_1}{2D_2}\right)\exp(2D_2\,t) - \frac{D_1}{2D_2}. \tag{82}$$

Whereof,

$$\left(\frac{\overline{\mathcal{J}}(t)}{\mathcal{J}_{\rm sep}}\right)^{(a)} = \left(\frac{\overline{\mathcal{J}}(0)}{\mathcal{J}_{\rm sep}}\right)\exp\left(+\frac{1}{2}\frac{\Omega_0^2\mathcal{P}^{(a)}(2\Omega_0)}{V_{\rm ext}^2}t\right);$$
$$\left(\frac{\overline{\mathcal{J}}(t)}{\mathcal{J}_{\rm sep}}\right)^{(\varphi)} = \frac{\pi}{6} + \left(\frac{\overline{\mathcal{J}}(0)}{\mathcal{J}_{\rm sep}} - \frac{\pi}{6}\right)\exp\left(-\frac{3}{8}\frac{\Omega_0^2\mathcal{P}^{(\varphi)}(\Omega_0)}{V_{\rm ext}^2}t\right). \tag{83}$$

The second of eqs.(83) implicates a seeming effect of the average longitudinal emittance to reach the steady-state value of $\overline{\mathcal{J}}/\mathcal{J}_{\rm sep} = \frac{\pi}{6} \simeq \frac{1}{2}$ as $t \to \infty$. Nevertheless, this conclusion obviously falls beyond applicability range of the short-bunch approximation. Indeed, realistic distributions show $\overline{\mathcal{J}}(t) \simeq (\frac{1}{3} \div \frac{1}{2})\mathcal{J}_{\rm b}(t)$, while by assumption (75) there should be $\mathcal{J}_{\rm b}(t) \lesssim \frac{1}{2}\mathcal{J}_{\rm sep}$. Therefore, eqs.(83) hold true until $\overline{\mathcal{J}}(t) \lesssim (\frac{1}{6} \div \frac{1}{4})\mathcal{J}_{\rm sep}$.

Suppose, quadratic term in diffusion coefficient $D^{(\varphi)}(\mathcal{J})$ from eq.(77) was ignored from the very beginning. Then, instead of the second of eqs.(83), one would have got its asymptote which is linear in $\overline{\mathcal{J}}(0), t \to 0$:

$$\left(\frac{\overline{\mathcal{J}}(t)}{\mathcal{J}_{\rm sep}}\right)^{(\varphi)} = \left(\frac{\overline{\mathcal{J}}(0)}{\mathcal{J}_{\rm sep}}\right) + \frac{\pi}{16}\frac{\Omega_0^2\mathcal{P}^{(\varphi)}(\Omega_0)}{V_{\rm ext}^2}t. \tag{84}$$

Take account of identities

$$\left(\frac{\overline{\mathcal{J}}(t)}{\mathcal{J}_{\rm sep}}\right) \simeq \frac{\pi}{16}\overline{(q\Delta\vartheta)^2_{\rm b}}(t); \qquad \frac{\Omega_0^2\mathcal{P}^{(\varphi)}(\Omega_0)}{V^2_{\rm ext}} = \Omega_0^2 P_{\Delta\varphi}(\Omega_0) = P_{\Delta\omega_{\rm rf}}(\Omega_0),$$

where $\overline{(q\Delta\vartheta)^2_{\rm b}}$ is a square of the bunch r.m.s. half-length measured in the RF-radian units; $P_{\Delta\omega_{\rm rf}}(\Omega)$ is the spectral density of RF-frequency noise. In passing to the last of these identities, the account is taken of a common relation $\Omega^2 P_a(\Omega) = P_b(\Omega)$ between spectral densities of random processes $a(t)$ and $b(t) = da(t)/dt$. Hence, eq.(84) entails

$$\overline{(q\Delta\vartheta)^2_{\rm b}}(t) = \overline{(q\Delta\vartheta)^2_{\rm b}}(0) + P_{\Delta\omega_{\rm rf}}(\Omega_0)\, t, \tag{85}$$

which is a "standard" diffusion law: the bunch size varies as $\sqrt{t}$. It is a classical result for harmonic oscillations cited as early as in ref.[5]. Eq.(83) for the phase noise offers a somewhat lower growth rate of longitudinal emittance which is caused by a fall-off of the diffusion coefficient at larger amplitudes.

A more detailed application of this — the "moment-of-distribution" — approach to solving diffusion equation with D-coefficient of eq.(77) extended over the entire half-infinite straight-line segment $0 \le \mathcal{J} \le \infty$ can be found in ref.[3]. The same paper as well offers fundamental solutions (the Green functions) for $D(\mathcal{J}) \sim \mathcal{J}$ and $\mathcal{J}^2$.

5. MOTION BEYOND STATIONARY SEPARATRICES

Diffusion Equation

Consider a storage operation mode. Phase-plane (ϑ, ϑ') beyond separatrices is split into two disjoint half-planes. The closed phase trajectories running there correspond to rotational (orbital) motion. On abandoning stationary buckets, the off-set particles would proceed to a periodic motion in the outer vicinity of separatrices building up the beam halo. This situation should be taken into account in the formulation of the boundary-value problem of the noise-induced diffusion.

Apply eqs.(12), (13) to untrapped particles moving outside stationary buckets. Use former notations to denote all the quantities and functions which can be interpreted as continuations beyond separatrix of those introduced so far.

Take $(\tilde\psi, \tilde{\mathcal{J}})$ as angle-action variables of the orbital motion:

$$\tilde{\mathcal{J}}(\mathcal{E}) = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta\, \sqrt{2\left(\mathcal{E} - \Omega_0^2 U(\vartheta)\right)} \overset{\mathcal{E}\to\infty}{\to} |\vartheta'|, \tag{86}$$

$$\frac{d\tilde\psi}{dt} = \frac{\partial\mathcal{E}}{\partial\tilde{\mathcal{J}}} = \tilde\Omega_{\rm s}(\mathcal{E}) \overset{\mathcal{E}\to\infty}{\to} |\vartheta'|; \qquad \frac{d\tilde{\mathcal{J}}}{dt} = -\frac{\partial\mathcal{E}}{\partial\tilde\psi} = 0. \tag{87}$$

The $\tilde\psi$-phase shift equal to $+2\pi$ corresponds to a complete turn around the orbit in a co-rotating frame (either in direction of the beam motion in half-plane $\vartheta' > 0$, or oppositely so if $\vartheta' < 0$). Moreover, as $\mathcal{E} \to \infty$ phase $\tilde\psi$ converts (up to its sign) into azimuth ϑ of a co-rotating coordinate frame. Thus, a natural transformation is provided of a multipole decomposition inside bunch into a plane-wave expansion peculiar to an unbunched beam.

There exists a formal analogy between a periodic motion of particles along phase-plane trajectories closed inside separatrices, and a periodic motion of the untrapped particles which follow the outer phase-plane trajectories closed orbitally in the storage mode. Hence, any further speculations would have been essentially just a word-by-word repetition of the scheme presented in Section 3 for the inside-bucket motion. (Angle-action pair $(\psi, \mathcal{J})$ of synchrotron oscillations is juxtaposed to variables $(\tilde\psi, \tilde{\mathcal{J}})$ of orbital motion, etc.)

The most dramatic simplifications are involved in passing from eq.(28) to eq.(29). Formerly, it was sufficient to introduce an assumption of no multipole modulation (ψ-uniformity) of a bunch at the initial moment of time. In many cases this suggestion does not conflict with the real situation.

However, $\tilde\psi$-uniformity of a beam-halo distribution also implies the beam to be made of $M = q$ identical bunches which suffer the similar noise effect. The latter is readily provided by a periodicity condition of eq.(39). The absence of empty buckets in the beam is a more serious simplification which, nevertheless, has a wide range of application. Withdrawal of this assumption results in an infinite set of partial differential equations for amplitudes of harmonics ($\sim \exp(im_1\tilde\psi)$) of the distribution function. This set is not amenable to a straightforward solution. (A similar problem arises in an attempt to study the noise effect on an un-matched bunch which exhibits a strong multipole perturbation.)

It is in a practically important case of $M = q$ identical bunches that only one closed equation would suffice. This is a diffusion equation for distribution function $\langle\tilde F_0\rangle(\tilde{\mathcal{J}}, t)$ averaged over angle variable $\tilde\psi$ (i.e. over the ring orbit):

$$\frac{\partial\langle\tilde F_0\rangle(\tilde{\mathcal{J}}, t)}{\partial t} = \frac{\partial}{\partial\tilde{\mathcal{J}}}\left(\tilde D(\tilde{\mathcal{J}})\frac{\partial\langle\tilde F_0\rangle(\tilde{\mathcal{J}}, t)}{\partial\tilde{\mathcal{J}}}\right). \tag{88}$$

Let us again restrict our consideration to perturbations of eqs.(39)-type. Then diffusion coefficient $\tilde D(\tilde{\mathcal{J}})$ acquires the form which coincides formally