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Discrete Coherent States and Probability Distributions in Finite-Dimensional Spaces

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Abstract

Operator bases are discussed in connection with the construction of phase space representatives of operators in finite-dimensional spaces and their properties are presented. It is also shown how these operator bases allow for the construction of a finite harmonic oscillator-like coherent state. Creation and annihilation operators for the Fock finite-dimensional space are discussed and their expressions in terms of the operator bases are explicitly written. The relevant finite-dimensional probability distributions are obtained and their limiting behavior for an infinite-dimensional space are calculated which agree with the well known results.

1 Introduction

An elegant mathematical way of treating quantum systems characterized by a finite number of states has long been proposed by Weyl in his description of the quantum kinematics as an Abelian group of ray rotations in the

system space[1]. One of the important results of this approach, in the particular case of finite spaces, is that pairs of unitary rotation operators obey special commutation relations –entailing roots of unity– which are the unitary counterparts of the fundamental Heisenberg relations. From a mathematical point of view, the ray representation of the Abelian group of rotations can be associated to the representations of generalized Clifford algebra and, in this connection, some physical problems have been extensively studied by Ramakrishnan and coworkers in this context [2]. On the other hand, the same results have also been obtained in a different approach by Schwinger [3] who has shown in a particularly transparent way that a set of unitary operators, defined through cyclic permutations of finite state vector spaces coordinate systems, can be constructed such that they will be identified as the generators of a complete operator basis and that will also obey the same commutation relations obtained by Weyl. As a basis, this set will suffice to construct all possible quantities related to the physical system characterized by those finite state vector spaces.

Although Weyl and Schwinger have shown that in the case of a system exhibiting an infinite number of states one gets back the well-known pair of canonical complementary quantum variables with continuous spectra, it has been also emphasized that the operator bases are of special interest for the description of any physical system characterized by discrete finite state vector spaces. In this sense, this approach permits one to treat in the same footing quantum systems described by a two-dimensional space (spin one-half) as well as the canonical q - p variables.

In recent years some authors have addressed the problem of discussing finite-dimensional quantum mechanics [4] and the related commutation relations associated to the pair of variables playing the role of coordinate and momentum. The problem of finite-dimensional spaces has been also treated alternatively by an approach which aims the construction of finite-dimensional phase spaces, which are connected to the original vector state spaces through a discrete Weyl-Wigner transformation [5, 6]. In this approach it was shown that the Schwinger unitary operators are of fundamental importance so as to define operator bases in the related operator spaces and, from this starting point, new bases were constructed such that they directly allow the mapping of operators to functions of discrete c-numbers, which are now the representatives of the operators in the discrete phase spaces. One of the important features of this phase space description is that it can also accomodate the dis-

crete cases without classical counterparts, as well as the continuous canonical case[5] and, in this form, it is possible to represent any physical system by its operator basis which is now dependent on the dimension of the space N. The expressions for the phase space representatives then constitute a powerful tool to the discussion of the behavior of quantum observables in the limit of large N without any direct reference to the underlying finite Hilbert space. In this connection, the classical limit can also be discussed as a special case; in particular this limit was studied in a soluble many-fermion model, namely, the Lipkin model[7], through the use of discrete angular momentum-angle variables[8]. It is important to stress that in this finite-dimensional phase space description, the Wigner function, being the Weyl-Wigner representative of the density operator, naturally emerges as a function which assumes values only at the points defining the discrete mesh of the phase space although preserving all the properties that make it a special object of quantum mechanics; in this sense it is always possible to construct a discrete Wigner function in the finite-dimensional phase spaces once the density operator is given. This kind of generalization for the Wigner function has been discussed also by Wooters in another context [9].

In the present paper we intend to show how the discrete phase space formalism can be used to discuss the connection between the defined finite discrete harmonic oscillator coordinate and momentum states and a finite space of Fock states; using the Schwinger construction scheme, a finite set of angle states also appears which are related to the finite Fock states by a discrete Fourier transformation. The previously defined harmonic oscillator-like coherent states, written in terms of the finite discrete coordinate and momentum variables[6], are reexamined in the present context and some important properties are discussed. The results thus obtained for the finite discrete Fock and angle states (and their connections to the finite discrete coordinate and momentum states) clearly indicate that the discrete phase space formalism is not only a valuable tool for the treatment of quantum systems described finite number of states but, in particular, it is also suitable for the discussion of the related Pegg and Barnett scheme of treating quantum optics [10]. In order to establish the basic results to construct the grounds for a future more detailed comparison we have also deduced the associated quantum probability distributions and we have discussed their continuum limits.

Our paper is organized as follows. In section 2 we present a review of

the fundamental concepts of operator bases and the transformations which allows for the phase space representation for finite dimensional cases. The discrete coherent states and some of their important properties are presented in section 3 where also the related creation and annihilation operators are presented. The probability distributions are presented in section 4 together with the Wigner function associated to the n-th finite Fock space state in the finite coordinate and momentum phase space representation. In section 5 we draw attention to the continuum limit which is obtained for large N for previously discussed distribution functions and in section 6 we present our conclusions. Finally in the Appendix we discuss some finite dimensional results which are related, in the continuum limit, to the finite version of the dispersion relations and we exhibit some interesting features which are directly connected to the finite character of the state space.

2 Operator Bases and Discrete Phase Spaces

As has been already shown [5, 6], when one is given a N-dimensional state vector space it is always possible to construct a finite discrete phase space associated to the physical system described by the starting state space. To this construction, it is of fundamental importance to have an operator basis in terms of which the decomposition of any operator acting on the state space can be directly given, being the coefficient function identified as the discrete Weyl-Wigner transform of that operator [5, 6]. Since it was also shown that the Schwinger unitary operator formalism [3] is the starting point for that description, it is important to present a brief review of its main aspects.

Let us consider the Schwinger unitary operators $\mathbf{U}$ and $\mathbf{V}$ defined by the general properties [3]

$$\mathbf{U}^k \mid u_j\rangle = u_j^k \mid u_j\rangle \ , \ u_j = \exp\left(\frac{2\pi i}{N} j\right) \ , \tag{1}$$

$$\mathbf{V}^k \mid v_l\rangle = v_l^k \mid v_l\rangle \ , \ v_l = \exp\left(\frac{2\pi i}{N} l\right) \ , \tag{2}$$

$$\mathbf{U}^k \mid v_l\rangle = \mid v_{l+k}\rangle \ , \tag{3}$$

$$\mathbf{V}^k \mid u_j\rangle = \mid u_{j-k}\rangle \ , \tag{4}$$

$$\mathbf{U}^N \mid v_l\rangle = \mid v_l\rangle \tag{5}$$

$$\mathbf{V}^N \mid u_j\rangle = \mid u_j\rangle \tag{6}$$

where the sets of orthonormal eigenvectors $\{\mid u_j\rangle\}$ and $\{\mid v_l\rangle\}$ are related to each other by the Fourier coefficients

$$\langle u_j \mid v_l\rangle = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} jl\right) . \tag{7}$$

The discrete labels j and l assume integer values in the definition domain of the state space which may assume values depending on the symmetry properties of the physical interpretation we assign to the labels. In the general discussion that follows, we will not especify the range of the labels unless necessary, being, however, the essentials of the related calculations independent of the particular choice.

The unitary operators $\mathbf{U}$ and $\mathbf{V}$ obey the fundamental relations

$$\mathbf{U}^N = 1 \ , \ \mathbf{V}^N = 1 \tag{8}$$

and

$$\mathbf{V}^l \mathbf{U}^j = \exp\left(\frac{2\pi i}{N} jl\right) \mathbf{U}^j \mathbf{V}^l \ , \ (j, l = 0, 1, ..., N-1) \ , \tag{9}$$

which show that they obey a generalized Clifford algebra [2]. The relevance of these ciclic relabelling unitary operators in the description of the physical system is that, as has also been shown by Schwinger, the set of N^2 operators

$$\mathbf{S}(j,l) = \frac{1}{\sqrt{N}} \exp\left(\frac{i\pi}{N} jl\right) \mathbf{U}^j \mathbf{V}^l = \frac{1}{\sqrt{N}} \exp\left(\frac{-i\pi}{N} jl\right) \mathbf{V}^l \mathbf{U}^j \tag{10}$$

built upon them constitute an operator basis in the operator space associated to the system described by any one of the two sets of N vector states $\mid u_j\rangle$ or $\mid v_l\rangle$. This symmetrized basis has some interesting properties which follow directly from those associated to the individual unitary operators, namely

i) Identity element

$$\sqrt{N}\ \mathbf{S}(0,0) = \mathbf{1} \tag{11}$$

ii) Inverse element

$$\mathbf{S}^\dagger(j,l) = \mathbf{S}^{-1}(j,l) = \mathbf{S}(-j,-l) \Rightarrow \left(\sqrt{N}\ \mathbf{S}(j,l)\right)\left(\sqrt{N}\ \mathbf{S}^\dagger(j,l)\right) = \mathbf{1} \tag{12}$$

iii) Behavior under a similarity transformation

$$\left(\sqrt{N}\ \mathbf{S}(j,l)\right) \mathbf{S}(r,s) \ \left(\sqrt{N}\ \mathbf{S}^\dagger(j,l)\right) = \exp\left[-\frac{2\pi i}{N}(js - lr)\right] \ \mathbf{S}(r,s) \tag{13}$$

iv) Associativity

$$[\mathbf{S}(j,l)\ \mathbf{S}(r,s)]\ \mathbf{S}(m,n) = \mathbf{S}(j,l)\ [\mathbf{S}(r,s)\ \mathbf{S}(m,n)] \tag{14}$$

v) Semi-periodicity

$$\mathbf{S}(N,p) = (-1)^p\ \mathbf{S}(0,p) \tag{15}$$

$$\mathbf{S}(p,N) = (-1)^p\ \mathbf{S}(p,0) \tag{16}$$

$$\sqrt{N}\ \mathbf{S}(N,N) = (-1)^N\ \mathbf{1} \tag{17}$$

vi) Action on the kets

$$\mathbf{S}(j,l) \mid u_k\rangle = \frac{1}{\sqrt{N}} \exp\left[\frac{2\pi i}{N} j\left(k - \frac{l}{2}\right)\right] \mid u_{k-l}\rangle \tag{18}$$

$$\mathbf{S}(j,l) \mid v_k\rangle = \frac{1}{\sqrt{N}} \exp\left[\frac{2\pi i}{N} l\left(k + \frac{j}{2}\right)\right] \mid v_{k+j}\rangle \tag{19}$$

vii) Product of T basis elements

$$\prod_{k=1}^{T} \mathbf{S}(j_k, l_k) = \frac{1}{\sqrt{N^T}} \exp\left(\frac{\pi i}{N} \sum_{k=1}^{T} j_k l_k\right) \prod_{k=1}^{T} \mathbf{U}^{j_k} \mathbf{V}^{l_k}$$

$$= \frac{1}{\sqrt{N^{T-1}}} \exp\left[\frac{\pi i}{N}\left(\sum_{k=1}^{T-1} l_k \sum_{p=k+1}^{T} j_p - \sum_{k=1}^{T-1} j_k \sum_{p=k+1}^{T} l_p\right)\right] \ \mathbf{S}\left(\sum_{k=1}^{T} j_k, \sum_{k=1}^{T} l_k\right) , \tag{20}$$

where use was made of the relation

$$\prod_{k=1}^{T} \mathbf{U}^{j_k}\mathbf{V}^{l_k} = \mathbf{U}^{\sum_{k=1}^{T} j_k}\, \mathbf{V}^{\sum_{k=1}^{T} l_k} \exp\left(\frac{2\pi i}{N}\sum_{k=1}^{T-1} l_k \sum_{p=k+1}^{T} j_p\right)$$

$$= \mathbf{V}^{\sum_{k=1}^{T} l_k}\, \mathbf{U}^{\sum_{k=1}^{T} j_k} \exp\left(-\frac{2\pi i}{N}\sum_{k=1}^{T} j_k \sum_{p=k}^{T} l_p\right) . \tag{21}$$

Now, it can be verified by direct inspection that this basis $\mathbf{S}(j,l)$ is invariant under the substitutions $\mathbf{U} \to \mathbf{V}$ and $\mathbf{V} \to \mathbf{U}^{-1}$ together with $j \to l$ and $l \to -j$, but, as has been recently shown [6], it is not invariant under a global mod N transformation. Thus, in order to implement this further invariance in a simple form, and for reasons that will be clarified below, it has been shown that a new basis can be rewritten as the Fourier transform of the original Schwinger basis [6]

$$\mathbf{G}(r,s) = \sum_{j,l} \frac{\mathbf{S}(j,l)}{N} \exp\left[i\pi\phi(j,l;N)\right] \exp\left[-\frac{2\pi i}{N}(rj+sl)\right] . \tag{22}$$

with the additional phase

$$\phi(j,l;N) = N\, I_j^N\, I_l^N - j I_l^N \; - l I_j^N \tag{23}$$

which guarantees the mod N invariance of the operator basis being

$$I_r^N = \left[\frac{r}{N}\right] \tag{24}$$

the integral part of r with respect to N.

The main advantage of using this operator basis, instead of Schwinger's, is that this new version can be directly used in the mapping of quantum operators – acting on the particular N - dimensional vector states space – onto functions of discrete c-numbers in a direct way through a trace operation. These functions are then the representatives of the operators in a discrete phase space characterized by a pair of discrete c-numbers for each degree of freedom of the physical system. In this way, the discrete Weyl-Wigner transformation of operators acting on finite-dimensional vector state spaces is given by

$$\mathbf{O} = \frac{1}{N}\sum_{r,s} O(r,s)\mathbf{G}(r,s) , \tag{25}$$

where, $O(r,s)$, the Weyl-Wigner equivalent of $\mathbf{O}$ in the discrete phase space, is given by the trace of that operator with the operator basis $\mathbf{G}(r,s)$

$$O(r,s) = Tr\left[\mathbf{G}^{\dagger}(r,s)\mathbf{O}\right] . \tag{26}$$

As was shown, this operator basis plays a fundamental role in the mapping procedure of quantum operators to their discrete Weyl-Wigner phase space representatives, so it is also useful to list some of its properties.

Due to the symmetric character of the underlying Schwinger basis $\mathbf{S}(j,l)$ discussed above, one gets

$$i)\ \mathbf{G}(r,s) = \mathbf{G}(s,-r) . \tag{27}$$

By a direct calculation one can also see that

$$ii)\ Tr\left[\mathbf{G}(r,s)\right] = 1 \tag{28}$$

$$iii)\ Tr\left[\mathbf{G}^{\dagger}(r,s)\mathbf{G}(a,b)\right] = N\delta_{r,a}\delta_{s,b} \tag{29}$$

and its generalization for the product of M operators

$$iv)\ Tr\left[\mathbf{G}^{\dagger}(r,s)\prod_{k=1}^{M}\mathbf{G}(a_k,b_k)\right] = \frac{1}{N^M}\sum_{\{j_k\}}\sum_{\{l_k\}} \exp\left[i\pi\Phi\left(\{j_k\},\{l_k\};N\right)\right]$$

$$\times \exp\left\{\frac{i\pi}{N}\left[\left(\sum_{k=1}^{M} j_k\right)\left(\sum_{k=1}^{M} l_k\right) + \sum_{k=1}^{M} j_k\left(l_k - 2\sum_{p=k}^{M} l_p\right)\right]\right\}$$

$$\times \exp\left\{\frac{2\pi i}{N}\sum_{k=1}^{M}\left[j_k\left(r-a_k\right) + l_k\left(s-b_k\right)\right]\right\} \tag{30}$$

where

$$\{j_k\} = j_1,\ldots,j_M\ ;\ \{l_k\} = l_1,\ldots l_M \tag{31}$$

and

$$\Phi\left(\{j_k\},\{l_k\};N\right)=\left(\sum_{k=1}^{M} j_k\right) I^{N}_{\sum_{k=1}^{M} l_k}+\left(\sum_{k=1}^{M} l_k\right) I^{N}_{\sum_{k=1}^{M} j_k}-N\, I^{N}_{\sum_{k=1}^{M} j_k}\, I^{N}_{\sum_{k=1}^{M} l_k}\,. \tag{32}$$

This result is directly related to the mapping of the product of M operators, as can be seen from Eq.(26), using Eq.(25)

$$\left(\prod_{k=1}^{M}\mathbf{O}_k\right)(r,s)=Tr\left[\mathbf{G}^{\dagger}(r,s)\prod_{k=1}^{M}\mathbf{O}_k\right]$$

$$=\frac{1}{N^M}\sum_{\{a_k\}}\sum_{\{b_k\}} O_1(a_1,b_1)...O_M(a_M,b_M)Tr\left[\mathbf{G}^{\dagger}(r,s)\prod_{k=1}^{M}\mathbf{G}(a_k,b_k)\right] \tag{33}$$

which is the discrete Weyl-Wigner representative of the product of M operators. It is also important to observe the presence of the phase Φ that guarantees the mod N invariance in the general mapping.

As a direct and fundamental result of these properties we exhibit the phase space representative of the commutator of two operators

$$\left([\mathbf{O}_1,\mathbf{O}_2]\right)(q,p)=2i\sum_{u,v,r,s}\sum_{a,b,c,d}\Gamma(u,v,r,s,a,b,c,d;N)\sin\left[\frac{\pi}{N}(bc-ad)\right] \tag{34}$$

where

$$\Gamma(u,v,r,s,a,b,c,d;N)=\frac{1}{N^4}O_1(u,v)O_2(r,s)\exp\left[i\pi\Phi(a,b,c,d;N)\right]$$

$$\times\exp\left\{\frac{2\pi i}{N}\left[a(q-u)+b(p-v)+c(q-r)+d(p-s)\right]\right\}\,. \tag{35}$$

Similarly, the discrete phase space expression for the anticommutator is written as

$$\left(\{\mathbf{O}_1,\mathbf{O}_2\}\right)(q,p)=2\sum_{u,v,r,s}\sum_{a,b,c,d}\Gamma(u,v,r,s,a,b,c,d;N)\cos\left[\frac{\pi}{N}(bc-ad)\right]\,. \tag{36}$$

It is important to note that these expressions already show the embryonic structure of the continuous sine and cosine functions as they must appear in the continuous Weyl-Wigner phase space descrition [11]. Here those functions are to be calculated only at the points associated to the discrete phase space labels. In fact, it has been shown that this discrete phase space is already endowed with a presymplectic structure of geometrical origin which manifests itself through the presence of the cosine function and that is responsible for the Poisson-like structure of the mapped commutator expression [12].

Another set of fundamental operators can also be constructed which act on the spaces of the orthonormal eigenvectors $\{|\,u_j\rangle\}$ and $\{|\,v_l\rangle\}$, namely the projectors $\mathbf{\Pi}(u_j,u_j)$, $\mathbf{\Pi}(v_l,v_l)$ within the spaces, the operators associated to transitions $\mathbf{\Pi}(u_j,u_k)$ and $\mathbf{\Pi}(v_l,v_k)$ and the operators that connect the two spaces $\mathbf{\Pi}(u_j,v_l)$. These operators can be expanded in the operator basis $\mathbf{S}(r,s)$ thus giving

$$\mathbf{\Pi}(u_j,u_j)=|\,u_j\rangle\langle u_j\,|=\frac{1}{N}\sum_r\exp\left(-\frac{2\pi i}{N}rj\right)\mathbf{U}^r=\frac{1}{\sqrt{N}}\sum_r u_r^{-j}\mathbf{S}(r,0)\,, \tag{37}$$

$$\mathbf{\Pi}(v_l,v_l)=|\,v_l\rangle\langle v_l\,|=\frac{1}{N}\sum_s\exp\left(-\frac{2\pi i}{N}sl\right)\mathbf{V}^s=\frac{1}{\sqrt{N}}\sum_s v_s^{-l}\mathbf{S}(0,s)\,, \tag{38}$$

$$\mathbf{\Pi}(u_j,u_k)=|\,u_j\rangle\langle u_k\,|=\frac{1}{N}\sum_r\exp\left(-\frac{2\pi i}{N}rj\right)\mathbf{U}^r\mathbf{V}^{k-j}=$$

$$=\frac{1}{\sqrt{N}}\sum_r\exp\left[-\frac{\pi i}{N}r(j+k)\right]\mathbf{S}(r,k-j)\,, \tag{39}$$

$$\mathbf{\Pi}(v_l,v_k)=|\,v_l\rangle\langle v_k\,|=\frac{1}{N}\sum_s\exp\left(-\frac{2\pi i}{N}sk\right)\mathbf{U}^{l-k}\mathbf{V}^s=$$

$$=\frac{1}{\sqrt{N}}\sum_s\exp\left[-\frac{\pi i}{N}s(l+k)\right]\mathbf{S}(l-k,s)\,. \tag{40}$$

Furthermore, the projectors satisfy the relations

$$i)\ \mathbf{\Pi}(u_j,u_j)\,|\,u_k\rangle=\frac{1}{N}\sum_r u_r^{-j}u_k^{r}\,|\,u_k\rangle=\delta_{k,j}\,|\,u_k\rangle\,, \tag{41}$$

$$\Pi(v_l, v_l) \mid v_k\rangle = \frac{1}{N}\sum_s v_s^{-l} v_k^s \mid v_k\rangle = \delta_{k,l} \mid v_k\rangle \tag{42}$$

where the sum

$$\frac{1}{N}\sum_r u_r^{-j} u_k^r = \delta_{j,k} \tag{43}$$

plays the role of a mod N Kronecker delta;

$$ii)\ \sum_j \Pi(u_j, u_j) = \sum_j \mid u_j\rangle\langle u_j \mid = 1\ , \tag{44}$$

$$\sum_l \Pi(v_l, v_l) = \sum_l \mid v_l\rangle\langle v_l \mid = 1\ , \tag{45}$$

$$iii)\ [\Pi(u_j, u_j), \Pi(v_l, v_l)] = \frac{1}{N^2}\sum_{r,s} \exp\left[-\frac{2\pi i}{N}(rj + sl)\right][\mathbf{U}^r, \mathbf{V}^s]$$

$$= -\frac{2i}{\sqrt{N^3}}\sum_{r,s} \exp\left[-\frac{2\pi i}{N}(rj + sl)\right]\sin\left(\frac{\pi}{N}rs\right)\mathbf{S}(r,s)\ . \tag{46}$$

and

$$\Pi(u_j, v_l) = \mid u_j\rangle\langle v_l \mid = \frac{1}{\sqrt{N}}\sum_k \exp\left(-\frac{2\pi i}{N}lk\right)\Pi(u_j, u_k)\ , \tag{47}$$

$$\Pi(v_l, u_j) = \mid v_l\rangle\langle u_j \mid = \frac{1}{\sqrt{N}}\sum_k \exp\left(\frac{2\pi i}{N}lk\right)\Pi(u_k, u_j)\ .$$

3 Construction and Properties of the Discrete Coherent States

3.1 Operator approach

In a similar way as that used to construct a continuous harmonic oscillator coherent state by the use of a displacement operator [13], one can also look for a similar procedure using instead a discrete generator operator such that

$$\mid r, s\rangle = \mathbf{D}(r, s) \mid 0\rangle \tag{48}$$

is a discrete coherent state, and $\mid 0\rangle$ is a reference state. Guided by the analogy with the continuous case and making use of the previously discussed operator basis, the discrete coherent state generator operator has been proposed to have the form [6]

$$\mathbf{D}(r, s) = \exp\left[-\frac{i\pi}{N}rs - i\pi\phi(r, s; N)\right]\mathbf{U}^r\mathbf{V}^{-s}$$

$$= \sqrt{N}\exp\left[-i\pi\phi(r, s; N)\right]\mathbf{S}(r, -s) \tag{49}$$

which also embodies the mod N invariance, associated to the cyclic character of the operators acting on finite-dimensional spaces through the phase $\phi(r, s)$. It must be stressed that the labels r and s are here associated to the "coordinate" and "momentum" variables of the discrete finite N^2-dimensional phase space and consequently $\mid 0\rangle$ must be interpreted as the associated "Fock vacuum". From these considerations we see that we must take the labels r and s to run from $(-t, t)$, where $t = \frac{N-1}{2}$. Being this coherent state labeled by coordinate and momentum variables, it is clearly differnt from those proposed recently which are written in terms of phase and number variables [14, 15, 16]

As a result of definition (49), a basic property associated to the cyclic action of the underlying generator operator is already evident, i.e.,

$$\mathbf{D}(0, 0) = \mathbf{D}(N, N) = 1\ . \tag{50}$$

With the definition of the displacement operator $\mathbf{D}(r, s)$, it is also direct to write the generalized composition law

$$\prod_{k=1}^{M}\mathbf{D}(r_k, s_k) = \exp\left\{-i\pi\left[\sum_{k=1}^{M}\phi(r_k, s_k; N) - \phi\left(\sum_{k=1}^{M}r_k, \sum_{k=1}^{M}s_k; N\right)\right]\right\}$$

$$\times\exp\left[\frac{i\pi}{N}\left(\sum_{k=1}^{M-1}r_k\sum_{p=k+1}^{M}s_p - \sum_{k=1}^{M-1}s_k\sum_{p=k+1}^{M}r_p\right)\right]\mathbf{D}\left(\sum_{k=1}^{M}r_k, \sum_{k=1}^{M}s_k\right)\ . \tag{51}$$

The additional phases appearing in this expression are related to the mod N invariance of the operator basis and to the area of the figure associated to

points in phase space respectively, being the latter the discrete counterpart of the similar continuous phase discussed in connection with the continuous coherent states [13].

In what refers to the vacuum state $|\,0\rangle$, it must be emphasized again that it does not belong to the N-dimensional "coordinate" or "momentum" spaces. In order to have our definition of the discrete coherent state, Eq. (48), completely meaningful, we must have the matrix elements connecting $|\,0\rangle$ with the states of one of the label spaces. Again we are guided here by the analogy with the continuous harmonic oscillator case for which it was shown that the number states are the eigenvectors of the Fourier transform operator connecting the $\{|\,q\rangle\}$ and $\{|\,p\rangle\}$ spaces [17], i.e.,

$$\mathcal{F}\ \ |\,n\rangle = i^n\,|\,n\rangle\ , \tag{52}$$

such that

$$\langle q\,|\,n\rangle = \left[\frac{1}{(2n-1)!!\sqrt{\pi}\,b}\right]^{\frac{1}{2}}\ \exp\left(-\frac{q^2}{2b^2}\right) H_n\left(\frac{q}{\sqrt{2}b}\right) \tag{53}$$

defines the matrix elements which connect the coordinate and number spaces and $H_n(x)$ are the usual Hermite polynomials. For the finite-dimensional spaces associated to the "coordinate" and "momentum" variables we can write the Fourier operator, as previously seen, as

$$\mathcal{F}\ = \sum_{k=-t}^{t}\,|\,v_k\rangle\langle u_k\,| = \sum_{k=-t}^{t}\mathbf{\Pi}(v_k,u_k) \tag{54}$$

which can still be rewritten as [5]

$$\mathcal{F}\ = \frac{1}{\sqrt{N}}\sum_{k,l=-t}^{t}\mathbf{\Pi}\,(u_l,u_k)\exp\left(\frac{2\pi i}{N}kl\right)\ . \tag{55}$$

This operator also satisfies the eigenvalue equation

$$\langle u_j\,|\,\mathcal{F}\ \ |\,n\rangle = i^n\langle u_j\,|\,n\rangle \tag{56}$$

or

$$\sum_{k=-t}^{t} A_{jk}\ f_{kn} = i^n\ f_{jn}\ , \tag{57}$$

where

$$A_{jk} = \frac{1}{\sqrt{N}}\exp\left(\frac{2\pi i}{N}jk\right)\ , \tag{58}$$

$$f_{kn} = \langle u_k\,|\,n\rangle\ .$$

Now, the matrix representing this operator, A_{jk}, and the eigenvalue problem associated to Eq.(39) have been discussed by Mehta [18] which has shown that

$$f_{kn} = \langle u_k\,|\,n\rangle = \mathcal{N}\sum_{\alpha=-\infty}^{\infty}\exp\left[-\frac{\pi}{N}(\alpha N+k)^2\right] H_n\left[\sqrt{\frac{2\pi}{N}}\,(\alpha N+k)\right] \tag{59}$$

is a possible normalized eigenstate of the Fourier operator, where the Hermite polynomial is written as

$$H_n\,(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-)^k(n)!}{k!(n-2k)!}(2x)^{(n-2k)} \tag{60}$$

and

$$\mathcal{N}\ = \frac{1}{\sqrt{N}}\left[\sum_{\beta=-\infty}^{\infty}\exp\left(-\frac{2\pi}{N}\beta^2\right) H_n^2\left(\sqrt{\frac{2\pi}{N}}\beta\right)\right]^{-\frac{1}{2}} \tag{61}$$

is the proper normalization.

Furthermore, as an important result of the eigenvalue problem, it was also shown by Mehta that only N values of n give linearly independent solutions; the remaining linearly dependent solutions form a redundant space which we may neglect. For the sake of simplicity we will limit ourselves to the case of odd N for which the first values $n = 0, ..., N-1$ give the linearly independent solutions we are interested in [18]. It is this set of N states $\{|\,n\rangle\}$ which will play the role of a finite Fock space in the present descrition.

It is then clear that we can write the expansion of a "Fock state" in the vector state basis $\{|\,u_j\rangle\}$, associated to the discrete version of the "coordinate" representation, using the harmonic oscillator-like solution,

$$|\, n\rangle = \sum_{j=-t}^{t} f_{jn} \,|\, u_j\rangle \tag{62}$$

From this result we adopt the particular expression

$$|\, 0\rangle = \sum_{j=-t}^{t} f_{j0} \,|\, u_j\rangle$$

$$= \frac{1}{\sqrt{N}} \left[\sum_{\beta=-\infty}^{\infty} \exp\left(-\frac{2\pi}{N}\beta^2\right)\right]^{-\frac{1}{2}} \sum_{j=-t}^{t} \sum_{\alpha=-\infty}^{\infty} \exp\left[-\frac{\pi}{N}(\alpha N + j)^2\right] |\, u_j\rangle \tag{63}$$

for the vacuum of the finite-dimensional Fock space. This result then enables us to write the expression for our discrete coherent state as

$$|\, r, s\rangle = \exp\left[\frac{i\pi}{N} rs - i\pi\phi(r, s; N)\right] \sum_{j=-t}^{t} \exp\left(\frac{2\pi i}{N} jr\right) f_{j0} \,|\, u_{j+s}\rangle \,. \tag{64}$$

It is interesting to observe that the coefficient f_{j0} given by Eq. (59) can be identified as the Jacobi $\theta_3(z \mid w)$ function evaluated at integer arguments[19], i.e.,

$$\sum_{\alpha=-\infty}^{\infty} \exp\left[-\frac{\pi}{N}(\alpha N + j)^2\right] = \frac{1}{\sqrt{N}}\, \theta_3\left(\frac{\pi j}{N} \mid e^{-\frac{\pi}{N}}\right) \tag{65}$$

and

$$\sum_{\beta=-\infty}^{\infty} \exp\left(-\frac{2\pi}{N}\beta^2\right) = \theta_3\left(0 \mid e^{-\frac{2\pi}{N}}\right) \,. \tag{66}$$

With these expressions we can also write the "Fock vacuum" as

$$|\, 0\rangle = \frac{1}{\sqrt{N}} \sum_{j=-t}^{t} \frac{\theta_3\left(\frac{\pi j}{N} \mid e^{-\frac{\pi}{N}}\right)}{\sqrt{\theta_3\left(0 \mid e^{-\frac{2\pi}{N}}\right)}} \,|\, u_j\rangle \,. \tag{67}$$

Any arbitrary "Fock state" can also be written in terms of the Jacobi θ_3 function if we perform a direct calculation using the Hermite polynomial generating function

$$|\, n\rangle = \left(N \sum_{p} (-1)^p \binom{n}{p} \frac{(2n)!!}{(2p)!!} \left\{ \frac{d^{2p}}{dx^{2p}} \left[e^{x^2} \theta_3\left(\sqrt{\frac{2\pi}{N}}\, x \mid e^{-\frac{2\pi}{N}}\right)\right]\right\}_{x=0} \right)^{-1/2}$$

$$\times \sum_{j=-t}^{t} \left\{ \frac{d^n}{dx^n} \left[e^{x^2} \theta_3\left(\frac{\pi j}{N} - \sqrt{\frac{2\pi}{N}}\, x \mid e^{-\frac{\pi}{N}}\right)\right]\right\}_{x=0} |\, u_j\rangle \,. \tag{68}$$

In fact, the appearance of this Jacobi function in the present context can be directly understood by realizing that this function, calculated at those arguments, is a periodic eigenstate of the discrete Fourier operator with eigenvalue $(+1)$ and period N[19].

3.2 Creation and annihilation operators

After constructing the discrete coherent state in the finite dimensional coordinate and momentum representation through the use of the discussed operator basis and establishing its connection with the Jacobi $\theta_3(z \mid w)$ function, we must now turn our attention to still another operator basis, similar to that already constructed in the previous sections, but defined now by its action on the "Fock states".

So, to this end let us consider now the new finite-dimensional set of states associated to an "action" variable which we will consider as the finite harmonic oscillator-like Fock states. For this set of states we can define again, in the same form as we did before, a set of unitary operators $\{\tilde{\mathbf{U}}^l\}$ exhibiting the same properties as those already discussed for the "coordinate" space

$$\tilde{\mathbf{U}}^l \,|\tilde{u}_n\rangle = \tilde{u}_n^l \,|\, \tilde{u}_n\rangle \;;\; \tilde{u}_n = \exp\left(\frac{2\pi i}{N} n\right) \,. \tag{69}$$

Furthermore, we can also define the unitary operator which cyclically shifts these states

$$\tilde{\mathbf{V}}^k \,|\, \tilde{u}_n\rangle = |\, \tilde{u}_{n-k}\rangle \;; \tag{70}$$

as before this operator also has its set of eigenstates

$$\tilde{\mathbf{V}}^l \,|\, \tilde{v}_k\rangle = \tilde{v}_k^l \,|\, \tilde{v}_k\rangle \;;\; \tilde{v}_k = \exp\left(\frac{2\pi i}{N} k\right) \tag{71}$$

for which $\tilde{\mathbf{U}}$ acts as a cyclic shift operator

$$\tilde{\mathbf{U}}^l \mid \tilde{v}_k\rangle = \mid \tilde{v}_{k+l}\rangle \ . \tag{72}$$

We have also

$$\tilde{\mathbf{U}}^N = \tilde{\mathbf{V}}^N = \mathbf{1} \ . \tag{73}$$

This set of states, $\{\mid \tilde{v}_k\rangle\}$, has the interpretation of "angle" states and together with the "action" states they satisfy the relations

$$\sum_{j=0}^{N-1} \mid \tilde{u}_k\rangle\langle\tilde{u}_k \mid = 1 \ , \tag{74}$$

$$\sum_{k=0}^{N-1} \mid \tilde{v}_k\rangle\langle\tilde{v}_k \mid = 1 \ , \tag{75}$$

being the Fourier operator in this case defined by

$$\tilde{F} = \sum_{k=0}^{N-1} \mid \tilde{v}_k\rangle\langle\tilde{u}_k \mid \ , \tag{76}$$

where the labels must now run from 0 to $N-1$.

For the "Fock action states" we can now define the creation and annihilation operators, as already discussed in the literature [10]

$$\mathbf{a} = \sum_{n=0}^{N-2} \sqrt{n+1} \mid \tilde{u}_n\rangle\langle\tilde{u}_{n+1} \mid \ , \tag{77}$$

$$\mathbf{a}^\dagger = \sum_{n=0}^{N-2} \sqrt{n+1} \mid \tilde{u}_{n+1}\rangle\langle\tilde{u}_n \mid \ , \tag{78}$$

which give the following important results when acting on the finite "action" multiplet

$$\mathbf{a} \mid \tilde{u}_0\rangle = \sum_{n=0}^{N-2} \sqrt{n+1} \ \delta_{n,-1} \mid \tilde{u}_n\rangle = 0 \ , \tag{79}$$

$$\mathbf{a}^\dagger \mid \tilde{u}_{N-1}\rangle = \sum_{n=0}^{N-2} \sqrt{n+1} \ \delta_{n,N-1} \mid \tilde{u}_{n+1}\rangle = 0 \ . \tag{80}$$

The annihilation and creation operators acting k times on a "Fock state" give

$$\mathbf{a}^k \mid \tilde{u}_m\rangle = \sum_{\{n_j\}=0}^{N-2} \left[\prod_{p=2}^{k} \sqrt{n_p+1} \ \delta_{n_p,n_{p-1}-1}\right] \sqrt{n_1+1} \ \delta_{n_1,m-1} \mid \tilde{u}_{n_k}\rangle \tag{81}$$

and

$$(\mathbf{a}^\dagger)^k \mid \tilde{u}_m\rangle = \sum_{\{n_j\}=0}^{N-2} \left[\prod_{p=2}^{k} \sqrt{n_p+1} \ \delta_{n_p,n_{p-1}+1}\right] \sqrt{n_1+1} \ \delta_{n_1,m} \mid \tilde{u}_{n_k}\rangle \tag{82}$$

respectively.

From these two operators we can also construct the "number" operator

$$\mathbf{N} = \mathbf{a}^\dagger\mathbf{a} = \sum_{n=0}^{N-2} (n+1) \mid \tilde{u}_{n+1}\rangle\langle\tilde{u}_{n+1} \mid \tag{83}$$

such that

$$\mathbf{N} \mid \tilde{u}_m\rangle = \sum_{n=0}^{N-2} (n-1) \ \delta_{n,m-1} \mid \tilde{u}_{n+1}\rangle \ , \tag{84}$$

with

$$\mathbf{N} \mid \tilde{u}_{N-1}\rangle = (N-1) \mid \tilde{u}_{N-1}\rangle \ . \tag{85}$$

The commutation relations for these operators can be immediately calculated giving results that also have been already discussed in the literature [10]

$$\left[\mathbf{a}, \mathbf{a}^\dagger\right] = \sum_{n=0}^{N-2} (n+1) \left(\mid \tilde{u}_n\rangle\langle\tilde{u}_n \mid - \mid \tilde{u}_{n+1}\rangle\langle\tilde{u}_{n+1} \mid\right) = \ \mathbf{1} - N \mid \tilde{u}_{N-1}\rangle\langle\tilde{u}_{N-1} \mid \ , \tag{86}$$

$$[\mathbf{a}, \mathbf{N}] = \sum_{n=0}^{N-2} \sqrt{n+1} \mid \tilde{u}_n\rangle\langle\tilde{u}_{n+1} \mid \ = \mathbf{a} \ , \tag{87}$$

and

$$\left[\mathbf{a}^\dagger, \mathbf{N}\right] = -\sum_{n=0}^{N-2} \sqrt{n+1}\, |\, \tilde{u}_{n+1}\rangle\langle \tilde{u}_n \,| \;\; = -\mathbf{a}^\dagger \,. \tag{88}$$

We can now express these creation and annihilation operators in terms of the Schwinger operators $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ what can be readily achieved by a direct decomposition into the operator basis generated by those operators

$$\mathbf{a}^\dagger = \sum_{nm=0}^{N-1} a^*(m,n)\tilde{\mathbf{G}}(m,n) = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{n=0}^{N-2} \sqrt{n+1}\, \exp\left(-\frac{2\pi i}{N} jn\right) \tilde{\mathbf{U}}^j \tilde{\mathbf{V}} \tag{89}$$

and

$$\mathbf{a} = \sum_{nm=0}^{N-1} a(m,n)\tilde{\mathbf{G}}(m,n) = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{n=0}^{N-2} \sqrt{n+1}\, \exp\left[-\frac{2\pi i}{N} j\,(n+1)\right] \tilde{\mathbf{U}}^j \tilde{\mathbf{V}}^{-1} \,. \tag{90}$$

These results just tell us that the creation and annihilation operators can be easily rewritten and interpreted in terms of the cyclic operators of the basis wich acts on the finite Fock space; all the other properties expected from the annihilation and creation operators are guaranteed by the explicit form of $a(m,n)$.

Now, it is immediate to see that the set of states $\{|\, \tilde{u}_n\rangle\}$, from which we constructed the creation and annihilation operators $\mathbf{a}$ and $\mathbf{a}^\dagger$, are identical with the set $\{|\, n\rangle\}$, describing the "number" states, i.e., $|\, \tilde{u}_n\rangle \equiv |\, n\rangle$, which was discussed in the context of the eigenstates of the Fourier operator, in connection with the "coordinate" and "momentum" representation of the discrete coherent state. Based on this result, we remark again the fundamental role played by expression (59) which allows for the analytic expression for the matrix elements connecting the finite Fock and the discrete "coordinate" spaces.

Also, it is of great interest to express the creation and annihilation operators in terms of the initial Schwinger operator basis associated to the "coordinate" and "momentum" representation. To this end we need to establish a relationship between the "Fock states" and the elements of the initial Schwinger basis; in fact this can be accomplished by using some previous results, namely

$$|\, u_j\rangle\langle u_k \,| = \frac{1}{N}\sum_{l=-t}^{t} \exp\left(-\frac{2\pi i}{N} lj\right) \mathbf{U}^l \mathbf{V}^{k-j} \tag{91}$$

and

$$|\, n\rangle = \sum_{j=-t}^{t} f_{j,n}\, |\, u_j\rangle \tag{92}$$

so that

$$|\, n\rangle\langle n+1 \,| = \sum_{j,k=-t}^{t} f_{j,n} f^*_{k,n+1}\, |\, u_j\rangle\langle u_k \,| = \frac{1}{N}\sum_{j,k,l=-t}^{t} f_{j,n} f^*_{k,n+1} \times \exp\left(-\frac{2\pi i}{N} lj\right) \mathbf{U}^l \mathbf{V}^{k-j} \,, \tag{93}$$

$$|\, n+1\rangle\langle n+1 \,| = \sum_{j,k=-t}^{t} f_{j,n+1} f^*_{k,n+1}\, |\, u_j\rangle\langle u_k \,| = \frac{1}{N}\sum_{j,k,l=-t}^{t} f_{j,n+1} f^*_{k,n+1} \times \exp\left(-\frac{2\pi i}{N} lj\right) \mathbf{U}^l \mathbf{V}^{k-j} \,. \tag{94}$$

With these results it is a direct matter to write Eqs. (77), (78) and (83) in terms of the Schwinger operators associated to the "coordinate" and "momentum" representation

$$\mathbf{a} = \frac{1}{N}\sum_{n=0}^{N-2}\sum_{j,k,l=-t}^{t} \sqrt{n+1}\, \exp\left(-\frac{2\pi i}{N} lj\right) f_{j,n} f^*_{k,n+1} \mathbf{U}^l \mathbf{V}^{k-j} \,, \tag{95}$$

$$\mathbf{a}^\dagger = \frac{1}{N}\sum_{n=0}^{N-2}\sum_{j,k,l=-t}^{t} \sqrt{n+1}\, \exp\left(-\frac{2\pi i}{N} lk\right) f^*_{j,n} f_{k,n+1} \mathbf{U}^l \mathbf{V}^{j-k} \,, \tag{96}$$

and

$$\mathbf{N} = \frac{1}{N}\sum_{n=0}^{N-2}\sum_{j,k,l=-t}^{t} (n+1) \exp\left(-\frac{2\pi i}{N} lj\right) f_{j,n+1} f^*_{k,n+1} \mathbf{U}^l \mathbf{V}^{k-j} \,. \tag{97}$$

Using these relations we can also establish the action of these operators upon the "coordinate" and "momentum" states

$$\mathbf{a} \mid u_j\rangle = \sum_{n=0}^{N-2} \sqrt{n+1}\, f^*_{j,n+1} \mid n\rangle \,, \tag{98}$$

$$\mathbf{a}^\dagger \mid u_j\rangle = \sum_{n=0}^{N-2} \sqrt{n+1}\, f^*_{j,n} \mid n+1\rangle \,, \tag{99}$$

$$\mathbf{N} \mid u_j\rangle = \sum_{n=0}^{N-2} (n+1)\; f^*_{j,n+1} \mid n+1\rangle \tag{100}$$

thus showing the importance of the function $f_{j,n}$ in the connection of the finite Fock and "coordinate" spaces.

3.3 Some Properties of the Discrete Coherent States

The adopted definition of the discrete coherent states, Eq. (48), directly allows for the calculation of some interesting results regarding these states, namely, the overcompleteness and the scalar product relations. The first property can be easily verified by noting that

$$\frac{1}{N}\sum_{k,l=-t}^{t} \mid k,l\rangle\langle k,l \mid = \sum_{l=-t}^{t}\sum_{j,r=-t}^{t} f_{j0}\, f^*_{r0}\, \delta_{jr} \mid u_{j+l}\rangle\langle u_{r+l} \mid$$

$$= \sum_{l,r=-t}^{t} \mid f_{r0} \mid^2 \mid u_{r+l}\rangle\langle u_{r+l} \mid \tag{101}$$

and using

$$\mid u_{r+l}\rangle\langle u_{r+l} \mid = \mathbf{V}^{-r}\mathbf{\Pi}(u_l, u_l)\mathbf{V}^r \tag{102}$$

so that

$$\frac{1}{N}\sum_{k,l=-t}^{t} \mid k,l\rangle\langle k,l \mid = \sum_{r=-t}^{t} \mid f_{r0} \mid^2 \mathbf{V}^{-r}\left(\sum_{l=-t}^{t} \mathbf{\Pi}(u_l, u_l)\right)\mathbf{V}^r$$

$$= \left(\sum_{r=-t}^{t} \mid f_{r0} \mid^2\right)\mathbf{1} = \mathbf{1}\,. \tag{103}$$

The scalar product can also be directly calculated by the use of the definition of the discrete coherent states and reads

$$\langle k', l' \mid k, l\rangle = \exp\left\{\frac{\pi i}{N}\left[k\,(l'-l) - l'\,(k'-k)\right]\right\}$$

$$\times \exp\left\{\pi i\left[\Phi\,(k', l'; N) - \Phi\,(k, l; N)\right]\right\} \exp\left[-\frac{\pi}{N}\,(k'-k)^2\right]$$

$$\times \frac{\theta_3\left(\frac{\pi}{N}\left[(l'-l) - i\,(k'-k)\right] \mid e^{-\frac{2\pi}{N}}\right)}{\theta_3\left(0 \mid e^{-\frac{2\pi}{N}}\right)}\,. \tag{104}$$

The overlap probability immediately follows

$$\left|\langle k', l' \mid k, l\rangle\right|^2 = \left|\frac{\theta_3\left(\frac{\pi}{N}\left[(l'-l) - i\,(k'-k)\right] \mid e^{-\frac{2\pi}{N}}\right)}{\theta_3\left(0 \mid e^{-\frac{2\pi}{N}}\right)}\right|^2$$

$$\times \exp\left[-\frac{2\pi}{N}\,(k'-k)^2\right]\,, \tag{105}$$

from which we can see that the following inequality

$$0 < \left|\langle k', l' \mid k, l\rangle\right|^2 \leq 1 \tag{106}$$

holds. Again, in the discrete case we can also see that the overlap is equal to one along the diagonal and falls to zero as $(k'-k)^2$ becomes large in agreement with the continuous case.

Let us expand now the discrete coherent states $\mid k, l\rangle$ in the finite Fock space

$$\mid k, l\rangle = \sum_{n=0}^{N-1} \langle n \mid k, l\rangle \mid n\rangle = \sum_{n=0}^{N-1} \mathcal{A}_n\,(k, l) \mid n\rangle\,. \tag{107}$$

The coefficient $\mathcal{A}_n(k, l)$ is of fundamental importance in this context and can be easily obtained by the use of the definition of the coherent state, Eq. (64), thus giving

$$\mathcal{A}_n\,(k, l) = \mathcal{N} \exp\left[-i\pi\phi\,(k, l; N) + \frac{i\pi}{N} k\,(l + ik)\right] \sum_{\alpha=-\infty}^{\infty} H_n\left(\sqrt{\frac{2\pi}{N}}\,\alpha\right)$$

$$\times \exp\left[-\frac{2\pi}{N}\alpha^2 - \frac{2\pi i}{N}\alpha\,(l+ik)\right] \tag{108}$$

where

$$\mathcal{N} = \left\{ \sum_{\alpha,\beta=-\infty}^{\infty} \exp\left[-\frac{2\pi}{N}\left(\alpha^2+\beta^2\right)\right] H_n^2\left(\sqrt{\frac{2\pi}{N}}\,\alpha\right)\right\}^{-\frac{1}{2}} . \tag{109}$$

It is immediate to see that the discrete coherent state satisfies the relation

$$\mathbf{a} \mid k,l\rangle = F(k,l) \mid k,l\rangle \tag{110}$$

such that

$$F(k,l) = \sum_{n=0}^{N-2} \sqrt{n+1}\,\mathcal{A}_{n+1}(k,l)\,\mathcal{A}_n^*(k,l) . \tag{111}$$

This expression for $F(k,l)$ is the discrete analog of the eigenvalue of the annihilation operator in the continuous case, but here we see that it is not merely a complex number so that we cannot assert that the discrete coherent state is an eigenstate of the annihilation operator in the same sense as in the continuous case. The difference comes from the finite-dimension character of the Fock space; in fact we can only expect to reobtain that result in the limit of large N.

As another important result, let us calculate the new operators

$$\mathbf{A} = \mathbf{D}^\dagger(k,l)\,\mathbf{a}\,\mathbf{D}(k,l) = \mathbf{a} + F(k,l) \tag{112}$$

and

$$\mathbf{A}^\dagger = \mathbf{D}^\dagger(k,l)\,\mathbf{a}^\dagger\,\mathbf{D}(k,l) = \mathbf{a}^\dagger + F^*(k,l) \tag{113}$$

such that it is immediate to verify the following relations

$$\langle 0 \mid \mathbf{A} \mid 0\rangle = F(k,l) \tag{114}$$

and

$$\langle 0 \mid \mathbf{A}^\dagger \mid 0\rangle = F^*(k,l) \tag{115}$$

which also show a close analogy with the corresponding results coming from the continuum case.

4 Probability Distributions

The mapping of a finite Fock space state – through the use of the proposed discrete coherent state written in terms "coordinate" and "momentum" variables – onto a positive discrete distribution function defined in the discrete phase space follows directly, by using the definition of the density operator $\rho_n = \mid n\rangle\langle n \mid$, from the statement

$$P_n(k,l) = \langle k,l \mid \rho_n \mid k,l\rangle = \mid \langle k,l \mid n\rangle \mid^2 = |\mathcal{A}_n(k,l)|^2 \tag{116}$$

where $\mathcal{A}_n(k,l)$ is given by the expression (108). The discrete ground state distribution is shown in figure 1.

Once we have the discrete distribution function $P_n(k,l)$, that is associated to the "coordinate" and "momentum" variables, we are able to extract the discrete marginal distributions by a simple summation

$$R_n(k) = \frac{1}{\sqrt{N}} \sum_{l=-t}^{t} P_n(k,l) =$$

$$= \mathcal{N}^2\, N^{\frac{1}{2}} \exp\left(-\frac{2\pi}{N}k^2\right) \sum_{\alpha=-\infty}^{\infty} \exp\left[-\frac{4\pi}{N}\alpha\,(\alpha-k)\right] H_n^2\left[\sqrt{\frac{2\pi}{N}}\,\alpha\right] , \tag{117}$$

where the normalization constant $\mathcal{N}$ is that defined in Eq. (109); and

$$Q_n(l) = \frac{1}{\sqrt{N}} \sum_{k=-t}^{t} P_n(k,l) =$$

$$= \mathcal{N}^2\,\frac{1}{N^{\frac{1}{2}}} \sum_{\alpha,\beta=-\infty}^{\infty} \exp\left[-\frac{2\pi}{N}\left(\alpha^2-\alpha\beta+\beta^2\right) + \frac{2\pi i}{N} l\,(\alpha-\beta)\right]$$

$$\times H_n\left(\sqrt{\frac{2\pi}{N}}\,\alpha\right) H_n\left(\sqrt{\frac{2\pi}{N}}\,\beta\right) \theta_3\left(\frac{i\pi}{N}(\alpha-\beta) \mid e^{-\frac{2\pi}{N}}\right) . \tag{118}$$

It is direct to write a discrete Wigner distribution for any finite-dimensional space whenever the density operator and a corresponding operator basis for that space are given, but it is also interesting to calculate the discrete Wigner distribution associated to the state $|n\rangle$ of the finite-dimensional Fock space by using the operator basis written in terms of "coordinate" and "momentum" variables, Eq. (22), i.e.,

$$W_n(k,l) = Tr\left[\mathbf{G}^\dagger(k,l) \mid n\rangle\langle n \mid\right] = \langle n| \mathbf{G}^\dagger(k,l) |n\rangle$$

$$= \mathcal{N}^{-1} \frac{1}{N} \sum_{r,s=-t}^{t} \exp\left[\frac{2\pi i}{N}(rk+sl) - \frac{\pi}{N} r(r-is)\right]$$

$$\times \sum_{\alpha=-\infty}^{\infty} \exp\left[-\frac{2\pi}{N}\alpha^2 + \frac{2\pi}{N}\alpha(r-is)\right] H_n\left(\sqrt{\frac{2\pi}{N}}\,\alpha\right) H_n\left(\sqrt{\frac{2\pi}{N}}\,(\alpha-r)\right), \tag{119}$$

where the normalization constant reads

$$\mathcal{N} = \sum_{\beta=-\infty}^{\infty} \exp\left(-\frac{2\pi}{N}\beta^2\right) H_n^2\left(\sqrt{\frac{2\pi}{N}}\,\beta\right). \tag{120}$$

An example of such a discrete Wigner distribution is shown in figure 2.

Although this expression seems somewhat entangled when compared to the continuous case, it has all the desired properties associated to a Wigner distribution function, including now the periodicity required by the finite character of the discrete phase space. In particular we can also calculate the marginal distributions for this discrete Wigner function

$$R_n(k) = \frac{1}{\sqrt{N}} \sum_{l=-t}^{t} W_n(k,l)$$

$$= \frac{1}{\sqrt{N}} \mathcal{N}^{-1} \sum_{r=-t}^{t} \exp\left(\frac{2\pi i}{N}kr - \frac{\pi}{N}r^2\right) \sum_{\alpha=-\infty}^{\infty} \exp\left[-\frac{2\pi}{N}\alpha(\alpha-r)\right]$$

$$\times\ H_n\left(\sqrt{\frac{2\pi}{N}}\,\alpha\right) H_n\left[\sqrt{\frac{2\pi}{N}}\,(\alpha-r)\right] \tag{121}$$

and

$$Q_n(l) = \frac{1}{\sqrt{N}} \sum_{k=-t}^{t} W_n(k,l)$$

$$= \sqrt{N}\,\mathcal{N}^{-1} \exp\left(-\frac{2\pi}{N}l^2\right) H_n^2\left(\sqrt{\frac{2\pi}{N}}\,l\right) \tag{122}$$

respectively.

5 Continuum limit of the distribution functions

One of the interesting aspects of the formalism describing discrete phase spaces is that the representative of quantum operators (or combinations of them such as commutators or anticommutators) are functions of integer c-numbers and also of the dimension of the state vector space, N. This constitutes a great advantage when one must consider the limit situation of state vector spaces with $N \to \infty$ because then one does not need to calculate that limit for expression giving the expectation value of the operators one is interested in. In other words, one takes the $N \to \infty$ limit for the functions which are the discrete phase space representatives of the operators without regarding the underlying Hilbert space. This procedure closely parallels the Pegg and Barnett prescription [10] in which only after calculating the expectation values of the operators we are allowed to perform that limit.

As examples of what can be obtained from this limiting procedure, in what follows we will calculate the expressions for the discrete distribution functions we have previously obtained for the case $N \to \infty$. To this end we will call $\epsilon^2 = 2\pi/N$ and, due to the expected symmetry, consider the interval of variation for the variables associated to the "coordinate" and "momentum" to range from $-(N-1)/2$ to $(N-1)/2$. Furthermore, we will take

$$q_k = p_k = \epsilon k \tag{123}$$

where

$$k = 0, \pm 1, \pm 2, ..., \pm\frac{N-1}{2} \tag{124}$$

and

$$\Delta q_k = \Delta p_k = \epsilon \tag{125}$$

so that $\epsilon \to 0$ when $N \to \infty$. In that limit the sums may be substituted by integrals $q_k \to q$, $p_k \to p$ and, when we consider the discrete distribution $P_n(k,l)$, one gets again the Poisson distribution exactly as in the continuous Glauber coherent states

$$P_n(p,q) = \frac{1}{n!}\left(\frac{p^2+q^2}{2}\right)^n \exp\left(-\frac{p^2+q^2}{2}\right) . \tag{126}$$

In a completely analogous form we can calculate the $N \to \infty$ limit for marginal distributions $R_n(k)$ and $Q_n(l)$ respectively and we get

$$R_n(p) = \exp\left(-\frac{p^2}{2}\right)\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}L_{n-j}^{-\frac{1}{2}}\left(\frac{p^2}{2}\right) , \tag{127}$$

$$Q_n(q) = \exp\left(-\frac{q^2}{2}\right)\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}L_{j}^{-\frac{1}{2}}\left(\frac{q^2}{2}\right) . \tag{128}$$

Such expressions are an alternative way of writting the marginal distributions in the continuum for the coherent states as can be readly verified by remembering that

$$R_n(p) = \exp\left(-p^2\right)\sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}\frac{p^{2(n-k)}}{[2(n-k)]!!} , \tag{129}$$

$$Q_n(q) = \exp\left(-q^2\right)\sum_{k=0}^{n}\frac{[2(n-k)-1]!!}{[2(n-k)]!!}\frac{q^{2k}}{(2k)!!} \tag{130}$$

and taking into account the relationship between the Hermite and Laguerre polynomials [20].

The Wigner function $W_n(k,l)$, Eq. (119), also can be obtained in the $N \to \infty$ limit and gives the standard expression as expected for a harmonic oscillator $|\, n\rangle$ state

$$W_n(p,q) = 2(-1)^n \exp\left[-\left(q^2+p^2\right)\right] L_n\left[2\left(q^2+p^2\right)\right] \tag{131}$$

while the associated discrete marginal distributions give

$$R_n(p) = \frac{\sqrt{2}}{(2n)!!}\exp(-p^2)\; H_n^2(p) \tag{132}$$

and

$$Q_n(q) = \frac{\sqrt{2}}{(2n)!!}\exp(-q^2)\; H_n^2(q) \tag{133}$$

respectively. From Eqs. (132-133) we would get

$$R_n(p) = \sqrt{2}\, e^{-p^2}\sum_{k=0}^{n}(-1)^k\,\frac{[2(n-k)-1]!!}{[2(n-k)]!!}\; L_k^{-\frac{1}{2}}(2p^2) \tag{134}$$

and

$$Q_n(q) = \sqrt{2}\, e^{-q^2}\sum_{k=0}^{n}(-1)^{n-k}\,\frac{(2k-1)!!}{(2k)!!}\; L_{n-k}^{-\frac{1}{2}}\left(2q^2\right) \tag{135}$$

These expressions can be directly seen to be equivalent to Eqs. (127) and (128) respectively.

6 Conclusions

In this paper we have addressed the problem of constructing coherent states and probability distributions in finite-dimensional spaces. To this end we have explored the fact that, starting from a finite N-dimensional "coordinate" space, it is always possible to associate a corresponding "momentum" space by using operator bases constructed out of the Schwinger unitary operators [3] and, as a fundamental consequence of the this construction, it results that those finite N-dimensional spaces are connected by a Fourier transform. Now, the general matrix representing the associated finite space Fourier operator has been studied by Mehta [18] who has shown that it also satisfies a property already presented and discussed in the continuous case [17], namely that a set of Fock states $\{|\, n\rangle\}$ exists such that they are the eigenvalues of that operator with eigenvalues i^n. However, in the finite case there are only N linearly independent solutions to the eigenvalue problem; they form then a finite N-dimensional Fock space which is directly connected to the original "coordinate" space. We explored this connection between the

two spaces to show that it is possible to generate an operator basis associated to a "coordinate-momentum" discrete phase space as well as another operator basis associated now to a "number-angle" discrete phase space. In this way, we have shown that, due to that important property of the Fourier transform, the introduction of a finite N-dimensional discrete "coordinate-momentum" phase space allows for the construction of a finite N-dimensional discrete "number-angle" phase space and vice-versa. In this connection, we have recalled the annihilation and creation operators acting on the finite-dimensional Fock space already described in the context of quantum optics [10] and we have shown how it is possible to express them also in terms of the "coordinate-momentum" operator basis.

Based on those results we have discussed the properties of a discrete harmonic oscillator-like coherent state constructed out of the "coordinate" and "momentum" discrete variables [6] and we have also presented the functions describing the finite-dimensional excitation number probability distribution in that discrete coherent state together with the marginal distributions directly obtained from it. A discrete Wigner pseudo-distribution in the "coordinate" and "momentum" variables is obtained for a Fock state and the marginal distributions are also explicitly written.

The limit of large N was carried out for the discrete probability distributions and their well known continuous counterpart expressions were reobtained thus showing the correct trend we would expect from such a construction.

In a future paper we will discuss the limiting behavior of the expressions giving the connections between the two phase spaces and we will also show how one can recover the Pegg and Barnett description of the phase distributions from the present scheme.

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7 Appendix

Let us define two hermitean operators in terms of the creation and annihilation operators which act on the finite number space

$$\mathbf{S} = \frac{\mathbf{a} + \mathbf{a}^\dagger}{\sqrt{2}} \text{ and } \mathbf{D} = \frac{\mathbf{a} - \mathbf{a}^\dagger}{\sqrt{2}}$$

so that

$$\mathbf{a} = \frac{\mathbf{S} + i\mathbf{D}}{\sqrt{2}} \text{ and } \mathbf{a}^\dagger = \frac{\mathbf{S} - i\mathbf{D}}{\sqrt{2}} .$$

With these expressions we have the matrix elements

$$\langle kl \mid \mathbf{S} \mid kl\rangle^2 = \frac{1}{2}\left(F^2(k,l) + F^{*2}(k,l) + 2\,|F(k,l)|^2\right)$$

$$\langle kl \mid \mathbf{S}^2 \mid kl\rangle = \frac{1}{2}\left(1 - NP_n(k,l) + F^2(k,l) + F^{*2}(k,l) + 2\,|F(k,l)|^2\right)$$

$$\langle kl \mid \mathbf{D} \mid kl\rangle^2 = \frac{1}{2}\left(2\,|F(k,l)|^2 - F^2(k,l) - F^{*2}(k,l)\right)$$

$$\langle kl \mid \mathbf{D}^2 \mid kl\rangle = \frac{1}{2}\left(1 - NP_n(k,l) - F^2(k,l) - F^{*2}(k,l) + 2\,|F(k,l)|^2\right) .$$

Thus, it is immediate to get the expressions for the variances of the operators $\mathbf{S}$ and $\mathbf{D}$ with respect to the discrete coherent state

$$\left\langle \Delta\mathbf{S}^2 \right\rangle = \langle kl \mid \mathbf{S}^2 \mid kl\rangle - \langle kl \mid \mathbf{S} \mid kl\rangle^2 = \frac{1}{2} - \frac{N}{2}P_n(k,l) \leq \frac{1}{2}$$

and

$$\left\langle \Delta\mathbf{D}^2 \right\rangle = \langle kl \mid \mathbf{D}^2 \mid kl\rangle - \langle kl \mid \mathbf{D} \mid kl\rangle^2 = \frac{1}{2} - \frac{N}{2}P_n(k,l) \leq \frac{1}{2}$$

respectively. These results clearly show the effects of the presence of the additional term, coming from the finite-dimensional character of the space, in the commutator of the creation and annihilation operators. In fact these results are directly related to the particular second order squeezing effect which appears in finite-dimensional Fock spaces[15, 16]. It is also evident that in the limit of large N we will get the standard dispersions results of the continuous case and the vanishing of the mentioned squeezing effect.

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8 Figure Captions

Figure 1: The discrete ground state harmonic oscillator distribution, $n = 0$, for a discrete space with $N = 15$ states.

Figure 2: The discrete Wigner distribution associated to the $n = 1$ harmonic oscillator state for a discrete space with $N = 9$ states.

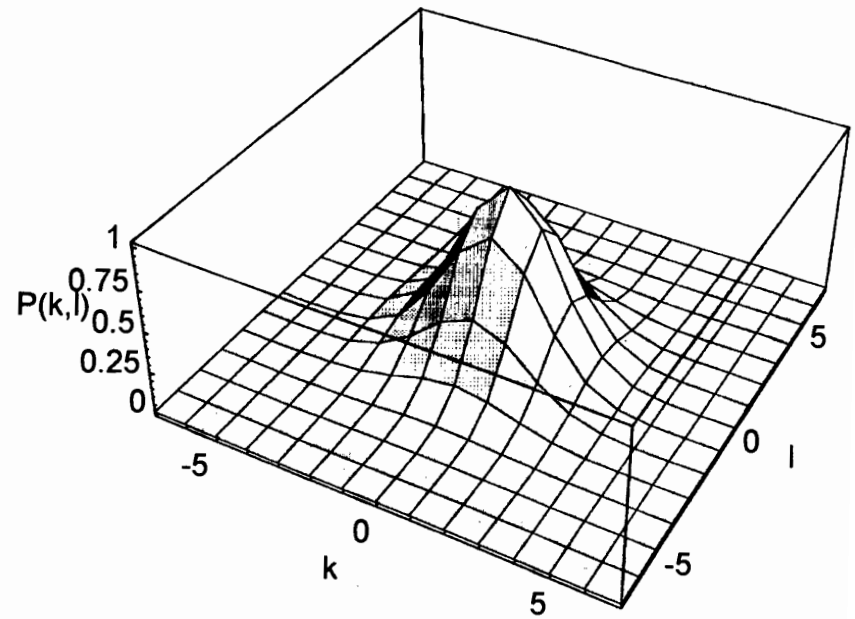

FIGURE 1

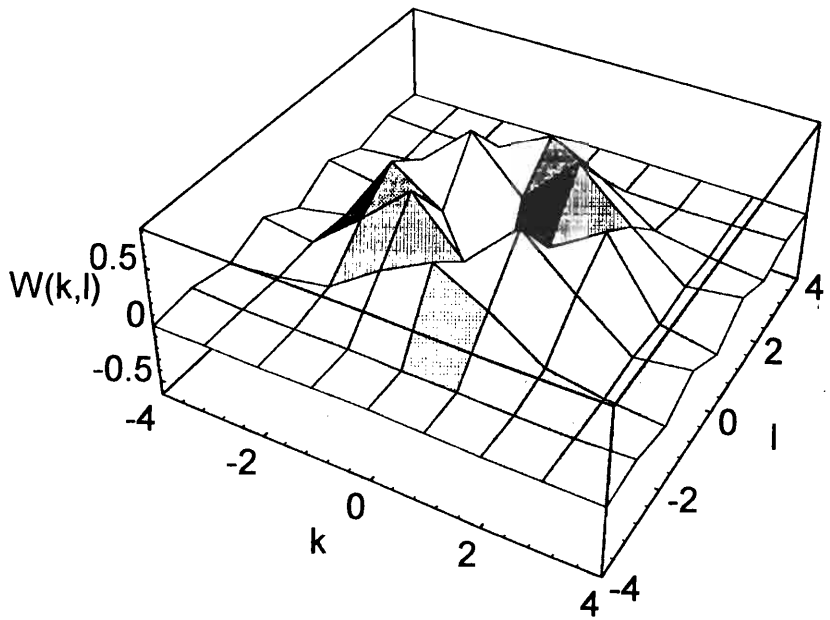

FIGURE 2