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Within the framework of scattering integral equations in momentum space we present numerical results of scattering of three-identical bosons at low energies in two dimensions for short-range separable potentials. An analysis of the present numerical results reveal the three-particle scattering observables to be independent of potential shape provided the low-energy two-particle binding energy and scattering length are held fixed throughout the investigation.

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I. INTRODUCTION

Over the past decade there has been a great deal of activity in studying the three particle problem in two space dimensions both theoretically[1–6] and experimentally[7–10] due to a variety of reasons. Experimentally, there has been considerable interest in two-dimensional systems such as helium adsorbed on graphaite[10] and spin-polarized hydrogen (H↓)[5,7–9] recombining on a helium film. Also in view of the possibility of detecting multiparticle bound states on monolayers of quantum gases[10], it is interesting to study numerically the quantum mechanical few-particle bound-state problem.

In the recent past there has been a significant amount of theoretical studies[11] in the quantum and statistical mechanics of anyons which are particles having continuous fractional spin and thus interpolating and simulating between boson and fermion properties. Anyons exist only in two-dimensions and this has increased the relevance of few-particle problems in two dimensions. As the anyons represent bosons in one of the extreme limits a study of the quantum mechanical three-boson system is likely to enhance the understanding of anyon properties.

A knowledge of the complete spectrum of the Hamiltonian for the few-particle systems in two dimensions is essential for studying the quantum-cluster coefficients in two dimensions.[12] For a confining potential the complete spectrum constitutes of bound states only, whereas in the nonconfining case one has to deal with both bound states and scattering. Such studies have been undertaken[11,12] for bosons, fermions, and anyons in general and also in the limits of high and low temperatures. A knowledge of the complete spectrum of the n-particle Hamiltonian is necessary for the study of the nth quantum-cluster coefficient.[12] At the low (high) temperature limit one needs such a knowledge at low (high) energies.

Hence, a systematic study of the quantum mechanical few-particle system in two space dimensions will shed light on all the above-mentioned problems. Here we report

numerical results on both scattering and bound state observables at low energies for the quantum mechanical three-boson system with a special objective in mind. We would like to investigate how sensitive are these observables to variations of the interaction potential while maintaining two-particle binding energy and scattering length fixed under certain restrictions. The restrictions are those of short-range potentials where the typical few-particle bound-state size is much larger than the range of the interaction potential. We would like to study the few-particle system under such restriction partly because of previous studies indicating that this is the situation in many atomic and chemical systems,[1,2] and partly because of the simplicity in the treatment of the few-particle problem in two dimensions via the momentum-space connected kernel integral equations using such short-range potentials. Also, under these restrictions and at low energies the result of such a study is expected to be universal and independent of the details of the potential model employed.[13]

There have been certain calculations[1–6] for few-particle bound states in two dimensions. Bruch and Tjon[1,2] were the first to investigate the model dependence of the three- and four-particle bound states on the shape of the interaction potentials. They compared the results of separable potential calculations with those employing Lennard-Jones potential. They concluded that the results were sensitive to the potential model employed as they varied the strength of the potential. Later on in relation to the study of Efimov and Thomas[14] effects in two dimensions Adhikari et al.[4] pointed out that the study of Bruch and Tjon is not entirely to the point and one should maintain the two-particle low-energy observables - both binding energy and scattering length - fixed in studying the above model dependence. Once this is done the above model dependence of few-particle bound state observables in two dimensions reduces significantly. This has been related to the absence[4] of Efimov and Thomas effects in two dimensions.

In this work we compliment the study of Ref. 4 in two dimensions on the model independence of three-particle bound states on potential models and present the first systematic study of the low-energy scattering of a boson on the bound state of two identical bosons. We consider three identical bosons interacting via pairwise S wave separable potentials with simple form-factors. We calculate the S wave scattering length and phase-shifts for the scattering of a boson on the bound-state of two others. We investigate the sensitivity of our result on variation of the shape of interaction potential. Presently, nothing is known about the scattering observables of three particles in two dimensions. Our study will provide some idea of the scattering in the three boson system.

Through our numerical study we establish that the three-particle bound state and scattering observables at low energies in two dimensions are quite insensitive to variations of the interaction potential provided that the two-particle binding energy and the scattering length are held fixed. This is not quite so in three dimensions where the low-energy scattering and bound state observables for three- and few-particle systems are sensitive to the details of the potential models employed or to the shape of the interaction potentials.

We have confined the present study to the S wave identical bosons but our conclusions are expected to hold true in the case of fermionic systems and higher partial waves also. This is because few-fermion systems obey scattering integral equations quite similar to the few-boson systems with a somewhat weakened effective interaction as a consequence of the Pauli principle between identical fermions. Because of the Pauli repulsion between identical fermions the few-fermion system is expected to be less sensitive to the interaction potential than the few-boson system and our conclusions should extrapolate to the fermionic case. Also, the higher partial waves of the three-particle system are expected to be weakly attractive because of the centrifugal

barrier, and so are expected to be insensitive to the two-particle potential.

The present result should have interesting consequence on the calculation of quantum-cluster coefficients, surface recombination rates of spin polarized hydrogen atoms, and the formation of helium-clusters in two dimensions. Our calculations and the general arguments of Ref. 4 indicate that the results of such calculations should be independent of potential shape provided that the low-energy two-particle observables are held fixed.

In Ref. 4 the model (in)dependence of few-particle observables in different space dimensions has been related to the divergence of the trace of the momentum space kernel of the scattering integral equation. This divergence is reduced as one moves from higher to lower dimensions as a result of the reduction of the momentum space phase space - $d^n k$ for n dimensions. In one dimension the bound state and scattering results are expected to be even more insensitive to potential shape. The calculation of Dodd[15] indicates this for three-particle bound states in one dimension.

In Sec. II we present the two-particle separable potential model which we employ for the calculation of the three-particle system and also some results for two-particle scattering. Then we develop the three-particle dynamical equations which we use for the numerical study of three-particle scattering in two dimensions. An account of the numerical method and a suitable definition of the scattering phase-shifts are also given in this case. In Sec. III we present and discuss our numerical results. In Sec. IV we present a summary and concluding remarks.

II. THE MODEL

Let us consider three identical bosons in two dimensional space interacting via the following S-wave separable potential in momentum space:

$$\langle \vec{p}|V|\vec{k}\rangle = -\lambda g(p)g(k), \tag{1}$$

with

$$g(p) = (\beta^2 + p^2)^{-m}. \tag{2}$$

One of our main interests in this work is to study the sensitivity of few-particle observables to variations of potential shape, and this will be achieved by varying m from 1 to 10. For potential (1) the scattering t matrix is given by

$$\langle \vec{p}|t(s)|\vec{k}\rangle = \tau(s)g(p)g(k), \tag{3}$$

with

$$\tau(s) = -\left[\frac{1}{\lambda} + 2\pi \int_0^\infty \frac{qdqg^2(q)}{k^2 - q^2 + i0}\right]^{-1} \tag{4}$$

where $s = \hbar^2 k^2/2\mu$, μ is the reduced mass of the two-boson system, s is the center of mass energy, and k is the on-the-energy-shell (on-shell) relative wave number. In this work we shall employ units $\hbar = 2\mu = 1$. The fully on-shell t matrix $\langle k|t(k^2)|k\rangle = 2\pi\langle \vec{k}|t(k^2)|\vec{k}\rangle$, $k^2 = s$, is given by

$$\langle k|t(k^2)|k\rangle = \left[-\frac{(\beta^2 + k^2)^{2m}}{2\pi\lambda} + ln(\frac{\beta}{k}) - \sum_{j=1}^{2m-1} \frac{(\beta^2 + k^2)^j}{2j\beta^{2j}} + \frac{i\pi}{2}\right]^{-1}. \tag{5}$$

The two-particle bound states at negative energies are given by the vanishing of the quantity in the square bracket. Using the effective range expansion, this t matrix is usually parametrized by[16]

$$\langle k|t(k^2)|k\rangle = \frac{2}{\pi(-cot\delta_2 + i)} \tag{6}$$

with

$$cot\delta_2 = a_2 + (1/\pi)ln(k^2) + bk^2 + ck^4 + ... \tag{7}$$

Here δ_2 is the scattering phase shift, and a_2 is the two-particle scattering length. There is some ambiguity in defining a_2 in two dimensions because of the $ln(k^2)$ term;

the numerical value of a_2 depends on the scale used to measure the energy k^2. In our study we shall be measuring energy in units of the two-particle binding energy B_2 and the scattering length $\bar{a}_2$, defined by

$$cot\delta_2 = \bar{a}_2 + \frac{1}{\pi}ln\frac{k^2}{B_2} + b'\frac{k^2}{B_2} + ..., \tag{8}$$

is related to that of Eq. (7) by

$$\bar{a}_2 = a_2 + \frac{1}{\pi}lnB_2. \tag{9}$$

The scattering length $\bar{a}_2$ will be useful in our study of model independence of three-particle bound state and scattering observables in two dimensions. When the two-particle binding energy $B_2 = 1$, $\bar{a}_2$ reduces to a_2. In the low energy limit, for the t matrix of Eq. (5), $cot\delta_2$ is given by

$$cot\delta_2 = \frac{\beta^{4m}}{\pi^2\lambda} + \frac{1}{\pi}ln\frac{k^2}{\beta^2} + \frac{1}{\pi}\sum_{j=1}^{2m-1}\frac{1}{j}, \tag{10}$$

so that the scattering length $\bar{a}_2$ is written as

$$\bar{a}_2 = \frac{\beta^{4m}}{\pi^2\lambda} + \frac{1}{\pi}ln\frac{B_2}{\beta^2} + \frac{1}{\pi}\sum_{j=1}^{2m-1}\frac{1}{j}. \tag{11}$$

With this discussion of the two-particle system we now present a discussion of the three-boson Faddeev equations[17,18] in two dimensions for separable potentials (1). In this case the Amado-Lovelace-Mitra equation[18], which is the special case of Faddeev equations for separable potentials, is given by

$$\langle \vec{q}|T(s)|\vec{k}\rangle = \langle \vec{q}|V(s)|\vec{k}\rangle + \int d\vec{p}\langle \vec{q}|V(s)|\vec{p}\rangle\tau(s - 3p^2/4)\langle \vec{p}|T(s)|\vec{k}\rangle, \tag{12}$$

where

$$\langle \vec{q}|V(s)|\vec{p}\rangle = 2\frac{g(|\vec{q} + \vec{p}/2|)g(|\vec{p} + \vec{q}/2|)}{s - q^2 - p^2 - \vec{p}.\vec{q} + i0}. \tag{13}$$

Here $T(s)$ is the t matrix for scattering of a boson on the bound state of two others at a three-particle center of mass energy s, and $V(s)$ is the energy-dependent effective potential in this case. The on-shell wave number k in this case is given by

$$s = 3k^2/4 - B_2, \tag{14}$$

where $3k^2/4$ is the relative kinetic energy.

The partial S-wave projection of the Amado-Lovelace-Mitra equation (12) is written as

$$\langle q|T(s)|k\rangle = \langle q|V(s)|k\rangle + \int_0^\infty pdp\langle q|V(s)|p\rangle\tau(s-3p^2/4)\langle p|T(s)|k\rangle, \tag{15}$$

where

$$\langle q|V(s)|p\rangle = \int_0^{2\pi} d\theta\langle\vec{q}|V(s)|\vec{p}\rangle. \tag{16}$$

The equation for the partial wave t matrix (15) above has a singularity in the kernel in the $\tau(s-3p^2/4)$ term whenever p equals the on-shell wave-number k. In order to make this singularity more explicit we introduce a modified t matrix and interaction potential in the Amado-Lovelace-Mitra equation via

$$\langle q|\bar{T}(s)|p\rangle = -(k^2-q^2)\tau(s-3q^2/4)\langle q|T(s)|p\rangle, \tag{17}$$

and

$$\langle q|\bar{V}(s)|p\rangle = -(k^2-q^2)\tau(s-3q^2/4)\langle q|V(s)|p\rangle. \tag{18}$$

The condition for the two-particle system to have a bound state of binding energy B_2 is given by

$$\frac{1}{\lambda} = 2\pi\int_0^\infty pdp\frac{g^2(p)}{B_2+p^2}. \tag{19}$$

With this condition the expression (17) for $\bar{T}(s)$ above is rewritten as

$$\langle q|\bar{T}(s)|p\rangle = \left[\frac{3\pi}{2}\int_0^\infty p'dp'\frac{g^2(p')}{(B_2+p'^2)(s-3q^2/4-p'^2+i0)}\right]^{-1}\langle q|T(s)|p\rangle. \tag{20}$$

We have a similar equation for $\bar{V}(s)$.

In terms of these modified potential and t matrix the Amado-Lovelace-Mitra equation becomes

$$\langle q|\bar{T}(s)|k\rangle = \langle q|\bar{V}(s)|k\rangle + \int_0^\infty pdp\frac{\langle q|\bar{V}(s)|p\rangle\langle p|\bar{T}(s)|k\rangle}{(p^2-k^2-i0)}. \tag{21}$$

For energies below the three-particle breakup threshold the potential $\bar{V}(s)$ is real and the analytic structure of Eq. (21) is identical to that of the two-particle Lippmann-Schwinger equation. We employ a subtraction in the kernel of this equation in order to eliminate the singularity of the kernel. The nonsingular equation we consider for an auxiliary matrix $\Gamma(s)$ can be written as[17]

$$\langle q|\Gamma(s)|k\rangle = \langle q|\bar{V}(s)|k\rangle + \int_0^\infty pdp\frac{\langle q|\bar{V}(s)|p\rangle - \langle q|\bar{V}(s)|k\rangle}{(p^2-k^2)}\langle p|\Gamma(s)|k\rangle. \tag{22}$$

In Eq. (22) we have performed a specific subtraction in the kernel. This specific subtraction is a special case of a more general subtraction scheme for calculating the fully off-shell scattering t matrix in terms of the solution of an auxiliary nonsingular integral equation.[17] As the integral in Eq. (22) is nonsingular the $i0$ prescription in the denominator has been dropped. In the present study we shall be only concerned with the on-shell t matrix which is expressed in terms of the auxiliary matrix $\Gamma(s)$ via[17]

$$\langle k|\bar{T}(s)|k\rangle = \frac{\langle k|\Gamma(s)|k\rangle}{1-\int_0^\infty qdq\frac{\langle q|\Gamma(s)|k\rangle}{(q^2-k^2-i0)}}, \tag{23}$$

which can be rewritten as

$$\langle k|\bar{T}(s)|k\rangle = \frac{\langle k|\Gamma(s)|k\rangle}{1-\int_0^\infty dq\frac{q\langle q|\Gamma(s)|k\rangle - k\langle k|\Gamma(s)|k\rangle}{(q^2-k^2)} - \frac{i\pi\langle k|\Gamma(s)|k\rangle}{2}}. \tag{24}$$

In arriving at Eq. (24) we have separated the principal value and the imaginary parts of Eq. (23) and have added a term to the principal value part which is zero. The resultant integral of Eq. (24) is not singular and hence does not need the principal value prescription for its evaluation.

Comparing with Eq. (6) we identify $\langle k|\bar{T}(s)|k\rangle$ as the physical t matrix for the scattering of one boson on the bound state of two others and the phase shift for this scattering, δ_3, is given by

$$cot\delta_3 = \frac{2}{\pi\langle k|\Gamma(s)|k\rangle}\left[1 - \int_0^\infty dq \frac{q\langle q|\Gamma(s)|k\rangle - k\langle k|\Gamma(s)|k\rangle}{(q^2-k^2)}\right]. \tag{25}$$

This expression for $cot\delta_3$ should diverge as $(1/\pi)ln(3k^2/4)$ for small k. In order to calculate the scattering length a_3 for scattering of one boson on the bound state of two others we have to extract this divergent part of $cot\delta_3$ and take the limit $k \to 0$. The integral in Eq. (25) is identically rewritten as

$$\int_0^\infty dq \frac{q\langle q|\Gamma(s)|k\rangle - k\langle k|\Gamma(s)|k\rangle}{(q^2-k^2)} = \int_0^\infty dq \frac{\langle q|\Gamma(s)|k\rangle}{q+k} + k\int_0^\infty dq \frac{\langle q|\Gamma(s)|k\rangle - \langle k|\Gamma(s)|k\rangle}{q^2-k^2}. \tag{26}$$

It is easy to see that the first term on the right hand side of this equation gives the lnk divergence for small k and the second term vanishes in this limit. In order to extract the divergent term for small k in expression (26) we rewrite the first term on the right hand side, for example, as

$$\int_0^\infty dq \frac{\langle q|\Gamma(s)|k\rangle}{(q+k)} = \int_0^\infty dq \frac{\langle q|\Gamma(s)|k\rangle(1+q) - \langle 0|\Gamma(s)|k\rangle}{(1+q)(q+k)} + \langle 0|\Gamma(s)|k\rangle \int_0^\infty \frac{dq}{(1+q)(q+k)}. \tag{27}$$

Equation (27) is just one way of separating the divergent part from the finite part of the first integral on the right hand side of Eq. (26) in the limit $k \to 0$. The first term on the right hand side of Eq. (27) is finite in this limit whereas the last term, which

can now be evaluated analytically, diverges in this limit. This last term of Eq. (27) is immediately evaluated to yield

$$\langle 0|\Gamma(s)|k\rangle \int_0^\infty \frac{dq}{(1+q)(q+k)} = \frac{\langle 0|\Gamma(s)|k\rangle}{k-1} lnk. \tag{28}$$

Now the expression (25) for $cot\delta_3$ is rewritten as

$$cot\delta_3 = \frac{2}{\pi\langle k|\Gamma(s)|k\rangle}[1 - \int_0^\infty dq \frac{\langle q|\Gamma(s)|k\rangle(1+q) - \langle 0|\Gamma(s)|k\rangle}{(1+q)(q+k)} - k\int_0^\infty dq \frac{\langle q|\Gamma(s)|k\rangle - \langle k|\Gamma(s)|k\rangle}{q^2-k^2} - \frac{\langle 0|\Gamma(s)|k\rangle}{k-1} lnk]. \tag{29}$$

The analogue of the effective range expansion (7) in this case is given by

$$cot\delta_3 = a_3 + (1/\pi)ln(3k^2/4) + b(3k^2/4) + c(3k^2/4)^2 + ... \tag{30}$$

with $3k^2/4$ playing the role of the relative kinetic energy, which in the two-particle case given by Eq. (7) is k^2.

The scattering length for scattering of one boson on the bound state of two others, a_3, is now defined as

$$a_3 = \frac{2}{\pi\langle 0|\Gamma(0)|0\rangle}[1 - \int_0^\infty dq \frac{(1+q)\langle q|\Gamma(0)|0\rangle - \langle 0|\Gamma(0)|0\rangle}{q(1+q)} - \frac{\langle 0|\Gamma(0)|0\rangle}{2} ln(3/4)]. \tag{31}$$

We can redefine a_3 in units such that $B_2 = 1$ as in Eq. (9):

$$\bar{a}_3 = a_3 + \frac{ln(B_2)}{\pi}. \tag{32}$$

In Eq. (32) $\bar{a}_3$ so defined is the scattering length for $B_2 = 1$.

III. RESULTS AND DISCUSSIONS

In this section we present numerical results at low energies below the three-particle breakup threshold for scattering and bound states involving three identical bosons in

two dimensions. One of the purposes of this study is to investigate the sensitivity of these results for scattering and bound states on the shape of the two-particle interaction potential while the low-energy two-particle observables, the binding energy B_2 and the scattering length a_2, are maintained constant. In some areas, such as nuclear physics, such a variation of the potential is called an 'off-shell' variation, as the potential is varied in such a way that the (low-energy) two-particle on-shell observables are maintained constant. Usually, the features of the potential well known from physical considerations, e.g. the range, is held fixed in such a study while the shape is varied. We simulate such a variation by varying the constant m (=1,2,4, and 10) of the potential form-factor (2).

Because of considerations of Ref. 4 the results of three-boson bound-state and scattering calculations at low energies should depend only on the two-boson observables B_2 and a_2 for $B_2^{1/2} << \beta$, where β is the range parameter introduced in Eq. (2). Under this condition the behavior of the three-particle system is expected to be universal, independent of the detail of the interaction potential. Due to the low binding of the two-particle system, the three particles are expected to spend most of the time outside the range of the potential. This model independence of the three-particle system increases as the dimension of the space is reduced. In three dimensional space there is a reasonably strong model dependence, whereas in two-dimensional space this model dependence is non-existent at least at low energies and in the weak binding limit[4].

Another way to understand this model independence of three-particle observables in two dimensions is achieved through a consideration of the dynamical equations (15). In the three dimensional equivalent of Eq. (15) the convergence at the upper limit of the integral in this equation is achieved by the potential form-factors g's through Eq. (13). The integral in Eq. (15) diverges as the form-factors g's are

set equal to unity, which corresponds to the two-particle zero-range interaction; this divergence has been related to the appearance of Thomas effect in the case of zero-range potential.[4] Consequently, in three dimensions the solution of Eq. (15) depends on the shape of the potential form-factors. The integral in Eq. (15) in two space dimensions, because of the reduced phase-space, converges even when the potential form-factors g's are set equal to unity.[1,4] The solution of Eq. (15) in this case is expected to be insensitive, or at best very weakly sensitive, to the shape of the potential form-factors. It is important to note that in the limit $\beta \to \infty$ (potential range going to zero) Eq. (15) produces convergent result for three particles in two dimensions. Consequently, the Thomas effect is absent, which implies that the short-range or the large momentum parts of Eq. (15), expected to be the region most sensitive to the off-shell variation of the potential, have no significant relevance on the three-particle low-energy observables in two dimensions.

In our calculations we utilize $\beta = 1$ and vary λ, so that, $B_2 << 1$, which is expected to be the domain of model independence discussed above. We vary the two-particle scattering length a_2 and the two-particle binding energy B_2 during the actual calculation. However, we would like to have results for fixed a_2 and B_2 in order to study the model independence of three-particle observables. This is achieved in a simple fashion. The three-boson results for $\beta = 1$ and any B_2 can be transformed to the case with a fixed B_2 (=1) by considering $\bar{B}_3 = B_3/B_2$, $\bar{a}_3 = a_3 + (1/\pi)lnB_2$, and $\bar{a}_2 = a_2 + (1/\pi)lnB_2$. Here the barred quantities refer to the case with $B_2 = 1$. Equations (3), (4), and (19) exhibit this remarkable scaling property. Suppose that for a given $\lambda = \lambda_0$ and $\beta = \beta_0(= 1)$ we have the two-particle scattering length a_2 and a two-particle bound state of binding energy B_2. It is easy to see from Eq. (19) by scaling arguments that then one has a bound state of binding energy $B_2 = 1$ for $\beta = \beta_0/B_2^{1/2}$ and $\lambda = \lambda_0/B_2^{2m}$ and from Eqs. (9) and (11) that $\bar{a}_2$ is the corresponding

scattering length. A similar scaling argument applied to Eq. (15) reveals that under this modification the three-particle binding energy and scattering length get changed from B_3 and a_3 to $\bar{B}_3$ and $\bar{a}_3$, respectively.

In Ref. 4 it is demonstrated that the plot of $\bar{B}_3$ $(= B_3/B_2)$ versus $\bar{a}_2$ is independent of the potential form-factors. The plots for $m = 1, 2, 4$, and 10 yielded essentially the same result. This demonstrated the model independence of the three-boson bound states in two dimensions. We have reconfirmed this model independence for other values of m.

Here we shall mainly concentrate on the three-boson scattering observables. In Fig. 1 we plot the scattering length $\bar{a}_3$ versus $\bar{a}_2$ for m =1,2,4, and 10 in Eq. (2). For small $\bar{a}_2$ this plot is completely universal with all m yielding essentially the same result. For large $\bar{a}_2$ and $\bar{a}_3$ this model independence tends to break down as we start moving out of the weak binding limit $(B_2^{1/2}/\beta << 1)$. In the actual calculation for the extreme point given by $100\bar{a}_2 = 15$, $B_2^{1/2}/\beta \approx 0.35(m = 1), 0.24(m = 2), 0.01(m = 4), and 0.08(m = 10)$. We see that the smaller values of m have deviated more from the weak-binding limit compared to the larger values of m. This justifies the deviations of the $m = 1$ and 2 curves from the $m = 4$ and 10 curves for large $\bar{a}_2$. The $m = 10$ curve in this case is closer to the universal weak-binding limit. Even in the extreme point $100\bar{a}_2 = 15$ the dipersion in $\bar{a}_3$ is small and is of the order of ten per cent. This case is comparable to the three-dimensional three-nucleon system where the the range parameter $\beta = 1.4fm$ and square-root of the deuteron binding is $B_2^{1/2} = 0.23fm$.[17] However, in the three-nucleon system the dispersion in the value of the S wave neutron-deuteron scattering length a_3 is much larger, with a_3 varying from -$1fm$ to $1fm$. Hence, unlike in the three-dimensional case, in the weak binding limit $\bar{a}_2$, and $B_2(= 1)$ are enough to determine the scattering length a_3 of one boson on the bound state of two bosons independent of the shape of the interaction potential

as is obvious from Fig. 1.

In Fig. 2 we plot $\bar{a}_3$ versus $\bar{B}_3$ for $m = 1$ and 10. The plots for $m = 2 and 4$ lie between those for $m = 1$ and 10. For small $\bar{a}_3$ the model independence of the result is obvious. As in Fig. 1 this model independence starts to break down as $\bar{a}_3$ increases and we move out of the region of weakly bound two-particle bound states. This plot should be compared to the Phillips[19] plot of the three-nucleon system, which yields a linear correlation between the triton-binding energy and the S-wave spin-doublet neutron-deuteron scattering length while different potential models with fixed two-nucleon binding and scattering lengths are employed. However, there is an interesting difference between the Phillips plot and the present plot. The present variation of $\bar{B}_3$ and $\bar{a}_3$ arises due to the variation of $\bar{a}_2$ while in the correlation of the Phillips plot both two-nucleon binding and scattering length are fixed. From Figs. 1 and 2 we realize that once we hold both the two-particle scattering length a_2 and binding energy B_2 fixed this will correspond to a unique scattering length a_3 and binding energy B_3 with an estimated dispersion of ten per cent. In other words, in two dimensions with fixed a_2 and B_2, the $\bar{a}_3$ versus $\bar{B}_3$ plot reduces practically to a point with some dispersion, in contrast to the three dimensional three nucleon system. Consequently, the weakly bound three dimensional three-nucleon system is model dependent or sensitive to the off-shell behavior of two-nucleon interaction potential, whereas the two dimensional three boson system in identical situations is virtually model independent and insensitive to the off-shell behavior of two-particle interaction.

Finally, we see that this model independence of three-boson binding energy and scattering length in two dimensions can be extended to the scattering phase shifts δ_3 below the breakup threshold of one boson on two-boson bound state. In order to test this we calculated δ_3 for $k = 1$, and $B_2 = 1$. This corresponds to a three-particle

center of mass energy s given by Eq. (14). In this case we plot $tan\delta_3$ versus $\bar{a}_3$ in Fig. 3. As in Figs. 1 and 2 $\bar{a}_2$ is varying in this calculation. For small $\bar{a}_3$ again the result is model independent. When $\bar{a}_3$ and $\bar{a}_2$ increases this model independence starts to break down as in Figs. 1 and 2. This plot should be compared to that of Frederico and Goldman[20] (FG) for the three-dimensional three-nucleon system, where they plotted $kcot\delta$ (δ being the nucleon-deuteron S wave scattering phase shift) versus nucleon-deuteron scattering length and obtained a linear correlation as in the Phillips plot. This study of FG is the three-dimensional analogue of the present study in two dimensions in Fig. 3. However, in the study of FG both the two-nucleon binding energy and scattering length are held fixed. In the three-nucleon system even in the weak binding limit, in contrast to the two-dimensional three-boson case, the result is model dependent. In the present study the two-particle binding energy is fixed but the two-particle scattering length is varying. Once the corresponding quantities (B_2 and a_2) are held fixed in the present study we have only a section of the present plot in Fig. 3, which is just a point with small dispersion. Again we reach the same conclusion of model independence (off-shell independence) of the three-particle scattering observables in two dimensions.

IV. SUMMARY

In this paper we have presented a systematic study of scattering and bound states of three identical bosons in two space dimensions at low energies using the Faddeev equations in the momentum space. The two-particle potential is taken to be of the S wave separable type with varying form-factors. In this case the Faddeev equations reduce to one variable integral equations, which we refer to as the Amado-Lovelace-Mitra equations. One of the purposes of the present study is to investigate the model (in)dependence of the results once the two-particle binding energies and scattering

lengths are held fixed. In the limit of weak two-particle binding ($B_2^{1/2} << \beta$) we find the result of both scattering and bound states of the three-boson system to be independent of the variation of potential shape provided that the two-particle scattering length and binding energy are held fixed. In the three dimensional three-particle system the results for three particle bound states and scattering are sensitive to the shape of the potential employed even when the two-particle binding energy and scattering lengths are held fixed. This behaviour of the three-particle system in two dimensions has been justified by considering the analytic behavior of the dynamical equation (15) and has been related to the absence of the Thomas effect in this case.[4] The present finding is expected to have interesting consequence in the study of several few-particle systems in two-dimensions. We have made a limited model study but we have argued that our conclusions are expected to be generally valid provided that we are limited to low energies and consider only the case of weak two-particle binding: $B_2^{1/2} << \beta$, where B_2 is the two-particle binding and β is the inverse range parameter of the two-particle interaction.

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Figure Caption

1. The three-boson scattering length $\bar{a}_3$ plotted versus the two boson scattering length $\bar{a}_2$ for the separable potential (2) with $m = 1, 2, 4$, and 10.

2. The three-boson scattering length $\bar{a}_3$ plotted versus the the three-boson binding $\bar{B}_3(= B_3/B_2)$ for the separable potential (2) with m= 1 and 10.

3. The tangent of the three-boson phase-shift $tan\delta_3$ of Eq. (31) plotted versus the three-boson scattering length $\bar{a}_3$ for the separable potential (2) with m= 1 and 10.

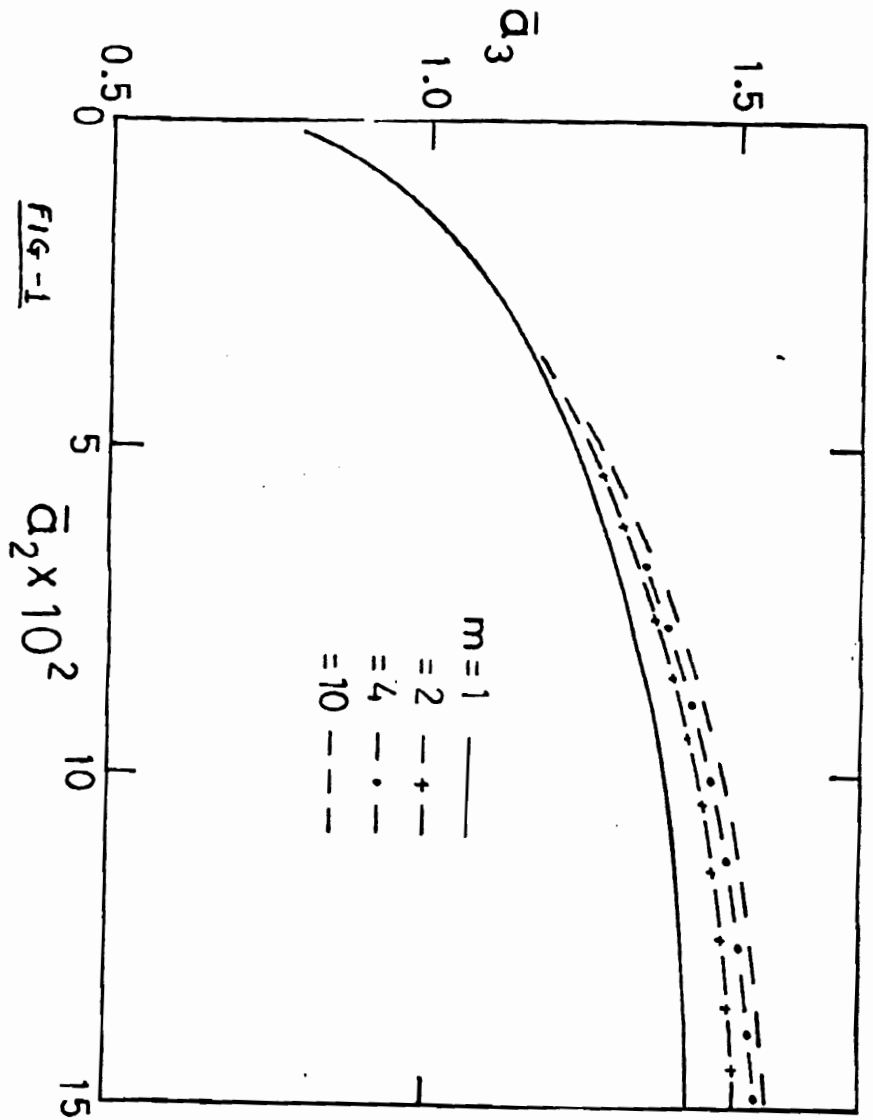

FIG-1

Phys. Rev. A. Adhikari et al

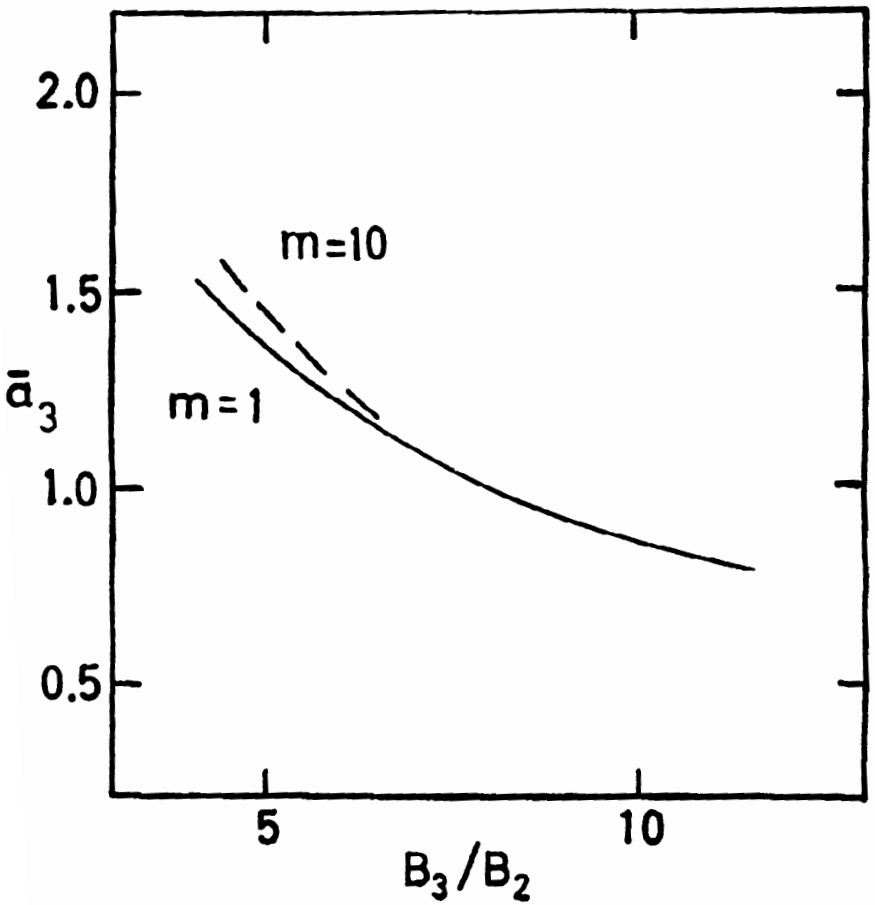

FIG 2

Phys. Rev. A. Adhikari et al

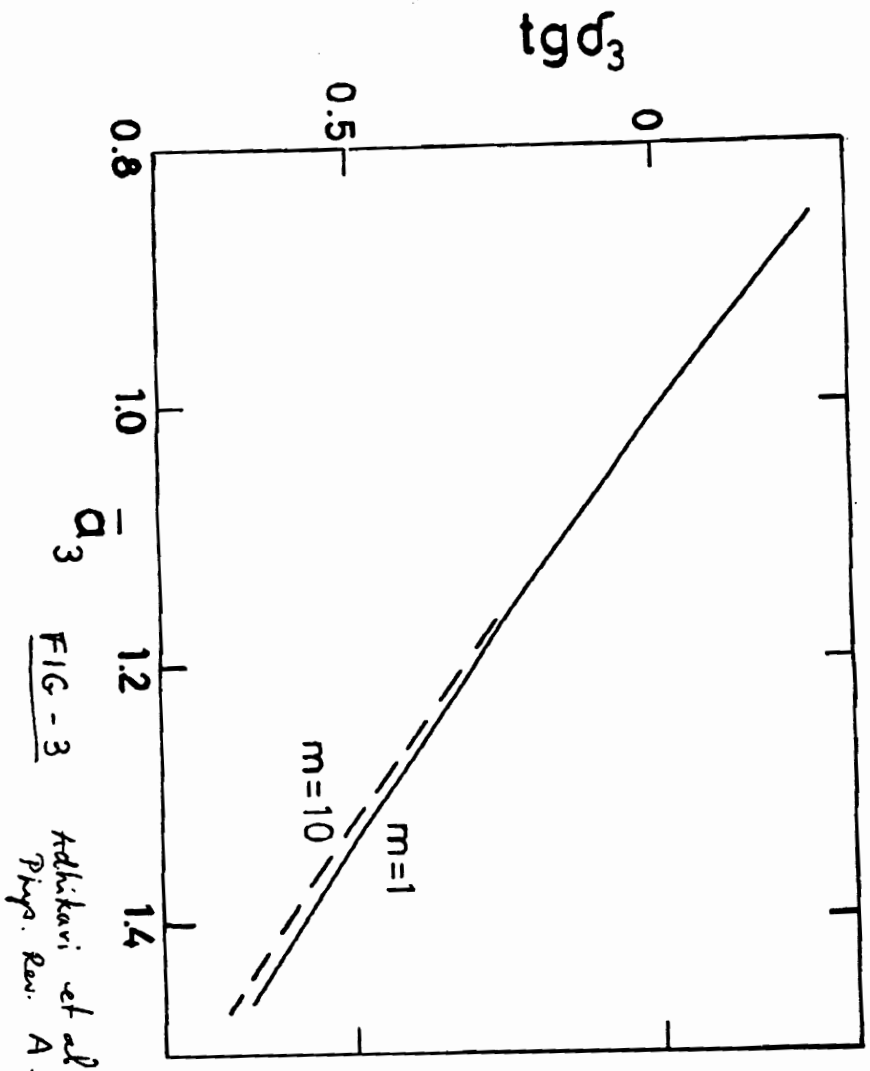

FIG - 3 Adhikari et al Phys. Rev. A.