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The Lipkin model second order phase transition in the deformed $su(2)$ algebra description

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Abstract

The standard Lipkin model is studied in the context of the $su(2)$ deformed algebra and a new spectrum is obtained with which a new energy gap curve is calculated. A suppression of the second order phase transition originally present is observed for the deformed case.

1 Introduction

The standard Lipkin model (SLM) [1],has been widely used to test different formalisms and approximations techniques in many body physics [2] because it is an exactly solvable model and adequate for those purposes.

In its original form, the SLM was more easily solved due to the realization that the starting many fermion Hamiltonian can be rewritten in terms of quasi - spin operators obeying a standard $su(2)$ algebra. In this context the collective excitations are more conspicuous and as a fundamental feature this model exhibits a second order phase transition [3].

As a result of the study of two - dimensional statistical models and in quantum field theory also [4,5], there appeared a generalization of the Lie algebra, called quantum or deformed algebra. This mathematical structure is expressed in terms of new commutation relations involving a free parameter, denoted by q, which characterises the deformation of the original algebra and when $q \longrightarrow 1$, one recovers the starting algebra, and, in particular, the deformed $su(2)$ algebra has been the object of extensive studies [6].

These ideas has been used in nuclear and molecular physics [7,8] to describe collective rotations and also applied to the Jaynes - Cummings model in order to study the effects of the deformation of the population invertions [9].

In the present work we applied the same ideas to study the SLM under the $su(2)_q$ description and, in particular, the behaviour of the second order phase transition in terms of q. A short review of the SLM and its q-version are presented in section 2. The results for the gap solutions are obtained exactly and in perturbation theory up to fourth order in section 3, while section 4 is devoted to the final comments and conclusions.

2 The Lipkin model (the $su(2)$ and $su(2)_q$ algebras)

For our purposes, we will briefly review here the essential features of the SLM only.

In this model a collection of N fermions is distributed in two N - fold degenerate levels separated by the energy ϵ. Each level is characterized by a quantum number σ which is equal to +1(-1) for the higher (lower) level and a quantum number p associated with the degeneracy in each level. The Hamiltonian of the model is given by

$$H = \frac{\epsilon}{2}\sum_{p,\sigma}\sigma\hat{a}^{\dagger}_{p,\sigma}\hat{a}_{p,\sigma} + \frac{V}{2}\sum_{p,p',\sigma}\hat{a}^{\dagger}_{p,\sigma}\hat{a}^{\dagger}_{p',\sigma}\hat{a}_{p',-\sigma}\hat{a}_{p,-\sigma} \ . \tag{1}$$

The collective excitations in this many fermion model is more clearly studied in the quasi - spin formalism, where the corresponding Hamiltonian is exactly diagonalized. To this end one introduces the quasi - spin operators

$$J_+ = \sum_p \hat{a}^{\dagger}_{p,+1}\hat{a}_{p,-1} \tag{2}$$

$$J_- = \sum_p \hat{a}^{\dagger}_{p,-1}\hat{a}_{p,1} \tag{3}$$

$$J_z = \frac{1}{2}\sum_{p,\sigma}\sigma\hat{a}^{\dagger}_{p,\sigma}\hat{a}_{p,\sigma} \tag{4}$$

with which the Hamiltonian is rewritten as

$$H = \epsilon J_z + \frac{V}{2}(J_+^2 + J_-^2) \ , \tag{5}$$

where the operators obey the standard $su(2)$ commutation relations, i.e.,

$$[J_+, J_-] = 2J_z \tag{6}$$

and

$$[J_z, J_\pm] = \pm J_\pm \ . \tag{7}$$

Since $[H, J^2] = 0$ therefore the Hamiltonian, equation (5), can be diagonalized within each multiplet. We want to note that the ground state belongs to the multiplet $J = \frac{N}{2}$ which is an eigenstate of J^2 and J_z, with eigenvalues $\frac{N}{2}(\frac{N}{2}+1)$ and $-\frac{N}{2}$ respectively.

The exact solution and standard second and fourth order results for different N's were obtained in Ref.(1). From these results it is a simple task to calculate the energy gap, $(E_1 - E_0)/\epsilon$, which is a very important quantity to verify the existence of a second order transition in this model [3], being the effective interaction written in this case as $\chi = NV/\epsilon$. The question one can pose here is that if this second order transition is preserved when the underlying $su(2)$ algebra is deformed into $su(2)_q$, where the new commutation relations are defined by [6]

$$[J_+, J_-] = [2J_z] \tag{8}$$

$$[J_z, J_\pm] = \pm J_\pm \ , \tag{9}$$

where

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{10}$$

and q is the parameter characterizing the deformation of the original algebra.

To answer the above questions we will consider the same formal Hamiltonian, equation (5), but now the operators involved obey equations (8) and (9) instead of (6) and (7). The great advantage of this choice is that we can apply straightforwardly the calculation techniques developed previously by Lipkin et al. [1].

3 Gap solutions of the Lipkin model with the $su(2)_q$

Once given eqs. (8) and (9) it exists one representation, as shown by Jimbo [6], for each j, $j = 0, 1/2, 1, ...$, with basis $\mid jm\rangle$,$-j \leq m \leq j$, such that

$$J_z \mid jm\rangle = m \mid jm\rangle \tag{11}$$

$$J_\pm \mid jm\rangle = \sqrt{[j \mp m][j \pm m + 1]} \mid jm \pm 1\rangle \tag{12}$$

$$\mathcal{C} \mid jm\rangle = [j + \frac{1}{2}]^2 \mid jm\rangle \ , \tag{13}$$

where $\mathcal{C}$ is the Casimir operator of $su(2)_q$ defined as

$$\mathcal{C} = [J_z + \frac{1}{2}]^2 + J_- J_+ = [J_z - \frac{1}{2}]^2 + J_+ J_- \ . \tag{14}$$

Since this Casimir operator commutes with the new Hamiltonian in the weak sense, $\langle jm' \mid [H, \mathcal{C}] \mid jm\rangle$, for all m, m', the property of the exact diagonalization within each multiplet is preserved in the same fashion as in the standard Lipkin model. In what follows we restrict ourselves to the ground state multiplet, $j = \frac{N}{2}$.

For the same symmetry reasons, we can also obtain analytical solutions for the energy levels up to $N = 8$, and, when $q \longrightarrow 1$, we get back the same exact solutions as the SLM. Here the energy levels are also characterized by two parameters V/ϵ and $[N]$ instead of N, and, in order to study the behaviour of the gap, $(E_1 - E_0)/\epsilon$, we introduced an effective interaction defined by

$$\chi_{eff} = \frac{[N]V}{\epsilon} \tag{15}$$

which now replaces the SLM 's χ; we want to note that $\chi_{eff} \longrightarrow \chi$ when $q \longrightarrow 1$. Using the equations (11) and (12) we numerically calculated the energy spectrum of the ground state multiplet. With these results we straightforwardly obtained the behaviour of the gap for $N = 8, 30$ and 50 and for two different values of q, namely, 1.05 and 1.10 in order to see the

$q-$dependence of the second order transition as described in the SLM. The corresponding results are depicted in figure 1(a,b,c).

To test the validity of the simplest approximation method we adapted the standard perturbative teory up to second and fourth order for the deformed Lipkin model (DLM). As in the previous case we calculated the matrix elements for two fixed values of q only within the ground state multiplet to compare with the exact solutions and also study the behaviour of the approximations when $q \longrightarrow 1$.

Using the Hamiltonian given by equation (5), in which the J 's obey the commutation relations (8) and (9) and the eigenvalue equations (11), (12) and (13), we obtain

$$\frac{E_1 - E_0}{\epsilon} = 1 - \frac{1}{2}\lambda_{eff}^2 \frac{[2]\,[N-1]}{4[N]^2}\Big\{[3]\,[N-2] - [N]\Big\} \tag{16}$$

up to second order and

$$\frac{E_1 - E_0}{\epsilon}\Big)_{4^{th}order} = -\frac{1}{8}\lambda_{eff}^4 \frac{1}{32[N]^4}\Big\{[2]\,[3]\,[4]\,[N-1]\,[N-3]\Big[[5] \times \times [N-4] - [N]\Big] - 2[2]^2[N-1]^2\Big[[3]^2[N-2]^2 - [N]^2\Big]\Big\} \tag{17}$$

is the fourth order correction term.

The effects of the q - deformation on the energy gap, up to fourth order, can be seen in figure 2(a,b,c) comparatively to the same order SLM results.

4 Comments and Conclusions

Here we have analysed the Lipkin model in the context of q - deformed $su(2)$ algebra in order to study the second order phase transition, already manifested in the SLM, as a function of the q - deformation. This was performed by using the exact solutions and the perturbative ones both obtained from the deformed model Hamiltonian. All this was possible due to the existence of one representation for the $su(2)_q$ algebra, as shown by Jimbo [6].

As is well known, for increasing N the gap behaviour in the SLM approaches the standard form [3]; this can be seen in figure 1(a,b,c). On the other hand, for the DLM exact results and a fixed q - deformation, the new energy gap curve deviates from the SLM one as N increases and as a general feature one notes that the second order phase transition is suppressed when q increases.

The same behaviour is exhibited for the energy gap curves in the DLM with perturbative solutions.

In conclusion we want to remark that the suppression of the second order phase transition in the DLM strongly resembles the vanishing of the revivals [9] in the deformed Jaynes - Cummings model [10] when q increases.

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Figure Captions

1. The exact gap curves for : (a) 8, (b) 30 and (c) 50 particles. The solid curve represents the $q = 1$ results while the dashed and dot-dashed curves are obtained for $q = 1.05$ and $q = 1.1$ respectively.

2. The perturbation theory results up to fourth order for : (a) 8, (b) 30 and (c) 50 particles. The solid curve represents the $q = 1$ results while the dashed and dot-dashed curves are obtained for $q = 1.05$ and $q = 1.1$ respectively.

FIGURE 1A

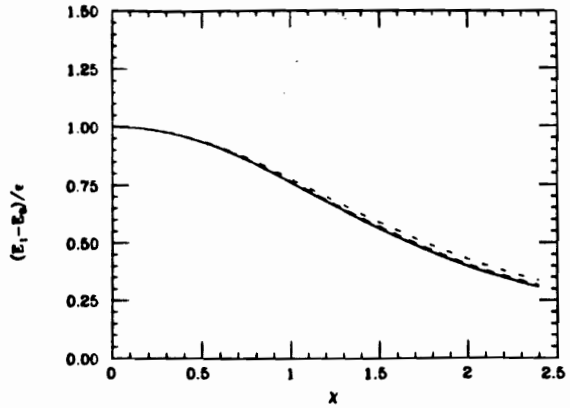

FIGURE 1B

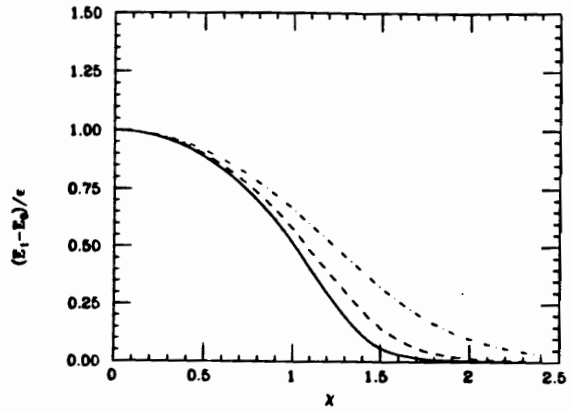

FIGURE 1C

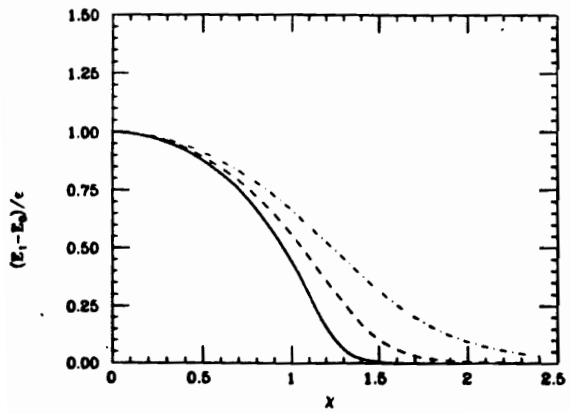

FIGURE 2A

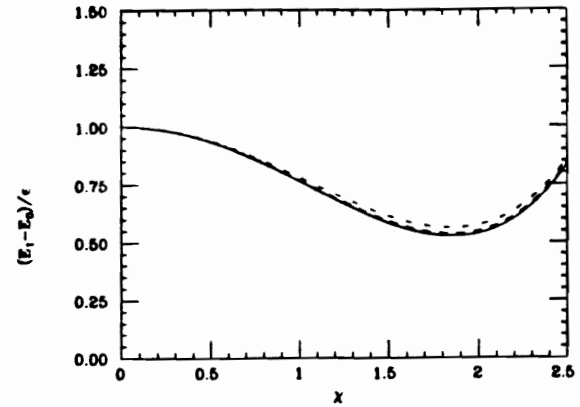

FIGURE 2B

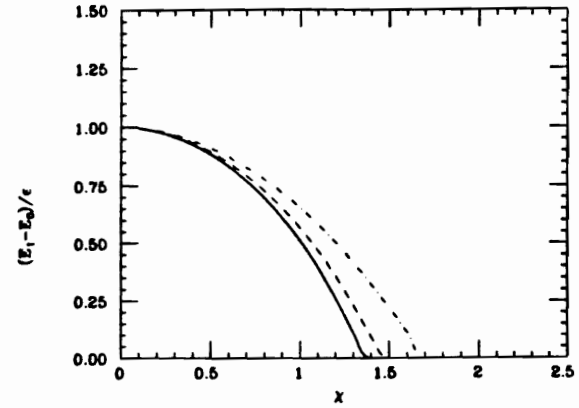
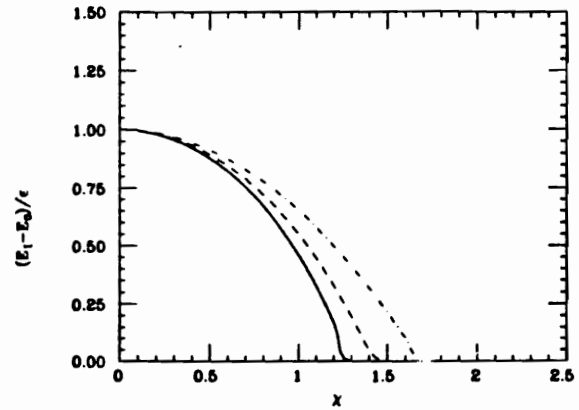
FIGURE 2C