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1 Work partially supported by CNPq
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Connection between the Affine and Conformal Affine Toda models and their Hirota's solution[1]

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ABSTRACT

It is shown that the Affine Toda models (AT) constitute a "gauge fixed" version of the Conformal Affine Toda model (CAT). This result enables one to map every solution of the AT models into an infinite number of solutions of the corresponding CAT models, each one associated to a point of the orbit of the conformal group. The Hirota's τ-function are introduced and soliton solutions for the AT and CAT models associated to $\hat{SL}(r+1)$ and $\hat{SP}(r)$ are constructed.

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1 Introduction

Two dimensional integrable non linear models have been studied quite extensively in the past years. More recently those models possessing conformal invariance have received special attention. One reason for that is the connection between integrability and conformal properties, which endow those theories with a very rich structure.

Among the models most studied are the Toda field theories. These can be classified in three hierarchies according to the algebraic structure underlying their associated linear system (zero curvature condition or Lax equation). First come the Conformal Toda models (CT) associated to the finite simple Lie algebras with the simplest example being the Liouville model (SL(2)). These are conformally invariant two dimensional field theories and their solutions have been constructed using highest weight representations of Lie algebras [1]. The symmetries of these models are described by the so called W-algebras [2,3]. The CT theories can be obtained via Hamiltonian reduction from WZNW models [3]. The one dimensional version of these models can also be obtained by Hamiltonian reduction from non compact symmetric spaces [4,5]. Then we have the Affine Toda models (AT) associated to the loop algebras. These are not conformally invariant but have been shown to be completly integrable [6]. Solutions to some classes of models of this hierarchy have been constructed using different methods [7,8]. However the analysis of Leznov and Saveliev [1] does not work in this case because of lack of highest weight representations. The most popular member of that hierarchy is the Sinh-Gordon model associated to the SL(2) loop algebra. Finally there are the recently proposed Conformal Affine Toda models (CAT) which are related to the Kac-Moody algebras [9,10,11]. These are conformally invariant field theories, and it has been shown that they can be obtained via Hamiltonian reduction from WZNW models associated to two-loop Kac-Moody algebras [9,12]. The symmetries of these models are described by some type of W-infinity algebra [13,14]. The solutions of the CAT models can be constructed using the Leznov and Saveliev analysis as it has been shown, for the $\hat{SL}(2)$ case, in ref. [10]. Here we give the corresponding solution to any Kac-Moody algebra. The solutions of the AT models can then be obtained from the solutions of the CAT models by setting a particular field to zero. However, finding the explicit space-time dependence of the solution is quite hard in pratice due to the fact one is dealing with representations of the Kac-Moody algebra with non vanishing central term.

The CAT model contains two extra fields with respect to its corresponding AT counterpart with one of them being a free field. In this note we show that the AT models constitute in fact a "gauge fixed" version of the CAT models. The space of regular solutions of this free field constitute an orbit of the conformal group. We then show that the free field can be "gauged" away by a conformal transformation leading the CAT model to the AT model together with the second extra field, lying in the transformed space time. As a consequence of this result any solution of the AT models can be mapped into an infinite number of solutions of the CAT model, each corresponding to a point of such orbit. We also introduce the Hirota's τ functions for the CAT model and show how to obtain soliton solutions for the cases of $\hat{SL}(r+1)$ and $\hat{SP}(r)$. Our results constitute in fact, a generalization of the analysis for $\hat{SL}(r+1)$ given in ref [8]. Explicit examples for $\hat{SL}(2)$ case are considered. The generalization for the remaining Kac-Moody algebras is now under

investigation.

2 The connection between the AT and CAT models

The equations of motion of the CAT model are given by [9,10]:

$$\partial_-\partial_+\phi^a = q^a e^{K_{ab}\phi^b} - q^0 l_a^\psi e^{-K_{\psi b}\phi^b + 2\mu} \tag{1}$$

$$\partial_-\partial_+\mu = 0 \tag{2}$$

$$\partial_-\partial_+\nu = \frac{2}{\psi^2} q^0 e^{-K_{\psi b}\phi^b + 2\mu} \tag{3}$$

where $K_{ab} = 2\alpha_a.\alpha_b/\alpha_b^2$ is the Cartan Matrix of a simple Lie algebra $\mathcal{G}$, $a, b = 1, ...,$ rank $\mathcal{G}$, ψ is the highest root of $\mathcal{G}$, $K_{\psi b} = 2\psi.\alpha_b/\alpha_b^2$, l_a^ψ are positive integers appearing in the expansion $\frac{\psi}{\psi^2} = l_a^\psi \frac{\alpha_a}{\alpha_a^2}$, where α_a are the simple roots of $\mathcal{G}$ and q^a, q^0 are coupling constants. The derivatives $\partial_\pm$ are w.r.t. the light cone coordinates $x^\pm = x \pm t$.

These equations can be written in the form of a zero curvature condition (associated linear system) [10,9]

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0 \tag{4}$$

where

$$A_+ = \partial_+\Phi + e^{ad\Phi}\mathcal{E}_+ \ , \ A_- = -\partial_-\Phi + e^{-ad\Phi}\mathcal{E}_- \tag{5}$$

and

$$\Phi = \frac{1}{2}\sum_{a=1}^{rank\mathcal{G}} \phi^a H_a^0 \ + \ \mu D + \frac{1}{2}\nu C \ . \tag{6}$$

$$\mathcal{E}_+ = \sum_{a=1}^{rank\mathcal{G}} E_{\alpha_a}^0 + E_{-\psi}^1 \quad , \quad \mathcal{E}_- = \sum_{a=1}^{rank\mathcal{G}} q^a E_{-\alpha_a}^0 + q^0 E_\psi^{-1} \tag{7}$$

These equations are invariant under the conformal transformations

$$x_+ \to \tilde{x}_+ = f(x_+) \ , \ x_- \to \tilde{x}_- = g(x_-) \tag{8}$$

if the new fields are defined as:

$$e^{-\tilde{\phi}^a(\tilde{x}_+,\tilde{x}_-)} = (\frac{df}{dx_+})^{r^a}(\frac{dg}{dx_-})^{r^a} e^{-\phi^a(x_+,x_-)} \tag{9}$$

$$e^{-\tilde{\mu}(\tilde{x}_+,\tilde{x}_-)} = (\frac{df}{dx_+})^{\frac{h}{2}}(\frac{dg}{dx_-})^{\frac{h}{2}} e^{-\mu(x_+,x_-)} \tag{10}$$

$$e^{-\tilde{\nu}(\tilde{x}_+,\tilde{x}_-)} = (\frac{df}{dx_+})^{B}(\frac{dg}{dx_-})^{B} e^{-\nu(x_+,x_-)} \tag{11}$$

where h is the Coxeter number of the algebra $\mathcal{G}$, B is arbitrary and r^a is defined as

$$r^a = \sum_{a=1}^{rank\mathcal{G}} K_{ab}^{-1} \tag{12}$$

and it satisfies

$$\sum_{b=1}^{rank\mathcal{G}} K_{ab} r^b = 1 \ , \ \sum_{b=1}^{rank\mathcal{G}} K_{\psi b} r^b = h - 1 \tag{13}$$

Therefore the exponential of the (negative of) fields ϕ^a, μ, ν are primary fields of dimensions $(r^a, r^a), (h/2, h/2), (B, B)$ respectively.

The general solution of the CAT model associated to $\hat{SL}(2)$ was constructed in ref [10] using the method of Leznov and Saveliev [1]. The generalization to any other algebra is quite straightforward and the result is

$$e^{-\phi^a(x_+,x_-) - l_a^\psi \nu(x_+,x_-)} = \langle \lambda_{(a)} \mid e^{K_+(x_+)} M_+(x_+) M_-^{-1}(x_-) e^{-K_-(x_-)} \mid \lambda_{(a)} \rangle \tag{14}$$

$$e^{-\nu(x_+,x_-)} = \langle \lambda_{(0)} \mid e^{K_+(x_+)} M_+(x_+) M_-^{-1}(x_-) e^{-K_-(x_-)} \mid \lambda_{(0)} \rangle \tag{15}$$

and since μ is a free field

$$\mu(x_+, x_-) = \mu_+(x_+) + \mu_-(x_-) \tag{16}$$

where $\mu_\pm(x_\pm)$ are arbitrary functions, $K_\pm(x_\pm)$ are elements of the Cartan subalgebra of the Kac-Moody algebra $\hat{\mathcal{G}}$ associated to $\mathcal{G}$, containing the parameters of the solution

$$K_\pm(x_\pm) = \sum_{a=1}^{rank\mathcal{G}} \theta_\pm^a(x_\pm) H_a^0 \mp 2\mu_\pm(x_\pm) D + \xi_\pm(x_\pm) C \tag{17}$$

$M_\pm$ are exponentiations of real linear combinations of the positive/negative root step operators of $\hat{\mathcal{G}}$. The parameters in $M_\pm$ are functions of x_+/x_- only and are determined in terms of the parameters of the solution through

$$\partial_+ M_+ M_+^{-1} = -e^{-adK_+}\mathcal{E}_+ \tag{18}$$

$$M_-\partial_- M_-^{-1} = e^{-adK_-}\mathcal{E}_- \tag{19}$$

The group elements $M_\pm$ appear in fact in the Gauss-type decomposition

$$g_1 = e^{K_-} N_+ M_- \ , \ g_2 = e^{K_+} N_- M_+ \tag{20}$$

where g_1 and g_2 are defined as

$$e^{-2\Phi} = g_2 g_1^{-1} \tag{21}$$

The states $\mid \lambda_{(a)} \rangle$, $a = 1, 2, ...rank\mathcal{G}$ and $\mid \lambda_{(0)} \rangle$ are highest weight states of representations of the Kac-Moody algebra $\hat{\mathcal{G}}$ where the highest weights are the fundamental weights $\lambda_{(a)}$ and $\lambda_{(0)}$ of $\hat{\mathcal{G}}$.

Eqs. (14)-(16) constitute the general solution for the CAT model equations of motion. We see it depends upon $2(\text{rank}\mathcal{G} + 2)$ chiral parameters. However, in practice, such result is not very useful when one wants to know the explicit space-time dependence of a given solution. The reason is, basically, that the quantities $M_\pm$ are exponentiations of an infinite number of generators. Unlike the finite dimensional case [1] they do not belong to a nilpotent subgroup. On the other hand, such form of the solutions are very closely related to

dressing transformations, exchange algebras and are useful in the study of the symmetries of the model [15,16].

We now show that the Affine Toda model can be understood as a CAT model when the conformal symmetry is in some sense "gauge fixed". The idea consists basically in conformally tranforming the field μ away for every solution of it. We start by redefining the fields as

$$\varphi^a = \phi^a - \frac{2r^a}{h}\mu \qquad \eta = \frac{2}{h}\mu \tag{22}$$

In terms of them the equations of motion become

$$\partial_-\partial_+\varphi^a = (q^a e^{K_{ab}\varphi^b} - l_a^\psi q^0 e^{-K_{\psi b}\varphi^b})e^\eta \tag{23}$$

$$\partial_-\partial_+\eta = 0 \tag{24}$$

$$\partial_-\partial_+\nu = \frac{2}{\psi^2} q^0 e^{-K_{\psi b}\varphi^b} e^\eta \tag{25}$$

and the quantity Φ in (6) becomes

$$\Phi = \frac{1}{2}\left(\sum_{a=1}^{rank\mathcal{G}} \varphi^a H_a^0 + \eta T_3 + \nu C\right) \tag{26}$$

where $T_3 = 2\hat{\delta}.H^0 + hD$, with $\hat{\delta} = \frac{1}{2}\sum_{\alpha>0}\frac{\alpha}{\alpha^2}$, is the generator used to perform the so called homogeneous grading of a Kac-Moody algebra. Notice that e^{φ^a} are scalars under conformal transformations . If we set the parameter B to zero in (11), e^ν is also scalar. On the other hand e^η is a $(1,1)$ primary field. Performing now a conformal transformation (9)-(11) with

$$f'(x_+) = e^{\eta_+(x_+)} \ , \ g'(x_-) = e^{\eta_-(x_-)} \tag{27}$$

where $\eta_\pm(x_\pm)$ are solutions of the η field, i.e., $\eta(x_+,x_-) = \eta_+(x_+) + \eta_-(x_-)$ (see (16)), one obtains

$$e^{-\tilde{\varphi}^a(\tilde{x}_+,\tilde{x}_-)} \to e^{-\varphi^a(x_+,x_-)} \ , \ e^{-\tilde{\eta}(\tilde{x}_+,\tilde{x}_-)} \to 1 \ , \ e^{-\tilde{\nu}(\tilde{x}_+,\tilde{x}_-)} \to e^{-\nu(x_+,x_-)} \tag{28}$$

Therefore the space of regular solutions of the η field constitute just one orbit of the conformal group. Consequently, for every solution of η, the equations of motion of the CAT model can be written as

$$\tilde{\partial}_-\tilde{\partial}_+\tilde{\varphi}^a = q^a e^{K_{ab}\tilde{\varphi}^b} - l_a^\psi q^0 e^{-K_{\psi b}\tilde{\varphi}^b} \tag{29}$$

$$\tilde{\partial}_-\tilde{\partial}_+\tilde{\nu} = \frac{2}{\psi^2} q^0 e^{-K_{\psi b}\tilde{\varphi}^b} \tag{30}$$

where the new space time coordinates are determined in terms of the old ones through the given solution of η, i.e. $\tilde{x}_+ = \int^{x_+} dy_+ e^{\eta_+(y_+)}$, $\tilde{x}_- = \int^{x_-} dy_- e^{\eta_-(y_-)}$. From now on we drop the tildes.

Eqs. (29) correspond to the Affine Toda equation, which for $\mathcal{G} = SL(2)$ become the Sinh-Gordon equation. However (29)-(30) can be written in a more compact form. Introduce the fields

$$\zeta^a = \varphi^a + l_a^\psi \frac{\psi^2}{2}\nu \ , \ \zeta^0 = \frac{\psi^2}{2}\nu \tag{31}$$

Using the fact that $K_{ab}l_b^\psi = 2\alpha_a.\psi/\overline{\psi^2} \equiv -K_{a0}$ and $K_{0b}l_b^\psi \equiv -2\psi.\alpha_b/\alpha_b^2 l_b^\psi \equiv -K_{00}$ one can write (29)-(30) as

$$\partial_-\partial_+\zeta^i = q^i e^{K_{ij}\zeta^j} \tag{32}$$

where $i,j = 0,1,2...rank\mathcal{G}$, and where we have introduced the extended Cartan matrix K_{ij} of the Kac-Moody algebra $\hat{\mathcal{G}}$, $K_{ij} \equiv 2\alpha_i.\alpha_j/\alpha_j^2$ with α_i being the simple roots of $\hat{\mathcal{G}}$, i.e. $\alpha_0 = -\psi$ and the remaining α_a, $a = 1,2,...rank\mathcal{G}$, as before. Eq. (32) is not just the Affine Toda model since it possesses one extra field. In the literature one finds the Affine Toda (or Toda Lattice) model written with one extra field as [6]

$$\partial_-\partial_+\rho^i = \sum_{j=0}^{rank\mathcal{G}} K_{ij} q^j e^{\rho^j} \tag{33}$$

The usual Affine Toda is then obtained by noticing that the extended Cartan matrix is singular and therefore has a null vector, $n^i K_{ij} = 0$ where $\psi = n_a^\psi \alpha_a$ and $n_0^\psi = 1$. So, the field $\rho \equiv n_i^\psi \rho^i$ satisfies $\partial_-\partial_+\rho = 0$. Restricting the model to the vacuum $\rho = 0$ one eliminates one field and obtains the usual Affine Toda as given by (29) [6]. However going from (32) to (33) would involve a singular transformation between the fields ζ^i and ρ^i and therefore they are not really the same model.

The conformal transformation (27) is closely related to the transformation performed in ref. [12] on the currents of the two-loop Kac-Moody algebra [9] in order to decouple the non-zero modes of a central current . Under (27), the two-loop Kac-Moody currents $J_R(x_+) \equiv k\hat{g}^{-1}\partial_+\hat{g}$ and $J_L(x_-) \equiv -k\partial_-\hat{g}\hat{g}^{-1}$, where $\hat{g}$ is an exponentiation of generators of an ordinary Kac-Moody algebra [9,12], transform as

$$J_R(x_+) \to e^{\int^{x_+} dy_+ C_R(y_+)} J_R(x_+) \ , \ J_L(x_-) \to e^{\int^{x_-} dy_- C_R(y_-)} J_L(x_-) \tag{34}$$

where $C_{R/L}$ are the components of the currents in the direction of the generator D and can be written as $C_{R/L}(x_\pm) = \partial_\pm\eta_\pm(x_\pm)$ with $\eta_\pm$ as in (27). These are the transformations performed in [12] except that there the zero modes of $C_{R/L}$ were excluded due to the periodic boundary condition imposed on $J_{R/L}$.

3 Hirota's method

We now describe how to use the Hirota's method [17] to construct solutions for the Affine and Conformal Affine Toda models. We introduce the τ-functions as

$$\zeta^j = l_j^\psi(-\ln\tau_j + \sigma) + \vartheta_j \tag{35}$$

where $l_0^\psi = 1$ and l_a^ψ, $a = 1,2,...rank\mathcal{G}$ as before and

$$\vartheta_0 = 0 \ , \ \vartheta_a = \sum_{b=1}^{rank\mathcal{G}} (RK)_{ab}^{-1} \ln\left(\frac{q^0 l_b^\psi}{q^b}\right) \tag{36}$$

where R is the matrix with entries $R_{ab} = \delta_{ab} + n_b^\psi$, with n_b^ψ being the integers in the expansion $\psi = n_a^\psi \alpha_a$, $l_a^\psi = \frac{\alpha_a^2}{\psi^2} n_a^\psi$ and so

$$(R^{-1})_{ab} = \delta_{ab} - \frac{n_b^\psi}{h} \tag{37}$$

where $h = \sum_{a=1}^{rank\mathcal{G}} n_a^\psi + 1$ is the Coxeter number of $\mathcal{G}$. The ϑ_a are associated to the vacuum of the fields φ_a and were introduced to make Hirota's equations independent of the coupling constants q^j. We have the freedom to introduce the field σ because it will drop from the exponential interaction term, since l_i^ψ is a null vector of the extended Cartan matrix $K_{ij} l_j^\psi = 0$. In addition, it is related to the vacuum of the field ν, and it will help us in decoupling Hirota's equations into a suitable form.

To simplify the notation we introduce the operator

$$\Delta(F) \equiv \partial_+ \partial_- \ln F = \frac{\partial_+ \partial_- F}{F} - \frac{\partial_+ F \partial_- F}{F^2} \tag{38}$$

which obviously satisfies

$$\Delta(FG) = \Delta(F) + \Delta(G) \ , \ \Delta(F^\gamma) = \gamma \Delta(F) \tag{39}$$

The equations one obtains by substituting (35) into (32) can be decoupled as

$$\Delta(\tau_j) = -\beta \left(\prod_{k=0}^{rank\mathcal{G}} \tau_k^{-K_{jk} l_k^\psi} - 1 \right) \tag{40}$$

$$\partial_+ \partial_- \sigma = \beta \tag{41}$$

where

$$\beta = \frac{q^j}{l_j^\psi} e^{K_{jk} \vartheta_k} \quad \text{for any } j = 0, 1, ..., rank\mathcal{G} \tag{42}$$

which, using (37), one can easily show that it is a constant independent of the index j. The solution for σ is therefore

$$\sigma(x_+, x_-) = \beta x_+ x_- + F(x_+) + G(x_-) \tag{43}$$

with F and G arbitrary functions. From (31) we get the relation between τ-functions and the original CAT model fields

$$\varphi^a = -l_a^\psi \ln \frac{\tau_a}{\tau_0} + \vartheta_a \qquad \nu = \frac{2}{\psi^2} (\sigma - \ln \tau_0) \tag{44}$$

In ref [8] the Hirota's solution for the Affine Toda models associated to $\hat{SL}(r+1)$ was constructed. In order to get the Hirota's equation the number of τ-functions introduced exceeded the number of fields by one. It is now clear that the extra τ- function, namely τ_0, corresponds to the ν field and that the Hirota's equation is intrinsically related to the structure of the CAT model.

Notice that the Hirota's equation (40) is invariant under [illegible] ations on the τ-functions, $\tau_j \to \lambda \tau_j$, since l_j^ψ is a null vector of the extended [illegible] This guarantees that when both sides of (40) are multiplied by $\tau_j^{K_{ij} l_j^\psi}$, all terms w[illegible] same order in τ. Obviously, the equations will be bilinear in τ only for those algebras wi[illegible] $l_j^\psi = 1$ for all j's. That happens for the algebras $\hat{SL}(r+1)$ and $\hat{Sp}(r)$. We next apply the Hirota method to the CAT model described by (29) and (30) for these two algebras. The application of such method to the remaining algebras is now under investigation and it will be published elsewhere.

3.1 $\hat{SL}(r+1)$

For $\hat{SL}(r+1)$ the N soliton solution from Hirota's method was obtained in [8]. The τ functions for the $\hat{SL}(2)$ CAT and AT models were also considered in ref.[15,18]. Here we give a short account of those results for completeness. The integers $l_a^\psi, a = 1, 2, ...r$ are all equal to unity in this case. The extended Cartan matrix is obtained from the usual one by adding an extra row and columm corresponding to an extra point in the Dynkin diagram. It is given by

$$K = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \tag{45}$$

The equations for the tau functions corresponding to (40) becomes

$$\tau_j^2 \Delta(\tau_j) = \beta \left(\tau_j^2 - \tau_{j+1} \tau_{j-1} \right) \tag{46}$$

$j = 0, 1, ...r$, where $\tau_{j+r+1} = \tau_j$ is understood from the periodicity of the extended Dynkin diagram. The above system admits N soliton solutions of the form

$$\tau_j = 1 + \epsilon \tau_j^{(1)} + ... + \epsilon^N \tau_j^{(N)} \tag{47}$$

We now seek for one soliton solution, i.e. $\tau_j = 1 + \epsilon \tau_j^{(1)}$, where

$$\tau_j^{(1)} = \exp(\gamma(x - vt - \xi)) \delta_j \tag{48}$$

The truncation of (47) in order ϵ is consistent if we impose the following recursion relations

$$\delta_j^2 = \delta_{j-1} \delta_{j+1} \tag{49}$$

An obvious solution satisfying the periodicity condition, $\tau_{r+1}^{(1)} = \tau_0^{(1)}$ is given in terms of the $(r+1)$-th root of unit, i.e.

$$\delta_j = \omega^j \tag{50}$$

where $\omega = \exp(\frac{2\pi i k}{r+1})$, $k = 0, 1, 2, ...r$. The wave parameters γ and v are related by ($4\partial_- \partial_+ = \partial_x^2 - \partial_t^2$)

$$\gamma^2 (1 - v^2) = 16 \beta \sin^2 (\frac{k\pi}{r+1}) \tag{51}$$

3.2 $\hat{SP}(r)$

Again in this case the integers $l_j^\vee = 1, j = 0, 1, ...r$. The extended Cartan matrix is given by

$$K = \begin{pmatrix} 2 & -2 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{pmatrix} \tag{52}$$

The system of equations for the tau functions corresponding to (40) can therefore be obtained by reading off the elements of the extended Cartan matrix, i.e.

$$\begin{aligned} \tau_0^2 \Delta(\tau_0) &= \beta\left(\tau_0^2 - \tau_1^2\right) \\ \tau_a^2 \Delta(\tau_a) &= \beta\left(\tau_a^2 - \tau_{a+1}\tau_{a-1}\right) \quad \text{for } a = 1, 2, ..., r-1 \\ \tau_r^2 \Delta(\tau_r) &= \beta\left(\tau_r^2 - \tau_{r-1}^2\right) \end{aligned} \tag{53}$$

The one soliton solutions is again obtained setting $\tau_j = 1 + \epsilon\tau_j^{(1)}$, where

$$\tau_j^{(1)} = \exp(\gamma(x - vt - \xi))\delta_j \tag{54}$$

This case leads to two solutions. One is obtained setting $\delta_0 = \delta_1 = ... = \delta_r$ yielding $\gamma \neq 0$ and $v^2 = 1$, i.e.,

$$\tau_j^{(1)} = \exp(\gamma(x \pm t - \xi))\delta_0 \tag{55}$$

implying from (44)

$$\varphi^a = \vartheta_a \ , \ \nu = \frac{2}{\psi^2}(-\ln(1 + \exp(\gamma(x \pm t - \xi))\delta_0) + \beta x_+ x_- + F(x_+) + G(x_-)) \tag{56}$$

The second solution is obtained when $\delta_0 = \delta_2 = ... = \delta_{2n}$ and $-\delta_0 = \delta_1 = \delta_3 = ... = \delta_{2n+1}$ yielding

$$\tau_j^{(1)} = (-1)^j \exp(\gamma(x - vt - \xi))\delta_0 \tag{57}$$

where the wave parameters are related as

$$\gamma^2(1 - v^2) = 16\beta\delta_0 \tag{58}$$

Again from (44) we get

$$\begin{aligned} \varphi^a &= \vartheta_a \quad \text{for } a \text{ even} \\ \varphi^a &= -\ln\left(\frac{1 - \exp(\gamma(x - vt - \xi))\delta_0}{1 + \exp(\gamma(x - vt - \xi))\delta_0}\right) + \vartheta_a \quad \text{for } a \text{ odd} \\ \nu &= \frac{2}{\psi^2}(-\ln(1 + \exp(\gamma(x - vt - \xi))\delta_0) + \beta x_+ x_- + F(x_+) + G(x_-)) \end{aligned} \tag{59}$$

where $a = 1, 2, ...r$. Notice that φ^a for a odd constitute copies of the Sinh-Gordon soliton.

3.3 The $\hat{SL}(2)$ Case

We now illustrate the procedure described in the previous sections where the solutions of the CAT model were shown to be related to those of the AT model with space time variables parametrized in terms of a free field η as follows

$$\begin{aligned} \tilde{x} &= \frac{1}{2}(\tilde{x}_+ + \tilde{x}_-) = \frac{1}{2}\left(\int^{x_+} dy e^{\eta_+(y)} + \int^{x_-} dy e^{\eta_-(y)}\right) \\ \tilde{t} &= \frac{1}{2}(\tilde{x}_+ - \tilde{x}_-) = \frac{1}{2}\left(\int^{x_+} dy e^{\eta_+(y)} - \int^{x_-} dy e^{\eta_-(y)}\right) \end{aligned} \tag{60}$$

Let us consider the CAT model described by equations (23)-(25) for $\hat{SL}(2)$. The solution given in terms of the τ-functions in (44) reads

$$\begin{aligned} \varphi(x,t) &= -\ln\left(\frac{1 - i\exp(\gamma(\tilde{x} - v\tilde{t}))}{1 + i\exp(\gamma(\tilde{x} - v\tilde{t}))}\right) + \frac{1}{4}\ln\frac{q^0}{q^1} \\ &= -2i\arctan(\exp(\gamma(\tilde{x} - v\tilde{t}))) + \frac{1}{4}\ln\frac{q^0}{q^1} \\ \nu(x,t) &= \sigma - \ln[1 + i\exp(\gamma(\tilde{x} - v\tilde{t})] \\ \eta(x,t) &= \eta_+(x + t) + \eta_-(x - t) \end{aligned} \tag{61}$$

where we have made the choice $\xi = \frac{\pi i}{2\gamma}$ in (48). Equations (60) stress the fact that there are infinite many space time new variables in correspondence with the choice of $\eta_\pm$. In particular for $\eta_+ = \eta_- = 0$ the solution for the CAT model field φ is the same as that for the Sinh-Gordon field .

Consider, for instance, the usual static (in $\tilde{t}$) soliton solution for the Sinh-Gordon model, i.e. $v = 0$ [19]. The corresponding solution for the $\hat{SL}(2)$ CAT model for given $\eta_\pm$ will not in general be static (in t). In fact, the solution can be completly different and in some cases the solitonic character is lost. This means its topological charge may not be preserved by the conformal transformation. We have checked that for $\eta_\pm = x_\pm$ the static Sinh-Gordon soliton is mapped into a solitonic solution of the $\hat{SL}(2)$ CAT model. It travels preserving its asymptotic behaviour, and so having a topological charge. One can describe its trajectory by giving, for instance, the position of the point $\varphi = \frac{3\pi}{4}$.

$$x = -\ln(\cosh t) + x_0 \tag{62}$$

So, it travels with velocity $\frac{dx}{dt} = -\tanh t$ coming from $x = -\infty$ (for $t = -\infty$) bouncing at the point $x_0 = \ln(\frac{1}{\gamma}\ln(\tan(\frac{3\pi}{4} - \frac{1}{4}\ln\frac{q^0}{q^1})))$ and then returning to $x = -\infty$ (for $t = \infty$).

Such procedure can be used to study the perturbation caused by the η field on the solitons of the Affine Toda models. It would be interesting to develop a systematic way of doing that.

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