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Singular potentials in Quantum Mechanics

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This paper is a review of some mathematical methods as recently developed and applied to deal with singular potentials in Quantum Mechanics. Regular and singular perturbative methods as well as variational treatments are considered.

1. Introduction

Since the papers by Klauder [1], Ezawa *et al.*[2], and Detwiler and Klauder [3] singular potentials have been receiving increased interested in the literature [4]-[18]. These potentials are present in a number of situations which are mathematically challenging and physically interesting . As is well known, standard Rayleigh-Schrödinger perturbative method are of very limited help in cases where singular potentials are present, failing definitely when the singularity is of high degree. In such cases, the potential is said to be *supersingular*— a quantitative classification of the potential singularity may be given in terms of the analytical results obtained by Harrell, as we shall discuss later in Section 2. In other words and in terms of perturbation theory, a potential is called supersingular if its matrix elements in the representation defined by the unperturbed eigenvectors are infinite or do not exist. In consequence, we must not expect to be able to find a conventional perturbation (power) expansion for either the eigenvectors or eigenvalues. As a matter of fact, logarithmic term and fractional powers of λ may appear in the perturbative expansion. Such nonpower expansions are called *singular perturbative expansions.*

The Hamiltonian we will be mostly dealing with is given by

$$H = -\frac{d^2}{dx^2} + x^2 + \frac{\lambda}{|x|^\alpha}\ , \quad \alpha > 0 \tag{1.1}$$

Due to the graphical appearance of the term $\lambda/|x|^\alpha$, the system described by Eq. (1.1) is usually called a *spiked oscillator*. Depending on the magnitude of λ, we can classify three kinds of regimes, namely, weak, [illegible] of the parameter λ, the spikelike term has little effect on the harmonic oscillator in the region away from the origin. However, for λ big enough it can temporarily dominate the oscillator in a large interval of x values. These situations require different perturbative approaches which will be discussed in Section 3.

One peculiar feature of the Hamiltonian (1.1), when α is sufficiently large, is that its perturbed solution, when it exists, may not converge to the harmonic oscillator solution as $\lambda \to 0$. This unexpected behavior, first mentioned by Klauder [1], was said to be a *vestigial effect* of singular potentials [2], and is presently known as *Klauder's phenomenon.* What this effect shows is that once the perturbation is turned on, it cannot be completely turned off any more. As commented by Harrell [4], from the work by Ezawa *et al.* [2] it follows that Klauder's phenomenon occurs only in one dimension.

In the next Section we discuss the essential features of singular perturbative expansions as studied by Harrell. In Section 3, we address the case of nonsingular perturbative expansions related to weak and strong couplings. Variational methods are discussed in Section 4. Finally, some works that are being developed by the authors and collaborators are discussed in Section 5.

2. Singular perturbative expansions

A variational treatment for the ground-state energy of the spiked oscillator made by Detwiler and Klauder [3] leads to upper bounds for the first energy corrections. These corrections are proportional to $\lambda \ln \lambda$ when $\alpha = 3$ and proportional to $\lambda^{1/(\alpha-2)}$ when $\alpha > 3$. These results were proved to be correct by Harrell, who also found the lowest-order corrections to the ground-state energy for all positive values of α [4]. Harrell elaborated on a special perturbation theory, named *singular perturbation theory*, and determined the leading contributions to the ground-state energy of the spiked harmonic oscillator described by the Hamiltonian Eq. (1.1) with small value of λ. Harrell's results are the following

$$E_0(\lambda) = 3 + \left[2\Gamma(1/4)/\sqrt{\pi}\right]\lambda + \left[16/\sqrt{\pi}\right]\lambda^2 \ln \lambda + O(\lambda^2), \quad \alpha = 5/2 \tag{2.1}$$

$$E_0(\lambda) = 3 + \left[2\Gamma((3-\alpha)/2)/\sqrt{\pi}\right]\lambda + \left[4\Gamma(1-\nu)\nu^{2\nu}/\sqrt{\pi}\Gamma(1+\nu)\right]\lambda^\nu + O(\lambda^2), \quad 5/2 < \alpha < 3 \tag{2.2}$$

$$E_0(\lambda) = 3 - \left[4/\sqrt{\pi}\right]\lambda \ln \lambda - \left[10\gamma/\sqrt{\pi}\right]\lambda + O(\lambda^2 \ln^2 \lambda), \quad \alpha = 3 \tag{2.3}$$

$$E_0(\lambda) = 3 + \left[4\Gamma(1-\nu)\nu^{2\nu}/\sqrt{\pi}\Gamma(1+\nu)\right]\lambda^\nu - \left[4\Gamma((5-\alpha)/2)/\sqrt{\pi}(\alpha-3)\right]\lambda + O(\lambda^{2\nu}), \quad 3 < \alpha < 4 \tag{2.4}$$

$$E_0(\lambda) = 3 + \left[4\Gamma(1-\nu)\nu^{2\nu}/\sqrt{\pi}\Gamma(1+\nu)\right]\lambda^\nu + O(\lambda^{2\nu}), \quad \alpha \geq 4 \tag{2.5}$$

where $\gamma = 0.577216...$ is Euler's constant, and $\nu = 1/(\alpha - 2)$.

Notice the presence of logarithmic and noninteger-power terms in the above singular perturbative energy expansions. The particular case $\alpha = 5/2$, for instance, the expansion

Eq. (2.1) shows that there exists at least one perturbative contribution between the first and second power of the coupling constant λ.

Examples of occurrences of logarithmic terms and fractional powers of the perturbative parameters can be found in physics. The low-density expansion of the energy per particle of a many-boson system at zero temperature is given by [19]

$$E/N = C_0 \rho a \left[1 + C_1(\rho a^3)^{1/2} + C_2 \rho a^3 \ln(\rho a^3) + C_3 \rho a^3 + ...\right] \tag{2.6}$$

where C_i are constants and the dimensionless smallness parameter, that plays de role of λ, is expressed in terms of the density ρ and the scattering length a as $\sqrt{\rho a^3}$. The electron gas system also admits of a weak as well as a strong coupling singular perturbation expansions [20].

It is very difficult to obtain the next terms in Harrell's singular expansions Eq. (2.1)-(2.5). Indeed, up to our knowledge, they are not known yet.

Other studies and approaches have been done for spiked oscillators. In the remaining sections, we will report some recent works treating this problem using perturbative and variational methods.

3. Nonsingular perturbative expansions

In this Section we will be mostly concerned with perturbative treatment of the spiked oscillator with $\alpha \leq 5/2$. For this range of values of the parameter α, there is some hope in using standard (nonsingular) perturbation theory.

Let us rewrite the Hamiltonian Eq. (1.1) in the form

$$H = -\frac{d^2}{dx^2} + x^2 + \lambda/x^\alpha \equiv H_0 + \lambda/x^\alpha, \quad x \geq 0 \tag{3.1}$$

and consider the (unperturbed) eigenfunctions $\psi_n^{(0)}(x)$ of H_0 that satisfy the Dirichlet boundary condition $\psi_n^{(0)}(0) = 0$. The eigenenergies associated with these eigenfunctions are given by

$$E_n^{(0)} = 3 + 4n, \; n = 0, 1, 2, \dots . \tag{3.2}$$

In order to get the perturbative corrections to the energy, two cases must be analysed separately. First, we consider the weak interaction regime (small values of λ) and then the strong interaction regime (large values of λ).

3.1. Weak coupling perturbation expansion

For small values of the perturbation parameter λ, we define the weak coupling perturbation expansion[11]

$$E_0 = E_0^{(0)} + S_1\lambda + S_2\lambda^2 + ... \tag{3.3}$$

By using standard perturbation theory, we obtain

$$S_1 = \frac{\Gamma((3-\alpha)/2)}{\Gamma(3/2)} \tag{3.4}$$

and

$$S_2 = -S_1^2 \sum_{n\neq 0} \frac{2^{2n}(\alpha/2)_n^2}{4n(2n+1)!} \tag{3.5}$$

where $(z)_n$ is the Pochhammer symbol[21]. These results for S_1 and S_2 are valid for $\alpha < 5/2$. For $\alpha = 5/2$, Eq. (3.4) is still valid and reproduces the result Eq. (2.1) obtained by Harrell [4]. The summation appearing in S_2 cannot be carried out exactly but for particular values of α. It is infinity and slowly convergent. A method for accelerating the convergence of that sum is described by Aguilera-Navarro and Guardiola [11]. These authors evaluate the energy Eq. (3.3) for $\alpha = 1/2$, 1, 3/2 and 2. They also show that for the very specific case $\alpha = 2$ the sum in Eq. (3.5) can be exactly evaluated giving $S_2(\alpha = 2) = -2$. Indeed, the case $\alpha = 2$ admits of exact solution [22], the corresponding energy being given by

$$E_n(\alpha = 2) = 2 + 4n + \sqrt{1 + 4\lambda}. \tag{3.6}$$

Unfortunately the perturbed eigenfunctions cannot in general be obtained in this same simple way because infinity terms arise in the evaluation of the corresponding expansion coefficients [11].

3.2. Strong coupling perturbation expansion

When the coupling parameter λ is large, both terms x^2 and λ/x^α in Eq. (3.1) are important. In order to define a strong coupling perturbation expansion [10], we first expand the potential

$$V(x) = x^2 + \lambda/x^\alpha \tag{3.7}$$

around its minimum $x_{\min} = (\lambda\alpha/2)^{1/(\alpha+2)}$. Now introduce

$$\rho = x - x_{\min} \tag{3.8}$$

and expand $V(x)$ around $\rho = 0$ getting

$$V(\rho) = V_{\min} + (\alpha+2)\rho^2 + \sum_{k=3}^{\infty}(-1)^k \frac{2}{\alpha}(\alpha)_k \mu^{k-2}\frac{\rho^k}{k!} \tag{3.9}$$

where $V_{\min} = V(x_{\min}) = \mu^{-2}(1 + 2/\alpha)$ and $\mu \equiv (2/\lambda\alpha)^{1/(\alpha+2)} = 1/x_{\min}$. The quantity μ will play the role of a perturbation parameter as it is shown in the following.

The Schrödinger equation is written as

$$H\psi = (H_0 + H_I)\psi = E\psi \tag{3.10}$$

where

$$H_0 = -\frac{d^2}{d\rho^2} + (\alpha+2)\rho^2 + V_{\min} \tag{3.11}$$

and

$$H_I = \sum_{n=1}^{\infty} \mu^n H_n \tag{3.12}$$

with

$$H_{k-2} = (-1)^k (2/\alpha)(\alpha)_k \rho^k / k! \tag{3.13}$$

Defining now the expansions for the energy and the eigenfunction as

$$E = E(\mu) = \mu^{-2} E_{-2} + E_0 + \mu E_1 + \mu^2 E_2 + \ldots \tag{3.14}$$

and

$$\psi = \psi(\mu) = \psi_0 + \mu\psi_1 + \mu^2\psi_2 + \ldots \tag{3.15}$$

From Eq. (3.11), we can see immediately that the first two contributions to the energy are given by

$$\mu^{-2} E_{-2} + E_0 = (1 + 2/\alpha)\mu^{-2} + (\alpha+2)^{1/2} \tag{3.16}$$

From parity considerations, it results that $E_{2n+1} = 0$, $n = 0, 1, 2, \ldots$. The higher order contributions can be gotten through a projection technique [10]. Numerical values of E_n, for $\alpha = 1/2$, 1, 3/2, 2 and 5/2; and $n = -2$ (2) 20 are given in Ref.[11] where also exact algebraic expressions in terms of rational functions in $\omega = (\alpha+2)^{1/2}$ are explicitly given for $n = -2$ (2) 10. As an example, E_4 is given by

$$E_4 = (\omega^8 - 10\omega^6 - 39\omega^4 + 320\omega^2 - 272)/1728\omega \tag{3.17}$$

Is there any path connecting the weak regime region with the strong one? This is a difficult problem to be solved for all values of α. However, in the region around $\alpha = 2$,a constructive method, based on two-parameter perturbation expansion, can be designed which leads to the solution to this problem for arbitrary value of the coupling parameter λ [11].

3.3. The charged harmonic oscillator

The so called charged harmonic oscillator, defined by the Hamiltonian

$$H = -\frac{d^2}{dr^2} + r^2 + \frac{\lambda}{r} \tag{3.18}$$

was considered by Aguilera-Navarro *et al.*[15]. This problem is analysed in three regimes: (a) coulombic, corresponding to large and negative values of the coupling parameter λ; (b) strong coupling, corresponding to large and positive value of λ; and, (c) weak coupling, corresponding to small (positive or negative) values of λ.

In the Coulombic regime, by using hypervirial relations and the Hellmann-Feynman theorem, the authors find that the ground-state energy is given by (they indeed explicitly give all terms up to λ^{-24})

$$E(\lambda) = \lambda^2 \left[-\frac{1}{4} + \frac{12}{\lambda^4} - \frac{1032}{\lambda^8} + \frac{348864}{\lambda^{12}} + \cdots \right], \ \lambda \text{ large and negative.} \tag{3.19}$$

For the strong coupling regime, the first few terms in the energy perturbative expansion are (the authors explicitly give all terms up to μ^{12})

$$E = \frac{3}{\mu^2} + \sqrt{3} + \frac{7\mu^2}{36} + \frac{37\mu^4}{432\sqrt{3}} + \cdots, \ \mu \equiv \left(\frac{2}{\lambda}\right)^{1/3}, \ \lambda \text{ large and positive.} \tag{3.20}$$

Finally, for the weak coupling regime, the energy is given by

$$E = 3 + \frac{2\lambda}{\sqrt{\pi}} - \frac{(2 + 2\ln 2 - \pi)}{\pi}\lambda^2 + \cdots, \ \lambda \text{ small.} \tag{3.21}$$

Those authors also found *elementary solutions* associated with the Hamiltonian (3.18). These solutions can be expressed as the product of an exponential function and a power expansion. For the present case, elementary solutions can be found of the form

$$\varphi(r) = \exp(-r^2/2) \sum_{n=0}^{\infty} p_n r^n \tag{3.22}$$

where the p_n coefficients satisfy the recurrence relation

$$(n+1)(n+2)p_{n+2} = \lambda p_{n+1} + (2n+1-E)p_n, \quad p_0 = 1 \tag{3.23}$$

Elementary solutions exist only for particular values of the coupling parameter λ.

In order to have a finite number of terms in the summation of Eq. (3.22), reducing it to a polynomial, it is necessary that the coefficients p_n satisfy the following two equations

$$p_{n+1}(E, \lambda) = 0 \tag{3.24}$$

and

$$E = 2n + 1, \quad n = 2, 3, 4, \ldots \tag{3.25}$$

Thus, for all odd and positive values of the energy $E > 3$, we can find values for λ, through Eq. (3.24), that cut the infinite summation in Eq. (3.22). In this way, an infinite number of elementary solutions can be found. The particular case $E = 3$ corresponds to $\lambda = 0$ and reduces our problem to the trivial case of the simple harmonic oscillator.

The simplest elementary solutions are found to be [15]

$$\varphi(r, E = 5) = r(1 + r)\exp(-r^2/2), \ \lambda = 2, \tag{3.26}$$

and

$$\varphi(r, E = 7) = r(1 + \sqrt{5}r + r^2)\exp(-r^2/2), \ \lambda = \sqrt{20}. \tag{3.27}$$

Those authors also use renormalization of series and Riccati-Padé summation techniques to deal with different regimes of the charged harmonic oscillator (3.18).

3.4. The Lennard-Jones-like Hamiltonian

It is worth mentioning the very extensive study of the anharmonic oscillator, Lennard-Jones-like Hamiltonian,

$$H = -\frac{d^2}{dx^2} + x^2 + \frac{b}{x^4} + \frac{c}{x^6} \tag{3.28}$$

that was carried out by Guardiola and Ros[13]. This Hamiltonian has many interesting and important properties examined and amply elaborated on by those authors. They discuss the large coupling regime, corresponding to b and/or c large, and the small coupling regime when both b and c are small. Guardiola and Ros[13] also found elementary solutions of the form

$$\varphi(r) = \exp\left[-\frac{x^2}{2} - \frac{\sqrt{c}}{2x^2}\right]\sum_n p_n x^n \tag{3.29}$$

associated with the Hamiltonian Eq. (3.28).

4. Variational approaches

Many efforts have been done to deal with spiked harmonic oscillators by carrying out variational analysis of the Hamiltonian in a space spanned by convenient basis. A concise description of the method used in most the cases we discuss in this Section can be phrased as follows. Given the Hamiltonian H defined in some region R, consider a set basis $\{\phi_n\}$ in R that satisfies appropriate boundary conditions at infinity and at the origin (usually the Dirichlet boundary condition $\phi_n(0) = 0$ is imposed at the origin). Then diagonalize H in the space spanned by the basis $\{\phi_n\}$. As it is well known, this procedure is equivalent to expanding the variational function ψ_{var} in the basis $\{\phi_n\}$

$$\psi_{var} = \sum_n a_n \phi_n \tag{4.1}$$

and then minimize the expectation value

$$\frac{\langle\phi_n|H|\phi_n\rangle}{\langle\phi_n|\phi_n\rangle} \tag{4.2}$$

with respect to the variational parameters a_n. The diagonalization process produce to the eigenvalues and eigenfunctions of H. One additional advantage of this variational analysis is that we can, in a systematic way, find the trend of the method by just changing the dimension of the variational space varying the number of elements in the basis $\{\phi_n\}$.

4.1. The potential λ/x^α

In this subsection, we will discuss the spiked oscillator described by the Hamiltonian [10]

$$H = -\frac{d^2}{dx^2} + x^2 + \lambda/x^\alpha \equiv H_0 + \lambda/x^\alpha, \quad x \geq 0 \tag{4.3}$$

The method used consists in diagonalizing H in a convenient basis. The problem requires that in choosing a variational basis, some care must be taken with its behavior at the origin. One appropriate basis set is formed by the normalized eigenfunctions $u_n(x)$ of the linear harmonic oscillator H_0 that satisfy the Dirichlet boundary condition $u_n(0) = 0$. With this prescription, apart a normalization constant, the functions $u_n(x)$ are given by the product of a Gaussian function and a Hermite polynomial of odd degree

$$u_n(x) = A_n e^{-x^2/2} H_{2n+1}(x), \quad n = 0, 1, 2, \ldots \tag{4.4}$$

These functions define a complete set of orthonormalized functions in the interval $0 \leq x < \infty$, with normalization constant given by

$$A_n^{-2} = 4^n (2n+1)!\sqrt{\pi}. \tag{4.5}$$

Obviously, H_0 is diagonal in this basis, its diagonal matrix elements being given by $(4n+3)$. The matrix elements of $x^{-\alpha}$ are found to be given by[10]

$$< m|x^{-\alpha}|n >= T_m T_n \frac{\Gamma(3/2)}{\Gamma(n+3/2)} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\Gamma(k+3/2-\alpha/2)\Gamma(n-k+\alpha/2)}{\Gamma(k+3/2)\Gamma(-k+\alpha/2)} \tag{4.6}$$

where

$$T_s \equiv (-1)^s \sqrt{(2s+1)!}/2^s s! \tag{4.7}$$

We can obtain information about the trend of the variational method by diagonalizing the Hamiltonian Eq. (4.3) in variational spaces of increasing dimension. This was done[10] for $\alpha = 5/2$ and λ varying from 0.001 to 1000. The resulting E_{var} energies were compared with the energy E_{exact} obtained from numerical integration of the corresponding Schrödinger equation. For a 20-dimensional variational space, the maximum relative error $|E_{exact} - E_{var}|/E_{var}$ occurs for $\lambda = 0.1$ and is of the order of 0.011 or 1.1%.

4.2. The potential $r^2 + \lambda/r^4$

The Hamiltonian

$$H = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + r^2 + \frac{\lambda}{r^4} \tag{4.8}$$

was studied in Ref.[17] where the authors develop a variational analysis of H using as trial function a linear combination of the non-orthogonal basis set

$$R_n(r) = A_n M(-n, 3/2, r^2)\exp(-ar^2 - b/r), \quad n = 0, 1, 2, \ldots \tag{4.9}$$

where $M(p, q, z)$ is the confluent hypergeometric function[21]. The analytical form of the exponential function is important to guarantee the appropriate behavior of the solution at infinity and at the origin. More specifically, the parameters a and b are determined such that $R_0(r)$ has the correct asymptotic form as $r \to \infty$ and $r \to 0$. These values were found to be $a = 1/2$ and $b = \sqrt{\lambda}$. Since the basis set Eq. (4.9) is not orthogonal, a special diagonalization procedure is required and detailed in the paper[17]. The variational results

are in excellent agreement with the corresponding exact values, *i.e.* those obtained by Fernández [12] through a numerical integration of the corresponding Schrödinger equation.

The Hamiltonian Eq. (4.8) allows us to study only the fundamental and excited s-states of the system. A global analysis of the ground and excited states for successive values of the orbital angular momentum was carried out by Aguilera-Navarro and Ley Koo[18]. These authors made a variational analysis of the Hamiltonian

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+\frac{\ell(\ell+1)}{r^2}+r^2+\frac{\lambda}{r^4} \tag{4.10}$$

using as variational basis the orthonormalized set of functions

$$\phi_{n\rho}(r)=N_{n\rho}r^{\rho}e^{-\omega r^2/2}M(-n,\rho+3/2,\omega r^2) \tag{4.11}$$

which are exact eigenfunctions of the auxiliary Hamiltonian

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+\frac{\ell(\ell+1)}{r^2}+\frac{B}{r^2}+\omega^2r^2 \tag{4.12}$$

with $B=\ell(\ell+1)-\rho(\rho+1)$. All required matrix elements of (4.10) in the basis (4.11) are determined in a closed form.

The analysis is carried out in two steps. Firstly, the Hamiltonian (4.10) is variationally analysed with respect to the two-parameter radially-nodeless state (N is a normalization constant)

$$\psi_{trial}=Nr^{\rho}e^{-\omega r^2/2} \tag{4.13}$$

producing two equations for ρ and ω involving the angular momentum ℓ and the spike intensity λ.

Then, for given values of λ and ℓ, the values of ω, ρ and B are determined and used to define completely the variational function (4.11) and the corresponding matrix representation of the original Hamiltonian (4.10). The diagonalization of the resulting matrix yields the eigenenergies and eigenfunctions with well defined angular momentum ℓ. Their treatment led to a reliable description of the vibrational-rotational energy spectra [18].

5. Other approaches

Fernández[12] considered the Hamiltonian

$$-\frac{d^2}{dr^2}+r^2+\frac{a}{r^2}+\frac{b}{r^4}+\frac{c}{r^6} \tag{5.1}$$

and look for a simple solution of the form

$$\psi(r)=r^{k+1}\exp(-s/2r^2-tr^2/2),\quad s\geq 0,\ t>0. \tag{5.2}$$

The idea here is to get the appropriate behavior of the solution at the origin by means of the factor function $\exp(-s/2r^2)$.

Solano-Torres *et al.*[14] consider the Hamiltonian defined in Eq. (3.18) for $\alpha=4$ and $\alpha=6$. Those authors evaluated the ground-state energy using two methods: (a) numerical integration of the associated Schrödinger equation, and (b) Padé summation of the strong coupling perturbation expansion described in the Subsection 3.2. They compare their results with those obtained with the methods of Detwiler and Klauder[3], Harrell[4] and Killingbeck[8].

A nonperturbative treatment was developed by Miller[23] to be applied in the determination of the eigenfunctions and eigenenergies of the Hamiltonian (3.18). The method is similar to the Lanczos algorithm[24] extended to self-adjoint operators. Miller applied his method to the case $\alpha=5/2$, and compare the ground-state energy obtained after 6 iterations with the corresponding perturbed energy E_{pert} obtained by Aguilera-Navarro *et al.*[10]. The relative "error" $|E_{Miller}-E_{pert}|/E_{pert}$ is of the order or 1%.

6. Works in process

The variational method is being applied by Aguilera-Navarro, Ley Koo and Mateos to study the fundamental and excited (vibrational-rotational) states of some spiked-oscillator-like systems. These authors use as variational set basis the exact orthonormalized Kratzer-Fues eigenfunctions[25]

$$\phi_n(r)=N\,r^{\rho}\exp(-\beta_n r)M(-n,2\rho+2,2\beta_n r) \tag{6.1}$$

which are solutions of the Hamiltonian

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+\frac{\ell(\ell+1)}{r^2}+a+\frac{b}{r}+\frac{c}{r^2}. \tag{6.2}$$

They have already considered the supersingular plus Coulomb potential $D/r^4-Ze^2/r$. This same basis is being used to study the supersingular Hamiltonian

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+\frac{\ell(\ell+1)}{r^2}+\frac{A}{r^4}-\frac{B}{r^3}. \tag{6.3}$$

Dedication

The authors want to dedicate this work to Professor Paulo Leal Ferreira with admiration and affection on the occasion of his seventieth birthday.

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