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*To appear in Journal of Physics A

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Renormalization in Nonrelativistic Quantum Mechanics *

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June 26, 1997

Abstract

The importance and usefulness of renormalization are emphasized in nonrelativistic quantum mechanics. The momentum space treatment of both two-body bound state and scattering problems involving some potentials singular at the origin exhibits ultraviolet divergence. The use of renormalization techniques in these problems leads to finite converged results for both the exact and perturbative solutions. The renormalization procedure is carried out for the quantum two-body problem in different partial waves for a minimal potential possessing only the threshold behavior and no form factors. The renormalized perturbative and exact solutions for this problem are found to be consistent with each other. The useful role of the renormalization group equations for this problem is also pointed out.

PACS Numbers 03.65.-w, 03.65.Nk, 11.10.Gh, 11.10.Hi

*To Appear in Journal of Physics A; hep-th preprint 9706193

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1 Introduction

The ultraviolet divergences in perturbative quantum field theory can be eliminated in man cases by renormalization to define physical observables, such as charge or mass, which are ofte termed the physical scale(s) of the problem [1, 2, 3, 4]. Ultraviolet divergences appear in exac as well as perturbative treatments of the nonrelativistic quantum mechanical two-body problem in momentum space interacting via two-body potentials with certain singular behavior at shor distances [5, 6, 7, 8, 9, 10, 11] in two and three space dimensions. Renormalization of th potential model leads to a scale(s) and a finite physical observable(s) [6].

Renormalization of a physical quantum mechanical model is essential in reproducing exper imental results irrespective of whether the original model exhibits ultraviolet divergence or no Renormalization removes the effect of different uncertainties and approximations of a physica model on the observables and brings some of the theoretical predictions in agreement wit experiment. Such uncertainties exist in all quantum mechanical models. Even in the mos well-understood quantum mechanical hydrogen-atom problem only the long distance behavio of the Hamiltonian can be considered to be known. For distances smaller than the radius of th proton, the electron-proton potential is not the bare Coulomb potential but some regularize Coulomb potential which, unlike the original Coulomb potential, does not diverge and lead to a constant value as the electron-proton separation r goes to zero. Also at this scale th effect of field theoretic corrections to the Hamiltonian is relevant. The detailed behavior o this regularized Hamiltonian for small r depends on the charge distribution of proton and not usually known. The role of renormalization is to remove the uncertainty of the regularize potential by fixing some of the observable(s). The effect of renormalization in the hydroge atom problem is small and not evident as the radius of the proton is very small. The effect indispensable in large atoms and specially in muonic atoms where the orbit of the mu meso could have a significant overlap with nuclear matter.

In the above-mentioned Coulomb problem the potential is divergent at $r = 0$. In spite of th singularity of the Coulomb potential at $r = 0$, both the scattering and bound state problem with the original potential are solvable and do not produce ultraviolet divergences. The ro of renormalization in this problem is to introduce a regularized well-defined potential whi reproduces some of the observables.

The situation is different for potentials with a stronger divergence at $r = 0$ than the Coulom potential. These are the potentials which lead to the above-mentioned ultraviolet divergenc in momentum space. If these divergent potentials are attractive at $r = 0$, the original proble does not permit convergent solution in either momentum or configuration space. The bou

state problem collapses and produces an infinite number of bound states with an accumulation point at infinite binding. The scattering Lippmann-Schwinger equation for these potentials possesses a noncompact kernel and hence is not amenable to numerical solution. Finite and meaningful physical solution is obtained only after renormalization. If the divergent potentials are repulsive at $r = 0$, in the configuration space treatment one can obtain a finite convergent solution essentially by imposing some constraints, such as the solution should vanish at some small r. In this way the trouble with integration over the singular potential near $r = 0$ is avoided. In case of many repulsive divergent potentials, this procedure works and produces physically meaningful results, some examples being the repulsive soft and hard core potentials exhibiting ultraviolet divergences. Even in these cases the momentum space treatment, after an appropriate truncation of the Hamiltonian at small r, may lead to a Lippmann-Schwinger equation with noncompact kernel. Renormalization is then necessary to produce finite and physically meaningful results.

The usual difficulty in momentum space treatment with the Coulomb potential is the large distance or the infrared divergence. This could be avoided with the usual Yukawa potential in nuclear and atomic physics. The Yukawa potential possesses the same large momentum or short distance behavior as the Coulomb potential but no infrared divergence. For a Yukawa potential without ultraviolet (and infrared) divergence(s), renormalization improves the large momentum or short distance convergence properties. In this problem renormalization is not necessary but is only desirable. Renormalization makes this potential smoother and hence easier for numerical and analytic treatment. However, renormalization is indispensable for problems with ultraviolet divergence in momentum space.

In this work we shall be limited to the study of renormalization of the three dimensional two-body problem possessing ultraviolet divergence in momentum space in close analogy with field theoretic problems. An account of parts of this work has recently appeared [6, 7]. Most of the present ideas can also be used in two dimensions and in configuration space treatments [6, 9]. We illustrate the present procedure for a minimal potential in different partial waves. In momentum space this potential possesses only the threshold behavior and is given by $V_L(p', p) = p'^L \lambda p^L$, where L is the angular momentum. As the scattering Lippmann Schwinger equation has the same generic form for all partial waves, the ultraviolet divergence of this potential model becomes stronger and stronger as L increases. The leading ultraviolet divergence of the momentum space integrals, encountered while solving the Lippmann-Schwinger equation with this potential, is linear (cubic,...) in nature for $L = 0$ (1,...). The renormalization of this potential model can be performed by fixing at least one observable, or equivalently, by

introducing at least one physical scale. It is also possible to renormalize by introducing more than one physical scale. We renormalize both the exact and the perturbative solutions and find that the renormalized exact and perturbative solutions are consistent with each other.

We also derive the renormalization group (RG) equations for this problem. These equations clearly exhibit the important scaling behaviors of the different renormalized solutions. The RG equations can be written for the scattering solutions expressed in terms of certain physical scales closely related to scattering observables. These equations are valid in general, independent of the existence of ultraviolet divergence in the original problem. Such RG equations and the associated scaling behavior involving observables are interesting from a physical point of view.

The ultraviolet divergence of the present problem for $L = 0$ can be compared to the ultraviolet structure and high energy behavior of the $\lambda\phi^4$ field theory [2, 4, 9, 11]. The super-renormalizable $\lambda\phi^4$ field theory in 1+1 dimensions possesses ultraviolet logarithmic divergence, requires regularization, and is perturbatively renormalizable [2, 4]. The nonrelativistic scattering problem with contact interaction in two dimensions also has similar logarithmic divergence [9]. The renormalizable $\lambda\phi^4$ field theory in 3+1 dimensions has both logarithmic and quadratic divergences [2, 4]. We have verified that the nonrelativistic scattering problem with the present minimal potential in two dimensions also has similar logarithmic and quadratic divergences. In the present study of scattering in three dimensions, although the divergent terms are of different nature, the renormalization can be performed in a similar fashion. In the field theoretic problem, one cannot go beyond few lowest orders of perturbation theory. On the other hand, the nonrelativistic scattering problem with the present minimal potential possesses stronger ultraviolet divergences than in the $\lambda\phi^4$ field theory and can be solved to find both the exact and the perturbative solutions analytically. In the present work we find that the exact renormalized solution is consistent with the perturbative one. The study of the present analytic model will allow us to understand most of the subtleties of renormalization and RG equations.

The plan of our work is as follows. In Sec. II we perform the renormalization of the exact solution for the minimal potential in different partial waves. In Sec. III the renormalization of the perturbative solution is carried out and consistency of the renormalized exact and perturbative solutions is demonstrated. In Sec. IV we derive the RG equations and discuss the scaling properties of the renormalized solution. In Sec. V a brief summary of the present work is presented.

2 Renormalization of the Exact Solution

The partial-wave Lippmann-Schwinger equation for the scattering amplitude $T_L(p,q,k^2)$ in three dimensions, at c.m. energy k^2, is given by

$$T_L(p',p,k^2) = V_L(p',p) + \frac{2}{\pi}\int q^2 dq V_L(p',q) G(q;k^2) T_L(q,p,k^2), \tag{1}$$

with the free Green function $G(q;k^2) = (k^2 - q^2 + i0)^{-1}$, in units $\hbar = 2m = 1$, where m is the reduced mass.

We discuss potential scattering with the minimal potential in different partial waves. The present minimal potential in the Lth partial wave is taken to be $V_L(p',p) = p'^L \lambda p^L$, which is the usual δ potential for $L = 0$. For increasing L this potential presents stronger and stronger ultraviolet divergence. The reason for studying this potential is that it is analytically tractable and presents arbitrarily strong ultraviolet divergence as L increases. It is not a priori clear that potentials with arbitrarily strong ultraviolet divergence can be meaningfully renormalized. Physically, this potential is one of arbitrary short range in higher partial waves and should be compared with the S wave δ function potential. If a meaningful solution of the problem could be found they could be of use in different areas of physics where the details of a potential is not of concern. (a) The renormalized solution of the minimal potential could be used in problems of statistical mechanics, such as, in Cooper pairing in superconductivity. There are evidences of pairing in higher partial waves [12]. In this case the details of the phonon induced short-range electron-electron potential is irrelevent. The Cooper and the Bardeen-Cooper-Schrieffer (BCS) equations in superconductivity, have been satisfactorily renormalized in S wave, but not in higher partial waves [12]. The present work should be of relevance to the renormalization of the Cooper and BCS equations in higher partial waves. (b) The present renormalization scheme is also of interest in deriving a nucleon-nucleon potential from an effective field theory [13] as suggested by Weinberg [5]. In this derivation one needs to sum an infinite series of Feynmann diagrams. In the lowest order, the nucleon-nucleon potential as derived from the effective field theory includes an attractive delta potential, the solution of which has been successfully renormalized [13]. However, in higher order one obtains a potential with stronger divergence involving powers of momentum (see, for example Eq. (3.4) of Ref. [14]). Such a potential should be renormalizable following the scheme presented here.

The present approach is also applicable to other potentials with weaker ultraviolet divergences and/or permiting only numerical solution. A numerical study of the renormalization scheme has recently been made [15].

For the above mentioned minimal potential, the t matrix of Eq. (1) permits the following

analytic solution

$$T_L(p',p,k^2) = p'^L \tau_L(k) p^L, \tag{2}$$

with the τ function defined by

$$\begin{aligned} \tau_L(k) &= [\lambda^{-1} - I_L(k)]^{-1}, & (3) \\ I_L(k) &= (2/\pi)\int q^2 dq q^{2L} G(q;k^2). & (4) \end{aligned}$$

As the τ function completely determines the t matrix, we shall consider only the renormalizatio of the τ function. Here the condition of unitarity is given by

$$\Im T_L(k) = -k|T_L(k)|^2, \tag{5}$$

where $T_L(k) = T_L(k,k,k^2)$ and $\Im$ denotes the imaginary part.

The integral $I_L(k)$ of Eq. (4) possesses ultraviolet divergence. For $L = 0$ (1,...) the leadin divergence of this integral is linear (cubic,...) in nature. Finite result for the t matrix of Eqs. (2 and (3) can be obtained only if λ^{-1} also diverges in a similar fashion and cancels the divergenc of $I_L(k)$. The function $\lambda I_L(k)$ is the trace of the kernel of the integral equation (1) and possesse ultraviolet divergence. The kernel of Eq. (1) is noncompact and it does not have scatterin solution.

Hence some regularization is needed to give meaning to Eq. (1). This can be achieved b using a regularized Green function involving a cut-off. One example is the following regularize Green function with a smooth cut-off Λ for $L = 0$ as in Ref. [6]

$$\begin{aligned} G_R(q,\Lambda;k^2) &= (k^2 - q^2 + i0)^{-1} + (\Lambda^2 + q^2)^{-1}, \\ &= \frac{k^2 + \Lambda^2}{(k^2 - q^2 + i0)(\Lambda^2 + q^2)}. & (6 \end{aligned}$$

However, in the present work we shall use the following regularized Green function with a shar cut-off

$$G_R(q,\Lambda;k^2) = (k^2 - q^2 + i0)^{-1}\Theta(q - \Lambda), \tag{7}$$

$\Theta(x) = 0$ for $x > 0$ and $=1$ for $x < 0$. In Eqs. (6) and (7) $\Lambda(>> k)$ is a large but finit quantity. The reason for choosing Green function (7) is that it is equally applicable for a types of ultraviolet divergences in all partial waves, whereas Green function (6) is only valid fc a linear divergence as encountered in the $L = 0$ treatment of Ref. [6] and requires modification the divergence is stronger as in this work. Though we use Green function (7) with the minim potential in the present treatment, the present idea of renormalization can be extended t

other (singular) potentials and to other regularized Green function(s). The imaginary part of the Green function is unaffected by this type of regularization, and this guarantees unitarity condition (5).

In the end, the limit $\Lambda \to \infty$ has to be taken, which will reduce the regularized Green function to the free Green function. Finite results for physical magnitudes, as $\Lambda \to \infty$, are obtained only if the coupling λ is also replaced by the so called bare coupling $\lambda_L(k,\Lambda)$. The choice of the bare coupling is different for different L and can be found by inspection of the following regularized expressions of the integral $I_L(k)$ of (4) for different L

$$\begin{aligned} I_{RL}(k,\Lambda) &\equiv \frac{2}{\pi}\int q^2 dq q^{2L} G_R(q,\Lambda;k^2), && (8)\\ &= -\frac{2}{\pi}\Big[\Lambda + \frac{k}{2}\ln|\frac{\Lambda-k}{\Lambda+k}|\Big] - ik, L=0, && (9)\\ &= -\frac{2}{\pi}\Big[\frac{\Lambda^3}{3} + k^2\Lambda + \frac{k^3}{2}\ln|\frac{\Lambda-k}{\Lambda+k}|\Big] - ik^3, L=1, && (10)\\ &= -\frac{2}{\pi}\Big[\frac{\Lambda^5}{5} + \frac{k^2\Lambda^3}{3} + k^4\Lambda + \frac{k^5}{2}\ln|\frac{\Lambda-k}{\Lambda+k}|\Big] - ik^5, L=2. && (11)\end{aligned}$$

Consistent with the large $\Lambda(>> k)$ limit, the logarithmic terms in the above expressions for $I_{RL}(k,\Lambda)$ tend to zero, and for a general L we have in this limit

$$I_{RL}(k,\Lambda) = -\frac{2}{\pi}\sum_{i=0}^{L}\frac{k^{2(L-i)}\Lambda^{2i+1}}{2i+1} - ik^{2L+1}. \quad (12)$$

All the terms in the summation in Eq. (12) diverges as $\Lambda \to \infty$. Except for $L=0$, these divergent terms are momentum (k) dependent. In the present work, the leading divergence is much stronger for a general L compared to the S wave case. In Eq. (12) the leading divergence is like Λ^{2L+1}. The S wave treatment of the δ potential in Ref. [6] had only an energy independent term diverging linearly as Λ. For a finite $k(<< \Lambda)$, the stronger divergence and the energy dependence of the divergent terms in the present case do not introduce any complication and the ideas of Ref. [6] can be generalized.

In order to obtain a finite renormalized τ function, the coupling λ should be replaced by the so called bare coupling defined, for example, by

$$\lambda_L^{-1}(k,\Lambda) = -\frac{2}{\pi}\sum_{i=0}^{L}\frac{k^{2(L-i)}\Lambda^{2i+1}}{2i+1} - \Lambda_{0L}(k^2), \quad (13)$$

where the function $\Lambda_{0L}(k^2)$ defines the physical scale(s) of the system and characterizes the interaction. In the end the physical scale(s) in $\Lambda_{0L}(k^2)$ should be identified with a physical observable(s). If the problem is characterized by a single physical scale, e.g., the scattering length a_L, it is appropriate to take $\Lambda_{0L}(k^2)$ to be independent of k^2: $\Lambda_{0L}(k^2) = -1/a_L^{2L+1}$. If

the problem is characterized by two physical scales, such as a scattering length a_L and another physical scale b_L, it is natural to take the following expansion

$$\Lambda_{0L}(k^2) = -1/a_L^{2L+1} - b_L^{1-2L}k^2. \quad (14)$$

We have taken both the scales a_L and b_L to have the dimension of length. A third scale c_L can be accommodated similarly through

$$\Lambda_{0L}(k^2) = -1/a_L^{2L+1} - b_L^{1-2L}k^2 - c_L^{3-2L}k^4, \quad (15)$$

where c_L has also been chosen to have the dimension of length. Equation (15) is just a Taylor series expansion of $\Lambda_{0L}(k^2)$ at low energies. It is realized that in the present renormalization the number of divergent terms and the number of scales are not related.

The regularized τ function of Eq. (3) can now be rewritten as

$$\tau_L(k,\Lambda) = [\lambda_L^{-1}(k,\Lambda) - I_{RL}(k,\Lambda)]^{-1}, \quad (16)$$

where for a finite Λ, $I_{RL}(k,\Lambda)$ is a convergent integral. As $\Lambda \to \infty$, however, this integral develops the original ultraviolet divergence. In this limit, the quantity $\lambda_L^{-1}(\Lambda,k)$ of Eq. (13) has the appropriate divergent behavior, that cancels the divergent parts of $I_{RL}(k,\Lambda)$. In Eq. (16) the explicit dependence of the τ function on Λ has been introduced.

Next the limit $\Lambda \to \infty$ has to be taken in Eq. (16). With this regularization, the renormalized τ function can be written as

$$\tau_{RL}(k,\lambda_R(k,\mu),\mu) = [\lambda_{RL}^{-1}(k,\mu) - I_{RL}(k,\mu)]^{-1}, \quad (17)$$

where μ is the scale of the problem and emerges as a result of renormalization. The renormalization scale μ should be contrasted with the physical scale(s) in $\Lambda_{0L}(k^2)$. The renormalized τ function will be independent of μ. In Eq. (17) the explicit dependence of the τ function on both μ and the renormalized coupling $\lambda_{RL}(k,\mu)$ has been exhibited. The limiting procedure implied by $\Lambda \to \infty$ in Eq. (16) leads to the following definition for the renormalized coupling $\lambda_{RL}(k,\mu)$

$$\lambda_{RL}^{-1}(k,\mu) = \lim_{\Lambda\to\infty}[\lambda_L^{-1}(k,\Lambda) - \{I_{RL}(k,\Lambda) - I_{RL}(k,\mu)\}]. \quad (18)$$

In Eq. (18), if the limit $\Lambda \to \infty$ taken, we get

$$\lambda_{RL}(k,\mu) = \lambda_L(k,\Lambda=\mu). \quad (19)$$

Relation (19) between the renormalized coupling and bare coupling depends on the regularization scheme used. Equations (13) and (19) lead to the following expression for the renormalized coupling

$$\lambda_{RL}^{-1}(k,\mu) = -\frac{2}{\pi}\sum_{i=0}^{L}\frac{k^{2(L-i)}\mu^{2i+1}}{2i+1} - \Lambda_{0L}(k^2). \tag{20}$$

The renormalized coupling for two renormalization scales μ and μ_0 are related by the following flow equation

$$\lambda_{RL}^{-1}(k,\mu) + \frac{2}{\pi}\sum_{i=0}^{L}\frac{k^{2(L-i)}\mu^{2i+1}}{2i+1} = \lambda_{RL}^{-1}(k,\mu_0) + \frac{2}{\pi}\sum_{i=0}^{L}\frac{k^{2(L-i)}\mu_0^{2i+1}}{2i+1}. \tag{21}$$

For a general L, this flow equation is energy dependent but independent of the regularization scheme. For $L = 0$, as in Ref. [6], the renormalized coupling and the flow equations are energy independent. The absolute value of the renormalized coupling $\lambda_{RL}(k,\mu)$ increases with μ. Thus if we start with a small $\lambda_{RL}(k,\mu_0)$ at a given renormalization scale μ_0, the effective coupling constant increases with μ as in the $\lambda\phi^4$ model [2]. With the increase of μ one can reach a large enough $\lambda_{RL}(k,\mu)$, where perturbative treatment is not valid. The energy dependence of the renormalized coupling (20) and the flow equation (21) for $L \neq 0$ does not create any complication and one can renormalize the results and write the RG equations.

The present scattering model permits analytic solutions for all L. The renormalized τ function is given by

$$\tau_{RL}(k,\lambda_{RL}(k,\mu),\mu) = \left[\lambda_{RL}^{-1}(k,\mu) + \frac{2}{\pi}\sum_{i=0}^{L}\frac{k^{2(L-i)}\mu^{2i+1}}{2i+1} + ik^{2L+1}\right]^{-1}, \tag{22}$$

Explicitly, using the renormalized coupling (20), the renormalized τ function can be written as

$$\tau_{RL}(k,\lambda_{RL}(k,\mu),\mu) = [ik^{2L+1} - \Lambda_{0L}(k^2)]^{-1}. \tag{23}$$

This τ function depends on the renormalized coupling $\lambda_{RL}(k,\mu)$, but not on μ, that is the explicit and implicit (through $\lambda_{RL}(k,\mu)$) dependences of the τ function on μ cancel. Physics is determined by the value of $\lambda_{RL}(k,\mu)$ at an arbitrary value of μ [9], or the following μ independent quantity

$$\lambda_{RL}^{-1}(k,\mu) + \frac{2}{\pi}\sum_{i=0}^{L}\frac{k^{2(L-i)}\mu^{2i+1}}{2i+1} = -\Lambda_{0L}(k^2), \tag{24}$$

as can be seen from Eqs. (20) and (22).

From Eq. (23) we find that the renormalized τ function is a function of $\Lambda_{0L}(k^2)$. It is convenient to express the renormalized τ function in terms of the physical scales a_L, b_L, and

c_L introduced in Eq. (15). Once this is done, τ_{RL} is determined by the physical scale(s) which are closely related to the observables of the system. Then Eq. (23) reduces to

$$\tau_{RL}(k,a_L,b_L,c_L) \equiv \tau_{RL}(k,\lambda_R(k,\mu),\mu) = [ik^{2L+1} + 1/a_L^{2L+1} + b_L^{1-2L}k^2 + c_L^{3-2L}k^4]^{-1}. \tag{25}$$

The name 'physical scale' given to a_L, b_L, and c_L, is justified as these quantities are a measure of low energy scattering in each partial wave.

We have here renormalized a divergent physical problem and obtained the well-defined solution (25). As the original problem is ill-defined, it is interesting to ask if this renormalized solution is physically acceptable or is just a finite answer obtained by a mathematical trick from the unregularized original problem. The fact, that the renormalized result is physically motivated, can now be established by a careful examination. For $L = 0$ Eq. (25) is just the usual effective range expansion for the t matrix [16]. The same is also true for higher partial waves [16]. Hence the renormalized solution (25) is the physically expected solution of the problem for a short-range potential and should lead to acceptable results for other observables. So the present renormalization scheme for $L \neq 0$ should be considered as a natural generalization of our previous results presented in Ref. [6].

To bring further evidence to the acceptability of the present result and to demonstrate the self-consistency of the present renormalization scheme, we perform perturbative renormalization of the same problem in the next section and establish the equivalence between the two approaches.

3 Perturbative Renormalization

In the last section we performed the renormalization of the exact analytic solution. In the simplest field theoretic $\lambda\phi^4$ model the exact solution is not known because of the creation and annihilation of particles and also because of the quartic nature of the interaction. In that case one usually performs perturbative renormalization. Though it is expected that the result of perturbative renormalization should be consistent with that of exact renormalization, there is no general proof in this regard. As the present problem is much simpler than the $\lambda\phi^4$ model, it is also illustrative to perform perturbative renormalization of the present problem and to show that the result is consistent with the exact renormalization of the last section. This consistency can also be established in the case of the exactly soluble Schwinger model for massless quantum electrodynamics in 1+1 dimensions.

From Eq. (3) the perturbative solution of the present problem is given by

$$\tau_L(k) = \lambda[1 + \lambda I_{RL}(k,\Lambda) + \lambda^2 I_{RL}^2(k,\Lambda) + \lambda^3 I_{RL}^3(k,\Lambda) + ...], \tag{26}$$

where we have used the regularized version of the integral $I_{RL}(k,\Lambda)$ given by (12).

Up to the first order in perturbation theory in Eq. (26), we take $\lambda_L^{(1)} = \bar{\lambda}_L$ and consequently to this order in the redefined coupling strength $\bar{\lambda}_L$, $\tau_L^{(1)}(k) = \bar{\lambda}_L$ and there is no divergence as $\Lambda \to \infty$. In order to find the result finite to first order one could have taken $\lambda_L^{(1)} = \lambda$. But if we would like to obtain finite results in all orders of perturbation theory, which are consistent with the exact renormalized result of the last section, a different finite coupling ($\bar{\lambda}_L$) has to be introduced in all orders as in this section.

The second order τ function with this $\bar{\lambda}_L$, given by,

$$\tau_L^{(2)}(k) = \bar{\lambda}_L[1 + \bar{\lambda}_L I_{RL}(k,\Lambda)], \tag{27}$$

however, diverges as $\Lambda \to \infty$, because in this limit the regularized integral $I_{RL}(k,\Lambda)$ of (12) diverges. A finite result for the τ function up to the second order in $\bar{\lambda}_L$ could be obtained by employing the following coupling

$$\lambda_L^{(2)} = \bar{\lambda}_L\Big[1 + \bar{\lambda}_L \frac{2}{\pi}\sum_{i=0}^{L}\frac{k^{2(L-i)}\Lambda^{2i+1}}{2i+1}\Big]. \tag{28}$$

With this modified coupling, the second order τ function (27) is given by

$$\tau_L^{(2)}(k) = \lambda_L^{(2)}[1 + \lambda_L^{(2)} I_{RL}(k,\Lambda)], \tag{29}$$

where $I_{RL}(k,\Lambda)$ is given by Eq. (12). With $\lambda_L^{(2)}$ given by (28), the second order τ function, $\tau_L^{(2)}(k)$, contains terms up to the fourth order in $\bar{\lambda}_L$. Up to the second order in $\bar{\lambda}_L$, $\tau_L^{(2)}(k)$ is finite in the limit $\Lambda \to \infty$ and is given by

$$\tau_L^{(2)}(k) = \bar{\lambda}_L[1 - \bar{\lambda}_L i k^{2L+1}]. \tag{30}$$

With the second order coupling constant given by Eq. (28), the third order τ function

$$\tau_L^{(3)}(k) = \lambda_L^{(2)}[1 + \lambda_L^{(2)} I_{RL}(k,\Lambda) + \lambda_L^{(2)^2} I_{RL}^2(k,\Lambda)] \tag{31}$$

diverges in the limit $\Lambda \to \infty$. In order to obtain a finite τ function in this limit up to the third order in $\bar{\lambda}_L$, one could employ the following third order λ

$$\lambda_L^{(3)} = \bar{\lambda}_L\Big[1 + \bar{\lambda}_L \frac{2}{\pi}\sum_{i=0}^{L}\frac{k^{2(L-i)}\Lambda^{2i+1}}{2i+1} + \bar{\lambda}_L^2\Big(\frac{2}{\pi}\sum_{i=0}^{L}\frac{k^{2(L-i)}\Lambda^{2i+1}}{2i+1}\Big)^2\Big] \tag{32}$$

in the following expression for the third order τ function

$$\tau_L^{(3)}(k) = \lambda_L^{(3)}[1 + \lambda_L^{(3)} I_{RL}(k,\Lambda) + \lambda_L^{(3)^2} I_{RL}^2(k,\Lambda)]. \tag{33}$$

This third order τ function now contains terms up to the sixth order in $\bar{\lambda}_L$. If the third order τ function is truncated up to third order terms in $\bar{\lambda}_L$, and the limit $\Lambda \to \infty$ is taken, we obtain

$$\tau_L^{(3)}(k) = \bar{\lambda}_L[1 - \bar{\lambda}_L i k^{2L+1} + \bar{\lambda}_L^2(-ik^{2L+1})^2]. \tag{34}$$

With the third order λ given by Eq. (32), the higher order τ functions $\tau_L^{(l)}, l > 3$ diverges in the limit $\Lambda \to \infty$. In order to obtain a finite $\tau_L^{(l)}, l > 3$, one should modify the coupling strength λ. A finite $\tau_L^{(l)}$ up to lth order, in the limit $\Lambda \to \infty$, can be obtained by employing the following coupling

$$\lambda_L^{(l)} = \bar{\lambda}_L\Big[1 + \sum_{j=1}^{l-1}\Big(\bar{\lambda}_L \frac{2}{\pi}\sum_{i=0}^{L}\frac{k^{2(L-i)}\Lambda^{2i+1}}{2i+1}\Big)^j\Big]. \tag{35}$$

With this λ, the lth order τ function is given by

$$\tau_L^{(l)}(k) = \lambda_L^{(l)}[1 + \sum_{j=1}^{l-1}\lambda_L^{(l)^j} I_{RL}^j(k,\Lambda)]. \tag{36}$$

Once the limit $\Lambda \to \infty$ is taken in Eq. (36) and terms up to the order of $\bar{\lambda}_L^l$ are maintained, the following result is obtained

$$\tau_L^{(l)}(k) = \bar{\lambda}_L[1 + \sum_{j=1}^{l}(-\bar{\lambda}_L i k^{2L+1})^j]. \tag{37}$$

Unlike in the case of $\lambda\phi^4$ field theory, one can calculate the result to an arbitrarily large order in perturbation theory. The summation in Eq. (37) is a geometric series and as $l \to \infty$ this series can be summed to yield

$$\tau_{RL}(k) = \frac{1}{1/\bar{\lambda}_L + ik^{2L+1}}, \tag{38}$$

which is the result of perturbative renormalization.

If we compare the result of perturbative renormalization (38) with the exact renormalized solution (23) we realize that these two are equivalent if $\Lambda_{0L}(k^2)$ of Eq. (23) is identified as $-\bar{\lambda}_L^{-1}$ of Eq. (38). If this identification is made, the result of perturbative renormalization is consistent with the exact renormalized result. Then we find that the parameter $\Lambda_{0L}(k^2)$ of exact renormalization is intimately related to the strength parameter $\bar{\lambda}_L$ of perturbative renormalization.

4 Renormalization Group Equations

The renormalized τ function is independent of μ, so is invariant under the group of transformations $\mu \to \exp(s)\mu$, which form the RG. In the present case, as in the $\lambda\phi^4$ model, it is

convenient to work in terms of the dimensionless coupling, $g_{RL}(\mu)$, defined by

$$g_{RL}(k,\mu) \equiv \mu^{2L+1}\lambda_{RL}(k,\mu), \tag{39}$$

The renormalization condition is given by

$$\mu\frac{d}{d\mu}\tau_{RL}(k, g_{RL}(k,\mu),\mu) = 0, \tag{40}$$

or,

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_L(k, g_{RL}(k,\mu),\mu)\frac{\partial}{\partial g_{RL}}\right]\tau_{RL}(k, g_{RL}(k,\mu),\mu) = 0, \tag{41}$$

where

$$\beta_L(k, g_{RL}(k,\mu),\mu) = \mu\frac{\partial g_{RL}(k,\mu)}{\partial\mu}. \tag{42}$$

Equation (41) is the RG equation.

As the present problem permits analytic solution, the constant β_L of Eq. (42) can be exactly calculated. From Eqs. (39) and (42) we have

$$\beta_L(k, g_{RL}(k,\mu),\mu) = (2L+1)g_{RL}(k,\mu) + \mu^{2L+2}\frac{\partial\lambda_{RL}(k,\mu)}{\partial\mu}. \tag{43}$$

With $\lambda_{RL}(k,\mu)$ defined by Eq. (20), we have from Eqs. (39) and (43)

$$\beta_L(k, g_{RL}(k,\mu),\mu) = (2L+1)g_{RL} + g_{RL}^2\frac{2}{\pi}\sum_{i=0}^{L}k^{2(L-i)}\mu^{2(i-L)}. \tag{44}$$

For $L=0$ the β function is energy independent and depends implicitly on μ through coupling g_{RL}, whereas for $L\neq 0$ the β function has explicit dependence on both energy and μ.

The following equation expresses the invariance of the τ function $\tau_{RL}(k, g_{RL}(k,\mu),\mu)$ under a change of momentum scale:

$$\tau_{RL}(\gamma k, g_{RL}(k,\mu),\mu) = \gamma^{-(2L+1)}\tau_{RL}(k, g_{RL}(k,\mu),\mu\gamma^{-1}). \tag{45}$$

Equations (22) and (39) are consistent with scaling (45). In Eq. (45) the change of scale is effected on the explicit momentum (k) dependence of the τ function and not on the implicit momentum dependence of the coupling constant g_{RL}. From Eq. (45) we obtain

$$\left[\gamma\frac{\partial}{\partial\gamma} + \mu\frac{\partial}{\partial\mu} + (2L+1)\right]\tau_{RL}(\gamma k, g_{RL}(k,\mu),\mu) = 0. \tag{46}$$

Eliminating the partial derivative $\mu(\partial\tau_{RL}/\partial\mu)$ between Eqs. (41) and (46) we have

$$\left[\gamma\frac{\partial}{\partial\gamma} - \beta_L(k, g_{RL}(k,\mu),\mu)\frac{\partial}{\partial g_{RL}} + (2L+1)\right]\tau_{RL}(\gamma k, g_{RL}(k,\mu),\mu) = 0, \tag{47}$$

with β_L given by Eq. (44). RG equation (47) expresses the effect on the τ function of scaling up momentum by a factor γ.

The RG equations (41) and (47) involve the renormalized coupling g_{RL} and the renormalization scale μ and are not closely related to the physical observables. However, one can write equivalent RG equations in terms of the physical scales a_L, b_L, and c_L of Eq. (25), which are closely related to experimental observables. From Eqs. (22), (25), and (39) one has the identity

$$\left[\gamma\frac{\partial}{\partial\gamma} - \beta_L\frac{\partial}{\partial g_{RL}}\right]\tau_{RL}(\gamma k, g_{RL},\mu) = \left[\gamma\frac{\partial}{\partial\gamma} - a_L\frac{\partial}{\partial a_L} - b_L\frac{\partial}{\partial b_L} - c_L\frac{\partial}{\partial c_L}\right]\tau_{RL}(\gamma k, a_L, b_L, c_L), \tag{48}$$

so that the RG equation (47) becomes

$$\left[\gamma\frac{\partial}{\partial\gamma} - a_L\frac{\partial}{\partial a_L} - b_L\frac{\partial}{\partial b_L} - c_L\frac{\partial}{\partial c_L} + (2L+1)\right]\tau_{RL}(\gamma k, a_L, b_L, c_L) = 0, \tag{49}$$

Equations (47) and (49) express the fact that the effect of a change in the momentum scale γ on τ_{RL} can be compensated by the effect of a change in g_{RL} or equivalently, in a_L, b_L, and c_L, respectively. In RG equation (49) a_L, b_L, and c_L are physical scales. RG equation (49) implies the following scaling

$$\tau_{RL}(\gamma k, a_L, b_L, c_L) = \gamma^{-(2L+1)}\tau_{RL}(k, \gamma a_L, \gamma b_L, \gamma c_L). \tag{50}$$

Hence from the knowledge of the τ function or the t matrix at a certain energy one can predict the τ function at another energy. RG equations allow one to extrapolate the τ function from one energy to another.

In principle, RG equations can be solved to yield the exact renormalized τ function. However, it is illustrative to obtain the asymptotic high-energy behavior of this τ function from RG equations (47) or (49). At high energies $\gamma\to\infty$, and Eqs. (47) or (49) reduces to

$$\gamma\frac{\partial\tau_{RL}(\gamma k)}{\partial\gamma} + (2L+1)\tau_{RL}(\gamma k) = 0. \tag{51}$$

This has the simple solution $\lim_{\gamma\to\infty}\tau_{RL}(\gamma k)\sim 1/\gamma^{2L+1}$ again consistent with the τ function o Eq. (25).

The RG equations of this section yield certain general scaling properties of the renormal ized τ function. Similar RG equations should be valid in general for potentials with certai renormalizable singular behavior at short distances. The RG equations in terms of the physic scales and the associated scaling relations, e.g. Eqs. (49), (50), and (51), should be valid i general independent of whether the original problem had ultraviolet divergence or not. Hen such equation should be useful in general. Obviously, such equation could now be generaliz to incorporate more physical scales.

5 Summary

We have emphasized the role of renormalization in nonrelativistic quantum mechanics. Renormalization is desirable in most of quantum mechanical bound state and scattering problems, if one is interested in comparing the result of a physical model with experimental observables. Renormalization is essential in some problems exhibiting ultraviolet divergence, as in quantum field theory, in order to yield well-defined and finite observables. In both cases the final renormalized results could be expressed in terms of certain physical scales which are closely related to physical observables.

We have renormalized a potential model exhibiting ultraviolet divergence in all partial waves. As this model permits analytic solution, we have renormalized the solution exactly and also perturbatively. In field theoretic model only perturbative renormalization is possible. As the present model permits both perturbative and exact solutions it gives us the unique opportunity to test the equivalence of the two. Such equivalence is established under very general conditions. The final renormalized result can be expressed equivalently, in terms of a renormalized coupling λ_{RL} and renormalization scale μ, or in terms of some physical scales related to observables.

Finally, we derived RG equations for the renormalized amplitudes expressed in terms of both renormalized coupling and physical scales. Though the physical content of both are identical, RG equations in terms of physical scales seem to be more useful from a practical point of view. Such RG equations in terms of physical scales are valid irrespective of the existence of the ultraviolet divergence in the original equation. These equations provide interesting scaling behavior of the physical scattering amplitude. The RG equations are expected to be very useful in situations where the analytic solution is not known, for example, in other few- and many-body problems. The study of renormalization and RG equations in these cases will be an interesting topic for future investigation.

We thank Dr. R. Banerjee, Dr. T. Frederico, Dr. B. M. Pimentel, and Dr. L. Tomio for discussions and the Conselho Nacional de Desenvolvimento Científico e Tecnológico, Fundação de Amparo à Pesquisa do Estado de São Paulo, and Financiadora de Estudos e Projetos of Brazil for partial financial support.

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