Causal approach to quantum electrodynamics in three-dimensional space-time

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Abstract

We study the dynamically generated photon mass, the electron self-energy and the vertex in the causal theory of quantum electrodynamics in three-dimensional space-time (QED3). We also provide a general discussion of the regularization and gauge (in)dependence of the photon mass. A last section is devoted to challenging open problems.

I. Introduction and Summary

Quantum electrodynamics in (2+1) space-time dimensions (QED3) [1] has attracted much interest in recent times. It is a successful model of both the integer and fractional quantum Hall effects [2]. Some aspects of the model are also of some interest for high-Tc superconductivity, as electrons seem to experience an attractive potential, with formation of bound states (B), (3), (4). The greatest interest lies, however, in serving as a laboratory for one of the most important conceptual problems of quantum field theory: the connection between dynamical mass generation, gauge invariance and regularizations. The classical Lagrangian density of a system of free electrons of mass and photons of mass is

$$\mathcal{L} = \overline{\psi} (i \gamma \cdot D - m) \psi - \frac{1}{4} F_{\mu \nu} F_{\mu \nu} + \frac{1}{4} \bar{\psi} \gamma^\mu F_{\mu \nu \alpha} \gamma^\nu A^\alpha - \frac{1}{2} (\partial_\mu A^\mu)^2. \tag{1}$$

Above, $$\psi$$ denotes the electron field, $$D_\mu = \partial_\mu + ieA_\mu$$ is the covariant derivative, $$A_\mu$$ the photon field with $$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$ the electromagnetic field tensor, $$\gamma_\mu = \gamma_\mu \gamma_5$$, $$\overline{\psi} = \psi^\dagger \gamma^0$$, and

$$f_1(\partial_\mu A^\mu)^2$$ is a gauge fixing term ($$a = 0$$ is the Landau gauge and $$a = 1$$, the Feynman gauge). The spinor and $$\gamma$$-matrix indices run from zero to three, with

$$\gamma^0 = i\sigma^3, \quad \gamma^1 = -i\sigma^1, \quad \gamma^2 = i\sigma^2$$

in terms of the usual Pauli matrices $$\sigma^i$$. In this paper, we adopt a two-dimensional representation of the Fermi fields in d=3 because it is irreducible and from this point of view most natural. Since the Pauli matrices form a complete basis, there is no $$\gamma^5$$ and thus chiral symmetry is not defined in the standard fashion. Other discrete symmetries are broken in connection with mass generation. An example of a parity transformation in d=3 is

$$x' = \Lambda_p x$$,

where $$x = (x_0, x_1, x_2)$$, and

$$\Lambda_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with det$$\Lambda_p = -1$$ as it should be. We might also let the $$x_2$$-component change sign, instead of the $$x_1$$-component, but not both $$x_1$$- and $$x_2$$-components, since, in two space dimensions, this is equivalent to a rotation over 180° with determinant +1. The effect of the parity transformation on the fermion and photon fields is [1]

$$\psi'(x') = \mathcal{P} \psi(x) \mathcal{P}^{-1} = \sigma^1 \psi(x')$$

$$A'^\mu(x') = \mathcal{P} A^\mu(x) \mathcal{P}^{-1} = \Lambda_p A^\mu(x)$$

and hence it is easy to see that both mass terms $$m^2 \psi$$ and $$f_1(\partial_\mu A^\mu)^2$$ change sign under $$\mathcal{P}$$. In the term $$\bar{\psi} \gamma^{\mu \nu} F_{\mu \nu} A_\alpha$$, $$\mu$$ may be considered to be a bare photon mass. Indeed, this term is bilinear in $$A_\alpha$$, and the bare photon propagator has a pole at $$k^2 = \mu^2$$. This term...
often called a topological or Chern-Simons mass term, because, for non-Abelian gauge fields, the corresponding mass term is a topologically non-trivial quantity, which must be quantized in terms of winding numbers on manifolds to maintain gauge invariance. Under the gauge transformation

$$\psi \rightarrow e^{i\alpha_\mu} \psi,$$
$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$$

this term changes by a total derivative, and hence the Lagrangian density

$$\mathcal{L} \rightarrow \mathcal{L} + \partial_\mu (D_\mu \Lambda).$$

If the fields vanish at infinity, i.e., are physical fields, the action $S = \int d^4x \mathcal{L}$ is thus gauge-invariant. Hence, physical quantities are gauge-invariant. The Lagrangian density (1.1) describes the so-called Maxwell-Chern-Simons theory, which has been studied extensively [5], most recently in the Coulomb gauge [6], which presents subtle problems. Another interesting direction of recent work was the construction of a pseudoclassical model for Chern-Simons particles ([7],[8]). In this introduction, we should like to explain some relevant issues, and summarize the content of the following sections.

In this paper we do not wish to consider (1.1) with the explicit Chern-Simons mass term, but rather we show how this term is generated from the theory with $\lambda$. This is the problem of dynamical mass generation. The question arises how mass is defined in the quantum theory. We assume a covariant quantization, which generally implies the use of an extended Hilbert space with indefinite metric (see, e.g., [9] for a concise description of this formalism). We may either restrict ourselves to the physical subspace of Fock space (with positive metric) or consider a non-self-adjoint $A_\mu$ operator.

Let $\psi$ and $A_\mu$ denote the canonically quantized fermion and photon fields, respectively. The corresponding propagators are defined as

$$S_{\psi}(x - y) = -i\langle 0 | T\psi(x)\psi(y) | 0 \rangle,$$
$$D_{\mu}(x - y) = -i\langle 0 | T\partial_\mu A(x) A(y) | 0 \rangle,$$

where $T$ is the time-ordering operator and $| 0 \rangle$ denotes the physical vacuum. The bare propagators $S_{\psi}(x - y)$ and $D_{\mu}(x - y)$ satisfy the equations (for the moment, $d$ is general and denotes the space-time dimension)

$$[i\gamma^\mu - \frac{1 - \gamma^5}{2}] \gamma^\mu D_{\mu}(x - y) = \delta^d(x - y),$$

where $\gamma^\mu = \left( \begin{array}{cc} 1 & \alpha \\ \bar{\alpha} & 0 \end{array} \right)$ for $d = 3$ is the metric tensor and $\delta^d$ is the $d$-dimensional delta function. In momentum space,

$$S(k) = \int d^dx e^{ikx} S^0(x) = \frac{1}{k^2 - m^2 - i\epsilon},$$
$$D_{\mu}(k) = -\frac{i}{k^2} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) - \frac{k_\mu k_\nu}{k^2},$$
II. The Infrared Problem

The method for dealing with non-stationary processes as electron scattering by an external potential with the emission of a finite number of low-energy photons, using the Dyson $S$ matrix expansion, leads to the so-called infrared divergences in the probability amplitudes in QED.

The program of taking into account the collective effect of low-energy photons was first accomplished by Bloch and Nordsieck [23], who showed that the probability of emission of a finite number of low-energy photons is zero and not infinite, as predicted by perturbation theory, and the transition probability must be extended to all possible final states to yield a finite result.

One can attribute to the infrared divergences an incorrect choice of the asymptotic state which can be compared with the experimental data, besides the ill-definition of the scattered state.

Consider a quantum system described by a Hamiltonian $H$ constituting by two terms

$$H = H_0 + V(t),$$

where $H_0$ is the free Hamiltonian and

$$\lim_{t \to \pm \infty} V(t) = 0.$$  

This implies that asymptotically the scattered particles are supposed to be free. Thus, since in scattering processes the particles are observed only in asymptotic regions ($t \to \pm \infty$), the asymptotic states are eigenstates of $H_0$.

Using these ideas, one defines the $S$ operator in perturbation theory assuming that the asymptotic dynamics is given by $H_0$:

$$S = W_{in} W_{out},$$

where $W_{in}$ are wave operators

$$W_{in} = \lim_{t \to \pm \infty} U(t, 0) e^{-iH_0 t}$$

and

$$U(t, \tau) = e^{-iH_0 (t-\tau)}.$$  

Using (II.5), one rewrites the $S$ matrix as

$$S = \lim_{t \to \pm \infty} e^{iH_0 t} U(t, \tau) e^{-iH_0 \tau}.$$
The transition probability from this state to another final asymptotic state $\psi$ is given by
\[ \langle \psi | S | \varphi \rangle^2, \] (II.7)
where the asymptotic states belong to the Fock space.

Solving the equation of motion for the time evolution operator, the $S$-matrix can be rewritten as a time ordered product
\[ S = T \exp \left\{ -i \int_{-\infty}^{\infty} V^{(1)}(t) dt \right\}, \] (II.8)
where $V^{(1)}(t)$ is the potential in the interaction picture. If in addition to (II.2) we require
\[ \int_{-\infty}^{\infty} \ln \left| V^{(1)}(s) \right| ds < \infty, \] (II.9)
then the Dyson series for the $S$ operator (II.8) is absolutely convergent. Thus, the $S$ operator is unitary in Fock space, provided the particles have free dynamics in asymptotic regions.

The above method is not suitable to describing scattering processes when we consider long-range potentials such as the Coulomb potential. In this case, even at regions very far this potential cannot be neglected.

Following [27], let us consider the scattering of a charged particle by a Coulomb potential in two dimensions. The Hamiltonian of the system has the following form:
\[ H = \frac{p^2}{2m} + g \ln r = H_0 + V \] (II.10)
where $m$ is the mass of the scattered particle and $g$ is the product of the charges of the particle and the scattering center.

First of all one constructs the potential shape in the asymptotic region in the interaction picture and then one obtains the wave packet which will represent the scattered particle in this region. For this purpose one considers the observables $\vec{r}$ and $\vec{p}$ as the position and momentum operators in the interaction picture. In this representation these operators satisfy the following equations of motion:
\[ \frac{d}{dt} \left[ \vec{r}, H_0 \right] = 0. \] (II.11)
Using the Hamiltonian (II.10) and the above equation, we see that the momentum of the scattered particle is a constant of motion:
\[ \frac{d}{dt} \vec{p} = \frac{1}{i} \left[ \vec{p}, H_0 \right] = 0. \] (II.12)
Similarly, the equation of motion for the coordinates
\[ \frac{d}{dt} \vec{r} = \frac{1}{i} \left[ \vec{r}, H_0 \right] = \frac{\vec{p}}{m}. \] (II.13)
whose solution
\[ \vec{r}(t) = \vec{r} + \frac{\vec{p}}{m} t, \] (II.14)
which describes the time evolution of the coordinate operator of the scattered particle, is identical to the classical trajectory of a particle in uniform rectilinear motion. From these results one can obtain the shape of the interaction potential at large distances, assuming that in this region the particles behave as classical particles with well defined trajectories. Thus, for $|t| \to \infty$
\[ V(t) = g \ln \left( \frac{|\vec{p}|}{m} t \right). \] (II.15)
This potential, which describes the interaction in the asymptotic region $|t| \to \infty$, is not absolutely convergent and its contribution to the asymptotic dynamics cannot be underestimated. In other words, the asymptotic dynamics is not governed by $H_0$ but by the operator
\[ H_{as}(t) = H_0 + V_{as}(t) = H_0 + g \ln \left( \frac{|\vec{p}|}{m} t \right). \] (II.16)
With this Hamiltonian describing the asymptotic dynamics and taking into account that $V_{as}(t)$ in the interaction picture is the same as in the Schrödinger picture, since
\[ [V_{as}(t), H_0] = 0, \] (II.17)
the wave function which describes the behavior of the particle in this region is obtained by solving the Schrödinger equation for $H_{as}$
\[ i \frac{d}{dt} \left[ \psi, H_0 \right] = H_{as}(t) \left[ \psi, H_0 \right], \] (II.18)
where $|\psi, t\rangle$ is the physical state of the scattered particle. In the momentum representation the above equation becomes
\[ i \frac{d}{dt} \left[ \psi, \vec{p}, t \right] = \left[ \frac{\vec{p}^2}{2m} + g \ln \left( \frac{|\vec{p}|}{m} t \right) \right] \psi \left( \vec{p}, t \right), \] (II.19)
whose solution
\[ \psi \left( \vec{p}, t \right) = \frac{1}{2\pi} \int d^n \vec{p} \psi \left( \vec{p}, t \right) e^{i\vec{p} \cdot \vec{r}} \]
\[ = \frac{1}{2\pi} \int d^n \vec{p} e^{i\vec{p} \cdot \vec{r}} \left( \frac{|\vec{p}|}{m} \right)^{-\frac{1}{2}} \exp \left\{ -i \frac{\vec{p}^2}{2m} t - ig \left[ t \ln \left( \frac{|\vec{p}|}{m} t \right) - t_0 \ln \left( \frac{|\vec{p}|}{m} t_0 - 1 \right) \right] \right\}. \] (II.20)
The choice of this solution is due to certain initial conditions for equation (11.19). These are determined by considering that the time variation of the coordinate and momentum distributions for the particle represented by the wave packet (11.20) for $|t| \to \infty$ must be governed by the classical dynamics.

The wave packet (11.20) can be written as

$$\Psi(t) = U_{as}(t) \Psi$$

$$= e^{-i\Omega t} \exp \left\{ -i \left[ t \ln \left( \frac{|p|}{m} - 1 \right) - t_0 \ln \left( \frac{|p|}{m} t_0 - 1 \right) \right] \right\} \Psi.$$  (11.21)

This means that the choice of the Hamiltonian, which describes the asymptotic dynamics, depends on the physical origin of the problem, in contrast with the usual definitions in the formal theory of scattering, where $H_0$ is taken as the asymptotic operator in the wave operators.

We then redefine the wave operators in the following way:

$$W_{m} = \lim_{|t| \to \infty} e^{-iHt} U_{as}(t).$$  (11.22)

We may derive the asymptotic operator $U_{as}(t)$ in quantum electrodynamics [27]. The interaction operator of the system of photons and charged particles, in the interaction representation, is given by

$$V_I = \int j^\mu(x) \cdot A_\mu(x) \, d^2 \vec{x},$$  (11.23)

where $j^\mu(x)$ is the current operator

$$j^\mu(x) = e \vev{\Psi(x) \gamma^\mu \Psi(x)},$$  (11.24)

The decomposition of $\Psi(x)$ into a creation part $\Psi^+(x)$, and an annihilation part $\Psi^-(x)$, separates $j^\mu(x)$ into four terms

$$j^{\mu+} = e \vev{\Psi^+(x) \gamma^\mu \Psi(x)},$$  (11.25)

$$j^{\mu-} = e \vev{\Psi^-(x) \gamma^\mu \Psi(x)},$$  (11.26)

$$j^{++} = e \vev{\Psi^+(x) \gamma^\mu \Psi^+(x)}$$  (11.27)

$$j^{--} = e \vev{\Psi^-(x) \gamma^\mu \Psi^-(x)}.$$  (11.28)

We can investigate the asymptotic behavior of this expression for $|t| \to \infty$. In this limit only $j^{++}$ and $j^{--}$ survive. In fact, since $V_I(t)$ contains the time $t$ in the form $e^{zt}$, only those terms for which $z$ approach zero within the range of integration over momentum space will survive. Thus, the terms $j^{++}$ and $j^{--}$ do not contribute in the asymptotic region.

Thus in (2+1) dimensions, the potential which describes the interaction between electron and positrons with the electromagnetic field in the asymptotic region is given by

$$V_{as}(t) = \frac{1}{(2\pi)^2} \int \frac{d^2 \vec{p}}{\sqrt{2\omega_p \rho(\vec{p})}} \int \frac{d^2 \vec{q}}{\sqrt{2\omega_q \rho(\vec{q})}} \left[ a_\mu(\vec{k}) e^{-i\omega t} + a_\mu^+(\vec{k}) e^{i\omega t} \right]$$

$$= \frac{1}{(2\pi)^2} \int \frac{d^2 \vec{p}}{\sqrt{2\omega_p \rho(\vec{p})}} j_{as}^{++}(\vec{k}, t) \left[ a_\mu(\vec{k}) e^{-i\omega t} + a_\mu^+(\vec{k}) e^{i\omega t} \right].$$  (11.28)

where $\rho(\vec{p})$ is the charge-density operator

$$\rho(\vec{p}) = e \left[ a_\mu(\vec{p}) b_\mu(\vec{p}) - a_\mu^+(\vec{p}) b_\mu^+(\vec{p}) \right],$$  (11.29)

and

$$j_{as}^{++}(\vec{k}, t) = \int \frac{d^2 \vec{q}}{\sqrt{2\omega_q \rho(\vec{q})}} \rho(\vec{q}) \frac{\vec{q} \times \vec{k}}{\rho(\vec{q}) \vec{q}}.$$  (11.30)

is an operator that has the shape of a current distribution of a particle with charge density $\rho(\vec{p})$ and uniform velocity $\vec{v}$. In fact, the eigenvalues of the operator (11.31), acting in a space of charged particles, are classical current densities due to the motion of these particles, showing that, asymptotically, the scattered particles behave as classical particles.

As in the non-relativistic case, the Hamiltonian $H_{as}$ that describes the asymptotic dynamics is given by

$$H_{as} = H_0 + V_{as}(t),$$  (11.32)

where $H_0$ is the free Hamiltonian. We can obtain $U_{as}(t)$ by solving the Schrödinger equation for the time evolution operator

$$i \frac{d}{dt} U(t) = H(t) U(t),$$  (11.33)

where $H$ is the Hamiltonian of the system. In our problem $H = H_{as}$ and, therefore, the above differential equation must be satisfied by the operator $U_{as}(t)$.

The asymptotic operator consists of two factors that commute. The first involves photons, and the second is a phase factor which can be written as

$$\phi(t) = \frac{1}{4\pi} \int d^2 \vec{q} \int d^2 \vec{p} \frac{p \cdot q}{\rho(\vec{q}) \rho(\vec{p})} \left[ 1 - \ln \left( \frac{\vec{p} - \vec{q}}{\rho(\vec{q})} \right) \right],$$  (11.34)

showing that this is a relativistic generalization of the above mentioned Coulomb phase. The spectrum of this operator acting in a space of charged particles derives from the Coulomb interaction among all the particles of the system.

The complete asymptotic operator is

$$U_{as}(t) = e^{-i\phi(t) R(t)} U(t),$$  (11.35)

where $R(t) = -i \int_0^t V_{as}(\tau) \, d\tau$, $V_{as}(t) = e^{iH_0 t} V_{as}(t) e^{-iH_0 t}$ and, following the non-relativistic generalization, the operator $S$ is expressed as

$$S = \lim_{t \to \infty} U_{as}(t) e^{-iH(t - \infty) t} U_{as}(t) = 0.$$  (11.36)
Thus, the asymptotic dynamics in QED in the infrared region is governed by an evolution operator which contains an infrared phase factor and a factored contribution of the asymptotic electromagnetic field. This leads to definition (II.30) for the S operator, which differs from the Dyson $S$-matrix (II.4) by the replacement

$$e^{-i\int dt' U_{\text{as}}(t')} \rightarrow U_{\text{as}}(t),$$

(II.37)

since the infinite range of the Coulomb potential destroys the behavior of free dynamics in asymptotic regions. According to Faddeev and Kulish, the new S operator maps spaces of coherent states of the electromagnetic field, instead of Fock spaces of initial and final states of free particles into one another. One may also embed the asymptotic dynamics in the S operator by performing an arbitrary splitting of the interaction hamiltonian into $H_1 + H_2$, where $H_1$ carries the contribution of the soft photons [28].

In spite of the conceptual relevance and elegance of the present approach, there remains the long-standing problem of turning the Faddeev-Kulish formalism into an algorithm for the effective computation of infrared-divergence free cross-sections.

### III. General Discussion of the Regularization Ambiguity

Gauge theories in (2+1)-dimensional space-time [1], may exhibit inconsistencies at one loop, due to the choice of the regularization method to evaluate ultraviolet divergent amplitudes such as the photon self-energy in spinor QED. In the latter, if we use analytic [15] or dimensional [18] regularization, the photon is induced a topological mass, in contrast with the result obtained through the Pauli-Villars-Rayski regularization, where the photon remains massless when we remove the regulators. Recently, alternative treatments have been given to the subject and, by using dispersion relations [29], it was shown that the photon indeed dynamically acquires a topological mass.

In order to get a new insight, let us reexamine the ordinary PVR prescription and analyse the conditions that must be imposed on the masses and coupling constants of the auxiliary fields such that a regularized closed fermion loop in 2+1 dimensions is rendered finite. Consider the integral corresponding to a fermion loop containing $n$ vertices with $n$ external photon lines attached, with momenta $k_i$, $(i=1,2,...,n)$. This integral is proportional to

$$\int \frac{d^2p}{(2\pi)^2} \frac{Tr[\gamma_0(m+p)\gamma_0(m+p+k_1)...\gamma_0(m+p+...+k_{n-1})]}{[m^2-(p+k_1)^2+i\epsilon][m^2-(p+k_1+k_2)^2+i\epsilon]...[m^2-(p+...+k_{n-1})^2+i\epsilon]}$$

(III.1)

so, for large $p$, its integrand behaves like $p^{-n}$ whereas for $n < 4$ the integral diverges as

$$\int_0^\infty \frac{p^{2n}}{p^n} \sim \int_0^\infty \frac{dp}{p^{n-2}}.$$  

The integrand $I$ in equation (III.1) behaves like

$$I \sim \sum_n n^k \alpha_{-(n+k)}(\mu),$$

(III.2)

where

$$\alpha_{-(n+k)}(\mu) \sim p^{-n-\epsilon}.$$  

(III.3)

Therefore, in making the substitution

$$I(k) \rightarrow n^I_I(M),$$

where $n_f$ is the number of auxiliary fermion fields, we must impose in the vacuum polarization tensor (III.4) the constraints

$$\sum_{i=1}^{n_f} c_i = 0,$$

(III.4)

$$\sum_{i=1}^{n_f} c_i M_i = 0,$$

(III.5)

in order to get rid of the linear and logarithmic divergences, respectively.

Having settled down the basis for the Pauli-Villars-Redski regularization method, we calculate the vacuum polarization tensor in spinor QED. In the standard notation, the regularized expression for the vacuum polarization tensor reads

$$\Pi_{\mu\nu}(k) = \frac{i\epsilon^2}{(2\pi)^3} \sum_{i=1}^{n_f} \int d^3p \frac{P(M_i)}{(M_i^2-p_\mu^2)(M_i^2-p_\nu^2)},$$

(III.6)

where

$$c_0 = 1, \quad M_0 = m, \quad M_i = m \lambda_i (i = 1,...,n_f)$$

(III.7)

and

$$P(M_i) = Tr[\gamma_m(m+iM_i)\gamma_n(m+iM_i)].$$

(III.8)

We choose both the electron mass and that of the auxiliary field $M_i$ to be positive quantities; the coefficients $\lambda_i$ ultimately go to infinity to recover the original theory. Using the Feynman parametrization

$$\frac{1}{(M_i^2-p_\mu^2)(M_i^2-p_\nu^2)} = \frac{1}{\int_0^1 d\xi \frac{1}{[M_i^2-p_\mu^2-(p_\nu^2-p_\mu^2)\xi]^2}},$$

(III.9)

we obtain

$$\Pi_{\mu\nu}(k) = [g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}] \Pi_{\mu\nu}^M(k^2) + int_{\mu\nu} k^2 \Pi_{\mu\nu}(k^2),$$

(III.10)

where

$$\Pi_{\mu\nu}^M(k^2) \equiv 4i\epsilon^2 k_\mu k_\nu \sum_{i=1}^{n_f} c_i \int_0^1 d\xi \frac{1}{[Q_i^2-P_i^2]^2 \xi^{1-\epsilon}} \frac{1}{(2\pi)^3} \frac{1}{[Q_i^2-P_i^2]^2},$$

(III.11)

$$\Pi_{\mu\nu}(k^2) \equiv -\frac{2ie^2}{m^2} \sum_{i=1}^{n_f} c_i M_i \int_0^1 d\xi \frac{dx}{(2\pi)^3} \frac{1}{[Q_i^2-P_i^2]^2},$$

(III.12)
with

\[ Q_1^2 = M_1^2 - \xi (1 - \xi) k^2 . \]  

(III.13)

The vacuum polarization tensor results to be transversal as expected by gauge invariance.

From the subsidiary conditions (III.4) and (III.5), we realize that the number of regulators must be at least two, otherwise we can't get the coefficients \( \lambda \) cannot become arbitrarily large.

So, let us take

\[ c_1 = \alpha - 1, \quad c_2 = -\alpha, \quad c_3 = 0 ; \quad j > 2, \]  

(III.14)

where the parameter \( \alpha \) can assume any real value except zero and the unity, so that condition (III.4) is satisfied. For \( |\lambda_1|, |\lambda_2| \to \infty \), \( \Pi^{(0)}(0) \to 0 \), whereas

\[ \Pi^{(2)}(0) = \frac{\alpha e^2}{4\pi m} (1 - s) \]  

(III.15)

\[ s \equiv \text{sign}(1 - \alpha^{-1}) . \]  

(III.16)

\[ \alpha \leq 0 < \alpha < 1, \]  

which corresponds to \( s = -1 \), \( \Pi^{(2)}(0) \leq 0 \); in this case the photon acquires a topological mass, proportional to \( \Pi^{(2)}(0) \), coming from proper insertions of the antisymmetric sector of the vacuum polarization tensor in the free photon propagator. If we assume that \( \alpha \) is outside this range, \( s = 1 \), \( c_1 \) and \( c_2 \) have opposite signs and \( \Pi^{(2)}(0) \) vanishes. We then conclude that this arbitrariness in the choice of the parameter \( \alpha \) reflects itself in different values for the photon mass. The new parameter \( s \) may be identified with the winding number of topologically nontrivial gauge transformations and also appears in lattice regularization [30].

Now we must look for the value of \( \alpha \) that leads to the correct photon mass. From equation (III.12) and we see that \( \Pi^{(2)}(k^2) \) is ultraviolet finite. We remind that a closed fermion loop must be regularized as a whole (III.7), (III.8) so to preserve gauge invariance. However, having done this, we have affected the finite antisymmetric sector of the vacuum polarization tensor and, consequently, the photon mass. The same reasoning applies when, using PVR regularization, we calculate the anomalous magnetic moment of the electron; again, if care is not taken, we obtain an incorrect physical result.

In order to solve this problem we should take the value of \( \alpha \) which cancels the contribution coming from the regulator fields. This occurs for \( \alpha = 1/2 \). Hence

\[ \Pi^{(2)}(0) = \frac{e^2}{4\pi m} , \]  

(III.17)

in agreement with the other approaches already mentioned. We should remember that PVR regularization violates parity symmetry in 2+1 dimensions. Nevertheless, for this particular choice of \( \alpha \), this symmetry is restored as the regulator masses get larger and larger. The result quoted above suggests that the ordinary parity-breaking PVR regularization, if carefully implemented, does not introduce any residual contribution to the photon topological mass.

### IV. General Aspects of The Causal Theory: the Phet and Superrenormalizability

Although the physics of the quantized electron-positron field in interaction with a classical electromagnetic field is well understood, some mathematical aspects of the theory have not established themselves as standard textbook material (an exception is [11]), in spite of its fundamental importance. We therefore review briefly one of these aspects, which we need in this section: the scattering operator \( S \) in Fock space for quantum electrodynamics in (2-dimensional) space-time, in an external time-dependent electromagnetic field \( A \).

We start from the one-particle Hamiltonian (II.1) with

\[ V(t) = e(V(t, \tilde{x}) - A, \tilde{x}, \tilde{\tilde{x}}, \tilde{\tilde{\tilde{x}}}) . \]  

(IV.1)

The potentials are assumed to vanish for \( t \rightarrow \pm \infty \) in such a way that the wave operator exist, together with a unitary \( S \)-matrix (II.3). The second-quantized free Dirac field is given in Fock space by

\[ \psi(f) = h(P_+ f) + d(P_- f)^t . \]  

(IV.2)

Here \( P_{\pm} \) are the projection operators on the positive and negative spectral subspaces of the one-particle free Dirac Hamiltonian \( H_0 \), respectively.

The second quantized \( S \)-matrix in Fock space is now defined by

\[ \psi(S f) = S^{-1} \psi(f) S \]  

(IV.3)

\[ \psi(S f)^t = S^{-1} \psi(f)^t , \quad \forall f \in \mathcal{H}_1 \]  

(IV.4)

if it exists, where \( \mathcal{H}_1 \) denotes the one-particle Hilbert space. We have taken the adjoint \( S^t \) the test functions since \( \psi f / \psi f \) is antilinear in \( f \).

In order to solve this problem we should take the value of \( \alpha \) which cancels the contribution coming from the regulator fields. This occurs for \( \alpha = 1/2 \). Hence

\[ \Pi^{(2)}(0) = \frac{e^2}{4\pi m} \]  

in agreement with the other approaches already mentioned. We should remember that PVR regularization violates parity symmetry in 2+1 dimensions. Nevertheless, for this particular choice of \( \alpha \), this symmetry is restored as the regulator masses get larger and larger. The result quoted above suggests that the ordinary parity-breaking PVR regularization, if carefully implemented, does not introduce any residual contribution to the photon topological mass.

\[ S = C e^{\gamma - S_2^1 S_3^1} ; \quad e^{(S_2^1 - S_2^1 S_3^1) \gamma} ; \quad e^{(1 - S_2^1 S_3^1) \gamma} ; \quad e^{S_2^1 S_3^1} . \]  

(IV.5)

where \( \gamma \) denotes Wick ordering.

\[ S_{ij} = P_i S P_j , \quad i, j = +, - \]  

and

\[ |C|^2 = \text{det}(1 - S_2 S_3^1) . \]  

(IV.6)

The first factor in (IV.5) describes electron-positron pair creation, the second one electron scattering, the third one positron scattering and the last one pair annihilation.
In the one-particle theory the condition that a change in the interaction law in any space-time region can influence the evolution of the system only at subsequent times can be translated into the factorization of the S-matrix [31]

\[ S[I] = S_S S_A, \quad S_A \triangleq S[A], \]  

where we have written the electromagnetic potential as

\[ A^{\mu}(x) = A^{\mu}_1(x) + A^{\mu}_2(x), \]

which is the sum of two parts with disjoint supports in time

\[ \text{supp} A_1 \subset [-\infty, t], \quad \text{supp} A_2 \subset [t, +\infty). \]

A similar factorization should hold for the S-operator S in Fock space,

\[ \langle \Omega, \Omega' \rangle = \langle \Omega, S_A S_F \rangle. \]

We call (IV.11) global causality condition for the Fock space S-operator in contrast to the differential condition

\[ \delta \frac{\delta \Omega}{\delta A_\mu(y)} = 0, \quad \text{for } x^0 < y^0. \]  

(IV.12)

We mentioned that the S-matrix in Fock space can be uniquely determined up to a phase,

\[ S = e^{i \Phi} S, \]

(IV.13)

where S is unitary, and given by expression (IV.3). Inserting (IV.13) into (IV.12) we obtain

\[ \frac{\delta}{\delta A_\mu(y)} \left( \Omega, \frac{\delta S}{\delta A_\mu(x)} \Omega \right) = i C^2 \sum_m \text{Tr} \left( S_{-1}^{\mu} \frac{\delta S}{\delta A_\mu(x)} \Omega \right). \]  

(IV.14)

It can be shown from the unitarity of \( S \) that the last term in (IV.14) is purely imaginary. Consequently, the real part of the causality condition (IV.12) is automatically satisfied while for the imaginary part we may choose \( \Phi \) conveniently such that (IV.12) holds.

We now turn to the determination of the causal phase in lowest order of perturbation theory. From (IV.5) we have

\[ \Phi[O] = C \Omega + \sum_m \left( S_{-1}^{\mu} A_\mu \Omega + \ldots \right), \]

(IV.15)

where we have put \( S_{-1}^{\mu} \) equal to the unity in lowest order. Taking the functional derivative of (IV.15) with respect to \( A_\mu(x) \) and keeping only terms of order \( O(A) \) in the resulting expression, we arrive at

\[ \left( \Omega, \frac{\delta S}{\delta A_\mu(x)} \right) = i C^2 \sum_m \text{Tr} \left( S_{-1}^{\mu} \frac{\delta S}{\delta A_\mu(x)} \Omega \right). \]  

(IV.16)

In lowest order we may set \( C^2 = 1 \).

The local causality condition (IV.12) together with expressions (IV.14) and (IV.16) yield

\[ F(x, y) \frac{\delta^2}{\delta \phi \delta \Omega} \left( \frac{\delta^2 \phi}{\delta \delta A_\mu(y) \delta \delta A_\mu(x)} + 2m \frac{\delta}{\delta A_\mu(y)} \text{Tr} \left( (S_{-1}^{\mu})^2 \frac{\delta S}{\delta A_\mu(x)} \right) \right) = \delta \]  

(IV.17)

for \( x^0 < y^0 \).

Next we calculate the second term in (IV.17). In lowest order of perturbation theory, we have

\[ S_{-1}^{\mu} = -(2\pi)^{-1} P_+(p) \gamma^\mu e_A(y + q + q P_-(q)), \]

(IV.18)

Using the following representation (1.2) for the Dirac matrices, we obtain from (IV.18)

\[ \text{Tr} \frac{\delta}{\delta A_\mu(y)} \left( S_{-1}^{\mu} \right) \delta \frac{\delta A_\mu}{\delta A_\mu(x)} = e^{i(2\pi)^{-2}} \int d^2 p d q e^{iy_q + i(x-y)q} \text{Tr} \left( P_-(q) \gamma^\mu e_A(y + q + q P_-(q)) P_+(p) \gamma^\mu e_A(x + p + p P_+(p)) \right) \]

\[ = -i \left( \frac{2}{(2\pi)^2} \right) \int d^3 k e^{i(x-y)k} \right) \]

(IV.19)

\( P^{\mu\nu}(k) \) is related to the tensor of pair creation in (2+1) dimensions.

Since the symmetric and the antisymmetric parts of \( P^{\mu\nu}(k) \) are, respectively, real and imaginary, we can write

\[ F(x, y) = \frac{\delta^2}{\delta \phi \delta \Omega} \left( \frac{\delta^2 \phi}{\delta \delta A_\mu(y) \delta \delta A_\mu(x)} \right) \]

\[ - \frac{1}{(2\pi)^2} \left[ \int_{k > 0} d^3 k \sin k(x - y) P_+^{\mu\nu}(k) - i \int_{k < 0} d^3 k \cos k(x - y) P_-^{\mu\nu}(k) \right]. \]  

(IV.20)

In order to write the last term in (IV.20) as a complex Fourier transform we must continue \( P^{\mu\nu}(k) \) and \( P^{\mu\nu}_+(k) \) antisymmetrically to \( k < 0 \)

\[ F(x, y) = \frac{\delta^2}{\delta \phi \delta \Omega} \left( \frac{\delta^2 \phi}{\delta \delta A_\mu(y) \delta \delta A_\mu(x)} \right) - \frac{1}{(2\pi)^2} \left[ \int d^3 k e^{i(x-y)k} [P^{\mu\nu}(k) - P^{\mu\nu}_+(k)] \right], \]  

(IV.21)

where

\[ d^{\mu\nu}_+(k) = (k^\mu k^\nu - k^2 g^{\mu\nu}) B(k^2), \]

(IV.22)

\[ d^{\mu\nu}_-(k) = i m e^{\mu\nu\nu} k^m B(k^2), \]

(IV.23)

and

\[ B(k^2) = \frac{-e^2}{2(4\pi)^2} \frac{k^2 + 4m^2}{k^2} \Theta(k^2 - 4m^2) \frac{\sinh(k_0)}{\sqrt{k^2}}, \]

(IV.24)

\[ |B(k^2)| = \frac{-e^2}{2(4\pi)^2} \Theta(k^2 - 4m^2) \frac{\sinh(k_0)}{\sqrt{k^2}}. \]  

(IV.25)
According to a theorem by Titchmarsh, the Fourier transform of a causal function vanishing for \( x^0 - y^0 = t < 0 \) satisfies a dispersion relation. Since \( \delta_+^e(k) \) and \( \delta_+^i(k) \) are real and purely imaginary, respectively, they cannot be the Fourier transform of a causal function. The lacking imaginary part of \( \delta_+^e(k) \) and the lacking real part of \( \delta_+^i(k) \) must be supplied by the first term containing the phase \( \phi(A) \).

\[
\frac{\delta^2 \phi}{\delta A_\mu(y) \delta A_\nu(x)} = -\frac{i}{(2\pi)^2} \int d^4k \, e^{-i k(x-y)} [q_+^e(k) - q_+^i(k)] ,
\]

where

\[
q_+^e(k) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dt \, \frac{d_+^e(k(t))}{t(1-t)} \equiv \frac{\alpha}{\pi} \left( k_+^2 - k^2 \right) \left[ \frac{1}{\sqrt{k_+^2}} \left( 1 + \frac{4m^2}{k_+^2} \right) \log \left( \frac{1 - \sqrt{4m^2/k_+^2}}{1 + \sqrt{4m^2/k_+^2}} \right) \right] \operatorname{sgn}(k_0) ,
\]

and

\[
q_+^i(k) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dt \, \frac{-i d_+^i(k(t))}{t(1-t)} \equiv -\frac{\beta}{\pi} m e^{\mu\nu} k_\mu \log \left( \frac{1 - \sqrt{4m^2/k_+^2}}{1 + \sqrt{4m^2/k_+^2}} \right) \operatorname{sgn}(k_0) .
\]

with \( \alpha \equiv -\epsilon^2/[2(1\pi)^2] \) and \( \beta \equiv -\epsilon^2/[2(2\pi)^2] \). In the above dispersion relations, \( \mathcal{P} \) denotes the principal value of the respective integral.

The causal phase is obtained by two integrations

\[
\phi(A) = \frac{1}{2} \int d^4x \int d^4y \frac{\delta^2 \phi}{\delta A_\mu(y) \delta A_\nu(x)} \lambda(y) A_\nu(x) + O(A^n) \equiv \frac{\pi}{2} \int d^4k \left[ \frac{k_+^2 - k^2}{k^2} \right] \Pi_1^{(k)}(k) A_\mu(k) A_\nu^*(k) ,
\]

where

\[
\Pi_1^{(k)}(k) = \alpha \left[ \sqrt{k^2} \left( 1 + \frac{4m^2}{k^2} \right) \log \left( \frac{1 - \sqrt{4m^2/k^2}}{1 + \sqrt{4m^2/k^2}} \right) + 4m \right] \operatorname{sgn}(k_0) ,
\]

and

\[
\Pi_2^{(k)}(k) = -\frac{\beta}{\pi \sqrt{k^2}} \log \left( \frac{1 - \sqrt{4m^2/k^2}}{1 + \sqrt{4m^2/k^2}} \right) \operatorname{sgn}(k_0) .
\]

If we decompose the electromagnetic fields which appear in the integrand of (IV.29) into the respective real and imaginary parts we see that \( \phi(A) \) is indeed real. The S-operator in Fock space \( S[A] \) is then completely determined.

The vacuum-vacuum amplitude is

\[
\langle \Omega, S \Omega \rangle = C e^{i\epsilon} (\Omega, e^{\mathcal{N} - S_0^{(1)} d^4 \Omega}) = C e^{i\epsilon} .
\]

The absolute square

\[
|\langle \Omega, S \Omega \rangle|^2 = C^2 = 1 - P
\]

must be equal to one minus the total probability \( P \) of pair creation,

\[
P = -2\pi \int d^4k \, \Pi_1^{(k)}(k) A_\mu(k) A_\nu^*(k) ,
\]

since the external field can change the vacuum state only into pair states.

The resulting expression for the vacuum-vacuum amplitude reads

\[
\langle \Omega, S \Omega \rangle = \exp \left( i \lambda \int d^4k \left[ \left( \frac{k_+^2 - k^2}{k^2} \right) \Pi_1^{(k)}(k) + i m e^{\mu\nu} k_\mu \Pi_2^{(k)}(k) \right] \right)
\]

where

\[
\Pi_1^{(k)}(k) = \Pi_1^{(1)}(k^2) - i \Pi_1^{(2)}(k^2) ,
\]

\[
\Pi_2^{(k)}(k) = \Pi_2^{(1)}(k^2) - i \Pi_2^{(2)}(k^2) ,
\]

and

\[
\Pi_1^{(1)}(k^2) = \alpha \sqrt{k^2} \left( 1 + \frac{4m^2}{k^2} \right) \Theta(k^2 - 4m^2) ,
\]

\[
\Pi_2^{(1)}(k^2) = \frac{\beta}{\pi \sqrt{k^2}} \left( 1 + \frac{4m^2}{k^2} \right) \Theta(k^2 - 4m^2) .
\]

For \( k^2 < 4m^2 \) the expression between square brackets in (IV.33) coincides up to a multiplicative factor with the complex conjugate of the two-point function that corresponds to the vacuum polarization tensor. The vacuum-vacuum amplitude is ultraviolet finite and exhibits an additional contribution from the antisymmetric part of the vacuum polarization tensor in \( (2+1) \)-dimensional space-time, which emerges from the topological structure of the theory. We should like to emphasize that our construction of the phase above is entirely perturbative.

The program of constructing the S-matrix by means of causality in quantum field theory goes back to Stueckelberg and Bogoliubov (see [32]). The S-matrix is completely determined by causality and translation invariance, supposed the coupling \( \lambda \) is given:

\[
S(g) = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, T(\mathcal{T}_1(x_1) \cdots \mathcal{T}_n(x_n)) g(x_1) \cdots g(x_n) ,
\]

where \( g(x) \in S(\mathbb{R}^4) \) is a C-number test function, assumed to be in the Schwartz space. We consider that the limit \( g \to 1 \) exists for the right physically measurable quantities. However (IV.39) contains ultraviolet divergences for \( n > 1 \), and, consequently, must be renormalized.
In the early 70's, Epstein and Glaser [10] proposed an axiomatic construct where ultraviolet divergences do not appear, leading directly to the renormalized perturbation series. They have shown that in the causal theory no UV problem arises if the causal distributions are correctly split (distribution splitting) [10].

In the causal theory the S-matrix is viewed as an operator-valued distribution and has the following form

\[ S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^3x_1 \ldots d^3x_n T_n(x_1 \ldots x_n) g(x_1) \ldots g(x_n). \] (IV.40)

The n-point operator-valued distributions \( T_n \) are the basic objects of the theory. They can be constructed inductively from \( T_1 \) through a number of physical requirements, the most essential one being causality. Let the operator-valued distributions \( \tilde{T}_n \) be defined by

\[ \tilde{T}_n = \sum_{P_2} \tilde{T}_{n-P_2}(Y, x_n) \tilde{T}_2(Y, x_1) \] (IV.42)

where the sums run over all partitions

\[ P_2 : \{x_1, \ldots, x_n\} = X \cup Y, \quad X \neq \emptyset \]

into disjoint subsets with \( |X| = n_1, |Y| \leq n - 2 \). We also introduce

\[ D_n(x_1 \ldots x_n) = \tilde{T}_n - \tilde{A}_n. \] (IV.44)

If the sums are extended over all partitions \( P_2 \), including the empty set \( X = \emptyset \), we obtain the distributions

\[ A_n(x_1 \ldots x_n) = \tilde{A}_n + T_n(x_1 \ldots x_n), \] (IV.45)

\[ R_n(x_1 \ldots x_n) = \tilde{R}_n + T_n(x_1 \ldots x_n). \] (IV.46)

These distributions are not known by the induction assumption because they contain the unknown \( T_n \). Only the difference

\[ D_n = R_n - \tilde{A}_n = R_n - A_n \] (IV.47)

is known. We can determine \( R_n \) or \( A_n \) separately by investigating the support properties of the various distributions. It turns out that \( R_n \) is a retarded and \( A_n \) an advanced distribution. Hence by causal distribution splitting of (IV.47) one gets \( R_n \) (and \( A_n \)), and \( T_n \) then follows from (IV.45) (or (IV.46)).

In QED, \( D_n \) is of the form

\[ D_n(x_1 \ldots x_n) = \sum_{k} \prod_i \psi^*(x_i) d_k(x_i \ldots x_n) \prod_i \psi(x_i) : \sum_{m} A(x_m) : , \] (IV.48)

where \( \psi (\bar{\psi}) \) are free fermion field operators and \( A \) the free radiation field operators. The double dots denote the usual normal ordering. In the above expression, \( \psi (\bar{\psi}) \) are tempered numerical distributions, which have causal support. They must be split as follows:

\[ d_k = r_n(x) - \bar{a}_n(x), \] (IV.49)

where \( r_n \) and \( \bar{a}_n \) have support in the forward and backward light-cone, respectively. The simplest way of splitting would be

\[ r_n (x) = \chi_n (x) d_k \] (IV.50)

with

\[ \chi_n (x) = \prod_{j=1}^{n} \Theta (c_j^2 - x_j^2). \] (IV.51)

This would lead to the usual UV divergent expression (IV.39). Since \( d_k \) are distributions they cannot, in general, be multiplied by discontinuous functions.

The causal splitting can directly be done in momentum space by means of the following dispersion formula

\[ i \omega = \frac{i}{2 \pi} \int_{-\infty}^{\infty} dt \frac{\hat{a}(tp)}{(t - i \varepsilon)^2 (1 - t + i \varepsilon)} \] (IV.52)

where \( \omega \) is the order of singularity of \( \hat{a} \). This last expression is called the symmetric splitting solution. We refer to [11] for further details on the splitting mechanism. In particular, the theory of quasi-asymptotics [32] has been used in [11] to simplify the analysis of distribution splitting given in [10].

The singular order of a generic graph in QED_2 is given by

\[ \omega = 3 - f - \frac{b}{2} - \frac{1}{2} n, \] (IV.53)

where \( f \) is the number of external fermions (bosons), and \( b \) is the order of perturbation theory. We see that the only singular graphs are the vacuum polarization (\( n = 2, \omega = 1 \)), the electron self-energy (\( n = 2, \omega = 3 \)) and the vacuum polarization in fourth order (\( n = 4, \omega = 0 \)). Therefore, we conclude that QED in (2+1) dimensions is superrenormalizable. See [29] for further details.

V. Results: Dynamical Photon Mass Generation, the Self-energy of the Electron and the Vertex

The first order term \( T_1 \) of QED is given by

\[ T_1 (x) = \frac{i e}{2} \bar{\psi} (x) \gamma^5 \psi (x) : A_n (x) = - \bar{T}_1 (x) \] (V.1)
On going from \( n = 1 \) to \( n = 2 \) the inductive method proceeds by forming

\[
\begin{align*}
A'_2(x_1, x_2) &= \hat{T}_1(x_1) T_1(x_2) = - T_1(x_1) T_1(x_2), \quad \text{(V.2)}
B'_2(x_1, x_2) &= \hat{T}_1(x_2) T_1(x_1) = - T_1(x_2) T_1(x_1), \quad \text{(V.3)}
\end{align*}
\]

and

\[
D_2(x_1, x_2) = B'_2 - A'_2 = T_1(x_1) T_i(x_2) - T_i(x_2) T_1(x_1). \quad \text{(V.4)}
\]

By using Wick’s theorem, the term due to vacuum polarization is obtained by two fermionic contractions

\[
\hat{P}_2(x_1, x_2) = A_2(x_1, x_2) = d_{\mu
u}(x_1, x_2) \cdot A_\mu(x_1) A_\nu(x_2), \quad \text{(V.5)}
\]

where

\[
d_{\mu
u}(x_1, x_2) = \Pi_{\mu\nu}(x_1 - x_2) - \Pi_{\mu\nu}(x_2 - x_1). \quad \text{(V.6)}
\]

with

\[
\Pi_{\mu\nu}(x_1 - x_2) = i^2 \text{Tr} \{ \gamma^\mu S^+(x_1 - x_2) \gamma^\nu S^-(x_2 - x_1) \}. \quad \text{(V.7)}
\]

In momentum space we find [20]

\[
d_\mu(k) = d^\mu(k) + d_{\mu A}(k), \quad \text{(V.3)}
\]

where

\[
\begin{align*}
\hat{d}^\mu(k) &= -i k^\mu k^2 B(k^2), \quad \hat{d}^\mu_{\mu A}(k) = \text{im} \varepsilon^{\mu\nu\rho\sigma} k_\rho \hat{B}(k^2). \quad \text{(V.9)}
\end{align*}
\]

In Eqs. (V.9) and (V.10), \( \hat{B}(k^2) \) and \( \hat{B}(k^2) \) are distributions such that \( d_{\mu A}^\mu(k) \) and \( d_{\mu A}^\mu(k) \) are of order \( \omega = 1 \) and \( \omega = 0 \), respectively.

From (IV.52) and the explicit form of \( \hat{B}(k^2) \) we obtain the antisymmetric splitting solution

\[
r_{\mu A}(k) = \text{im} \varepsilon^{\mu\nu\rho\sigma} k_\rho \Pi^{(2)}_{\nu\sigma}(k^2), \quad \text{(V.11)}
\]

where

\[
\Pi^{(2)}_{\nu\sigma}(k^2) = \frac{m^2}{2(2\pi)\sqrt{k^2}} \log \frac{1 - \sqrt{k^2}^\frac{1}{2}}{1 + \sqrt{k^2}^\frac{1}{2}}. \quad \text{(V.12)}
\]

The vacuum polarization tensor can be written in the form

\[
\Pi_{\mu\nu}(k) = (g_{\mu\nu} - k_\mu k_\nu k^2) \Pi^{(1)}_{\mu\nu}(k^2) + \text{im} \varepsilon_{\mu\nu\rho\sigma} k_\rho \Pi^{(2)}_{\nu\sigma}(k^2). \quad \text{(V.13)}
\]

From (IV.52) and the explicit form of \( \hat{B}(k^2) \) we also obtain the symmetric splitting solution \( s_{\mu A}(k) \). Then, in the limit \( k^2 \to 0 \) it follows that

\[
\Pi^{(1)}_{\mu\nu}(0) = 0, \quad \text{(V.14)}
\]

and, from (V.12),

\[
\Pi^{(2)}_{\mu\nu}(0) = \frac{e^2}{4\pi m}. \quad \text{(V.15)}
\]

The photon propagator modified by the proper vacuum polarization insertions is given

\[
\hat{D}_{\mu\nu}(k) = -\frac{1}{k^2 - \Pi(k^2)} [\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \text{im} \varepsilon_{\mu\nu\rho\sigma} \frac{k_\rho}{k^2} \Pi^{(2)}_{\rho\sigma}(k^2)], \quad \text{(V.16)}
\]

with

\[
\Pi(k^2) = \Pi^{(1)}(k^2) + \frac{m^2 \Pi^{(2)}(k^2)^2}{1 - \Pi^{(1)}(k^2)}. \quad \text{(V.17)}
\]

By (V.16) and (V.17), we see that the modified propagator have a pole displaced from origin and, thus, we conclude that the photon acquires a “dynamically generated” mass order \( e^2 \), in agreement with the usual procedure from (IV.39), using a gauge invariant analytic regularization.

We now discuss in more detail other second order diagram of QED3 with non-negative the electron self-energy. The corresponding c-number distribution is

\[
d(y) = -e^2 \gamma^\mu [S^{(-)}_{\mu}(y) D_{\mu}^{(n)}(-y) + S^{(+)}(y) D_{\mu}^{(n)}(y)] y_\mu, \quad \text{(V.18)}
\]

where

\[
\begin{align*}
d_{\mu}(y) &= S^{(-)}(y) D_{\mu}^{(n)}(-y),
d_{\mu}(y) &= S^{(+)}(y) D_{\mu}^{(n)}(y).
\end{align*}
\]

The Fourier transform of \( d_{\mu}(y) \) is

\[
\hat{d}_{\mu}(p) = -i(2\pi)^3 \int d^3 q \delta(p - q) \delta(q^0 - m) D_{\mu}^{(n)}(q), \quad \text{(V.19)}
\]

where

\[
\begin{align*}
l_1 &= \int d^3 q \theta(q^0 - p^0) \delta[l(p - q) m - q^0 - m^2],
l_2 &= \int d^3 q \theta(q^0 - p^0) \delta[l(p - q) m - q^0 - m^2].
\end{align*}
\]

Taking time-like \( p \) in the form \( p = (p_0, 0) \) yields

\[
l_1 = m \int d^3 q \theta(q^0 - p^0) \delta(-q^0) \delta(p_0^2 - 2q_0p_0 + m^2) \delta(q^2 - m^2) = m \int \frac{d^3 q}{(2\pi)^3} \delta(p_0^2 + 2q_0E_q + m^2) \theta(-E_q - p_0). \quad \text{(V.20)}
\]

It follows from (V.23) that \( p_0 < 0 \) and

\[
E_q = \frac{m^2 + p_0^2}{-2p_0} = \sqrt{m^2 + q^2}, \quad \text{(V.21)}
\]

...
so that
\[ |q| = \frac{n^2 - p_0^2}{2p_0} \leq m^2 - p_0^2 < 0. \]

Therefore,
\[ I_1 = \frac{2\pi}{2} \Im(p_0^2 - m^2) \theta(-p_0) \int_0^\infty dq |q| \left( \frac{1}{E_2} \right) \delta \left( E_2 - \frac{p_0^2 + m^2}{2|p_0|} \right) \]
\[ = \pi \Im(p_0^2 - m^2) \theta(-p_0) \int_0^\infty dE \frac{E}{E_2} \delta \left( E - \frac{p_0^2 + m^2}{2|p_0|} \right) \]
\[ = \frac{\pi}{2} \theta(p_0^2 - m^2) \theta(-p_0) \frac{m}{\sqrt{p_0^2}}. \]

We may write the integral (V.22) in the form $I_2 = I_2^* \gamma_w$. For $\nu \neq 0$, $I_2^*$ vanishes by symmetry.

For $I_2$, we have
\[ I_2^* = \int d^3q \delta(q^0 - p^0) \delta(p_0^2 - 2p_0q_0 + m^2) \frac{\delta(q^0 + E_q)}{2E_q} \]
\[ = \frac{\pi}{2} \theta(p_0^2 - m^2) \theta(-p_0) \frac{p_0^2 + m^2}{2p_0} \frac{p_0^2}{\sqrt{p_0^2}}, \]
\[ (V.27) \]

in the same way above for $I_1$. For generic $\nu$, we must replace $p_0^2$ by $p^2$ in $I_1$ and $I_2$ and the linear term $p_0^2 \gamma_w$ in $I_2^*$, with $p_0^2$ given by (V.27), by $\vec{p}$. We then obtain
\[ I_2^* = \int d^3q \delta(q^0 - p^0) \delta(p_0^2 - m^2) \theta(-p_0) \frac{1}{\sqrt{p_0^2}} \left[ m + \frac{\vec{p}}{2} \left( 1 + \frac{m^2}{p_0^2} \right) \right], \]
\[ (V.28) \]

which coincides with the distribution $\hat{r}(p)$. The distribution $\hat{d}_+(p)$, with support in the advanced light-cone, is obtained in an analogous manner from (V.19), or simply substituting $\theta(p_0)$ for $-\theta(-p_0)$ in (V.28).

It follows from (V.18), (V.28) that
\[ \hat{d}(p) = -e^i \gamma^a \hat{d}_-(p) + e^i \gamma^a \hat{d}_+(p) \gamma_a \]
\[ = e^i \gamma^a \frac{3\pi}{2} \theta(p^2 - m^2) \frac{\text{sgn} p_0}{\sqrt{|p|^2}} \left[ 3m - \frac{\vec{p}}{2} \left( 1 + \frac{m^2}{p^2} \right) \right]. \]
\[ (V.29) \]

From (V.29) we conclude that the singular order of $\hat{d}$ is $\omega = 0$. Therefore the symmetric solution of the splitting problem is
\[ \hat{f}(p) = e^i \gamma^a \frac{3\pi}{2} \frac{i}{2\pi} \int_0^\infty \frac{dt}{(t - i0)(1 - t + i0)} \theta(t^2p^2 - m^2) \]
\[ \times \frac{\text{sgn} p_0}{t|t|\sqrt{t^2}} \left[ 3m - \frac{\vec{p}}{2} \left( 1 + \frac{m^2}{t^2p^2} \right) \right]. \]
\[ (V.30) \]

Thus,
\[ \hat{r}(p) = e^i \gamma^a \frac{3\pi}{2} \frac{i}{2\pi} \int_0^\infty \frac{dt}{(t - i0)(1 - t + i0)} \theta(t^2p^2 - m^2) \]
\[ \times \frac{\text{sgn} p_0}{t|t|\sqrt{t^2}} \left[ 3m - \frac{\vec{p}}{2} \left( 1 + \frac{m^2}{t^2p^2} \right) \right]. \]
\[ (V.31) \]

which, from the tensor structure of $\hat{r}$, is the general solution.

The electron self-energy is defined as
\[ \Sigma(p) = -i(2\pi)^3 \hat{r}(p) - \hat{r}(p) \]
\[ = e^i \gamma^a \frac{3\pi}{2} \frac{i}{2\pi} \int_0^\infty \frac{dt}{(t - i0)(1 - t + i0)} \theta(t^2p^2 - m^2) \]
\[ \times \frac{\text{sgn} p_0}{t|t|\sqrt{t^2}} \left[ 3m - \frac{\vec{p}}{2} \left( 1 + \frac{m^2}{t^2p^2} \right) \right]. \]
\[ (V.32) \]

when $p$ is in the advanced light-cone. The logarithmic on shell singularity is more severe than in four dimensions. Anyway, $\Sigma(p)$ is given by remains well defined even in three dimensions, although in this case the adiabatic limit $g \to 1$ is more delicate.

The proper two-point insertions lead to the corrected second-order propagators. There remains another basic insertion in the theory, known as the vertex function, which, as the electron self-energy, also exhibits a singular behaviour in the infrared region. We construct the three-point function that corresponds to the vertex function from the retarded and advanced distributions
\[ R'_3(x_1, x_2, x_3) = \sum \mathcal{T}(Y, x_3) \mathcal{T}(X) \]
\[ = R_{31} \frac{T_3(x_1, x_2) T_1(x_3)}{R_{31}} + R_{32} \frac{T_3(x_2, x_3) T_1(x_1)}{R_{32}} + R_{33} \frac{T_3(x_1, x_3) T_1(x_2)}{R_{33}} \]
\[ (V.33) \]

and
\[ A'_3(x_1, x_2, x_3) = \sum \mathcal{T}(Y, x_3) \mathcal{T}(X) \]
\[ = A_{31} \frac{T_3(x_1, x_2) T_1(x_3)}{A_{31}} + A_{32} \frac{T_3(x_2, x_3) T_1(x_1)}{A_{32}} + A_{33} \frac{T_3(x_1, x_3) T_1(x_2)}{A_{33}} \]
\[ (V.34) \]

Each term in (V.33) and (V.34) corresponds to a certain decomposition of the vertex diagram in a vertex part ($T_1 = -T_1$) and a second-order diagram ($T_2, T_2$). The last may be a two-point function for Möller scattering,
\[ T^{(2)}_2(x_1, x_2) = -ie^2 \bar{\psi}(x_1) \gamma^a \psi(x_1) \bar{\psi}(x_2) \gamma_a \psi(x_2) \cdot D^a(p_1 - x_2). \]
\[ (V.35) \]
or that for Compton scattering,

\[ T_2(x_1, x_3) = -i e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = A_3(x_1) A_3(x_3) \]  

(V.36)

The inverse two-point functions are given by

\[ \tilde{T}_2(x_1, x_2) = -T_2(x_1, x_2) + T_2(x_1) T_1(x_2) + T_2(x_2) T_2(x_1) \]  

(V.37)

For the first term in (V.33) we get

\[ R';_{11} \equiv -T_2(x_1, x_3) H_{11}, \]

\[ = -i e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = A_3(x_1) A_3(x_3) \]  

(V.38)

such that to \( x_1 \) is associated an outgoing electron, to \( x_3 \) an incoming electron and to \( x_3 \) an external photon. Analogously,

\[ R';_{12} \equiv -T_2(x_2, x_3) H_{12}, \]

\[ = -i e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = A_3(x_1) A_3(x_3) \]  

(V.39)

The last term in (V.33) can be written as

\[ R';_{31} \equiv T_1(x_3) \tilde{T}_2^{(1)}(x_1, x_3), \]

\[ = -T_3(x_1) T_2^{(1)}(x_1, x_2) + T_1(x_1) T_1(x_3) T_2(x_2) + T_3(x_1) T_1(x_2) T_1(x_2), \]  

(R.40)

where

\[ R';_1 = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = A_3(x_1) A_3(x_3) \]  

(V.41)

\[ R';_2 = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = A_3(x_1) A_3(x_3) \]  

(V.42)

\[ R';_3 = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = A_3(x_1) A_3(x_3) \]  

(V.43)

The advanced distribution is obtained in a similar way,

\[ A';_1 = -T_1(x_2) \tilde{T}_2^{(2)}(x_1, x_3), \]

\[ = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = A_3(x_1) A_3(x_3) \]  

(V.44)

\[ A';_2 = -T_2(x_1) \tilde{T}_2^{(2)}(x_1, x_3), \]

\[ = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = A_3(x_1) A_3(x_3) \]  

(V.45)

\[ A';_3 = T_3(x_1, x_2) \tilde{T}_2^{(2)}(x_1, x_3), \]

\[ = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = A_3(x_1) A_3(x_3) \]  

(V.46)

The last term in (V.34) split into three pieces

\[ A';_{15} = T_5(x_1, x_2) T_1(x_3), \]

\[ = -T_5^{(1)}(x_1, x_2) T_1(x_3) + T_5(x_1) T_1(x_2) T_1(x_3) + T_5(x_2) T_1(x_2) T_1(x_3) \]  

(V.37)

with the following results:

\[ A';_1 = -e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = A_3(x_1) A_3(x_3) \]  

(V.47)

\[ A';_2 = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = A_3(x_1) A_3(x_3) \]  

(V.48)

\[ A';_3 = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = e^2 : \bar{\psi}(x_1) \gamma^\mu S^\mu(x_1 - x_3) \gamma^\nu \psi(x_3) = A_3(x_1) A_3(x_3) \]  

(V.49)

where all the 10 terms which contribute for \( D' \) have the same matrix structure. The following two pairs of terms combine: \( R';_{11} \) with \( R';_{21} \), \( R';_{22} \) with \( R';_{31} \) with \( A';_2 \) and \( -A';_2 \) with \( -A';_3 \). As a consequence the Feynman fermion propagators are converted into retarded and advanced propagators.

In momentum space we have

\[ \tilde{D}'(p, q) = (2\pi)^{-3} \int dy_1 dy_2 D'(x_1, x_2) e^{i y_1 y_2} = \tilde{R}' - \tilde{A}' \]

(V.50)

where

\[ y_1 = x_1 - x_3, \quad y_2 = x_2 - x_3. \]

The resulting retarded and advanced distribution resultants are given, respectively, by

\[ \tilde{R}'(p, q) = (2\pi)^{-3} \int d^3 k \gamma^\mu S^\mu(p - k) \gamma^\nu \psi(x_3) = \tilde{R}'(p) \]

(V.51)
\[
\begin{align*}
\hat{A}(p, q) &= -(2\pi)^{-3/2} \int d^3k \left[ -\gamma^\nu \hat{S}^\nu(p+k) \gamma^\alpha \hat{S}^\alpha(q-k) \gamma_\nu \hat{D}_0(k) \\
&\quad + \gamma^\nu \hat{S}^\nu(p-k) \gamma^\alpha \hat{S}^\alpha(q+k) \gamma_\nu \hat{D}_0(k) \right] \, , \\
\end{align*}
\]

(V.52)

After simplifying the matrix part, \( O^a \) can be written as

\[
D^a(p, q) = -3(\gamma^\nu \hat{D} + 3\gamma^\nu \gamma^\alpha \hat{S}_a + \gamma^\nu \hat{S}_a \gamma^\alpha \hat{D}) - 6\gamma^\nu \gamma^\alpha \hat{S}_a - 6i\gamma^\nu \gamma^\alpha \hat{D} \gamma^\rho \gamma^\alpha \gamma^\nu \hat{S}_a 
\]

when \( I, I^a \) and \( I^\nu \) are the following scalar, vector and tensor integrals

\[
I(p, q) \equiv \frac{1}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} [1, k^\nu, \hat{S}^\nu] \, , 
\]

(V.53)

and

\[
\begin{align*}
D^a(k) &= \theta(\pm k_0) \delta(k^2 - m^2) \, , \\
D^\nu(k) &= \frac{1}{k^2 - m^2 + i0} \, , \\
D^{\nu\mu}(k) &= \frac{1}{k^2 - m^2 - i0} \, , 
\end{align*}
\]

(V.54)

The quantities with index 0 correspond to \( m = 0 \).

We calculate the scalar integral

\[
I(p, q) = \frac{1}{(2\pi)^3} \sum_{j=1}^n \int d^3k \, , 
\]

(V.55)

with

\[
I_0 \equiv \int d^3k \theta(p^0 - k^0) \delta((k-p)^2 - m^2) \frac{1}{(k-q)^2 - m^2 + i0(k^0 - q^0)} \theta(k^0 - q^0) \delta(k^2 - m^2) 
\]

(V.56)

\[
I_1 \equiv \int d^3k (k-p)^2 - m^2 + i0(k^0 - p^0) \theta(k^0 - q^0) \delta((k-p)^2 - m^2) \frac{1}{(k-q)^2 - m^2 + i0(k^0 - q^0)} \theta(k^0 - q^0) \delta(k^2 - m^2) 
\]

(V.57)

The product of the first \( \theta \) and \( \delta \) functions in the integrand of (V.57) gives \( \delta(k_0 - E_k)/(2E_k) \), which implies

\[
k_0 = E_k = \sqrt{k^2 + m^2} \, , \quad m < k_0 < -m \, .
\]

(V.58)

Analogously, in (V.88) we have

\[
k_0 = -E_k = -\sqrt{k^2 + m^2} \, , \quad m < k_0 > m \, .
\]

(V.59)

\[
P_0^3 = -2E_k P_0 \, , \quad E_k = \frac{P_0}{2} = -\frac{1}{2}(p_0 - q_0) \, , \quad k_0 = -\frac{1}{2} P_0 \, ,
\]

(V.60)

For space-like \( P \) there exists a Lorentz frame with \( P_0 = 0 \), such that \( I_0 \) vanishes. Consequently \( P \) must be time-like. Then, there is a frame with \( P = p - q \) (center-of-mass system). The fast function \( \delta \) in (V.67) implies

\[
P_0^3 = -2E_k P_0 \, , \quad E_k = \frac{P_0}{2} = \frac{1}{2}(p_0 - q_0) \, , \quad k_0 = \frac{1}{2} P_0 \, .
\]

(V.61)

Analogously, in (V.88) we have

\[
k_0 = -E_k = -\sqrt{k^2 + m^2} \, , \quad m < k_0 > m \, .
\]

(V.62)

\[
P_0^3 = 2E_k P_0 \, , \quad E_k = \frac{P_0}{2} = \frac{1}{2}(p_0 - q_0) \, , \quad k_0 = \frac{1}{2} P_0 \, .
\]

(V.63)
Thus,

\[ I_1 = \int \frac{d^4k}{2E_k (E_k^2 - (k+p)^2 + i0)} \frac{\delta(E_k + P_0/2)}{2|P_0|} \]

for \(-P_0 - E_k > 0\).

(V.74)

\[ I_2 = -\int \frac{d^4k}{2E_k (E_k^2 - (k+p)^2 + i0)} \frac{\delta(E_k - P_0/2)}{2|P_0|} \]

for \(-P_0 - E_k > 0\).

(V.75)

If we use \(E = E_k\) as integration variable,

\[ \int \frac{d^4k}{E} = \int_0^{2\pi} d\varphi \int_0^\infty |k| d|k| = \int_0^{2\pi} E dE, \]

the integration in \(E\) can be easily performed due to the \(\delta\) function in (V.74) and (V.75). So, we obtain

\[ I = -\text{sgn} P_0 \frac{\theta(P_0^2 - 4m^2)}{4|P_0|} \int_0^{2\pi} d\varphi \frac{1}{\rho \omega - p^2 + m^2 - \sqrt{(\rho^2 - 4m^2)|p|\cos \varphi + 10}}. \]

(V.76)

We can rewrite the above integral in a covariant form if we notice that

\[ p\rho_0 - p^2 = p\rho_0 - p \cdot q = p \cdot q \]

and

\[ |p_0| = \sqrt{(p_0 - \rho_0)^2 p_0^2} = \sqrt{\rho_0^2 q^2 + q_0^2 p^2 - 2p\rho_0 p \cdot q} = \sqrt{(p\rho_0 - p \cdot q)^2 - (p_0^2 - p^2)(q_0^2 - q^2)} = \sqrt{(p \cdot q)^2 - p^2 q^2}. \]

In an arbitrary frame,

\[ I = -\text{sgn} P_0 \frac{\theta(p^2 - 4m^2)}{4\sqrt{P_0^2}} \int_0^{2\pi} d\varphi \frac{1}{a_3 + b_3 \cos \varphi + i0}. \]

(V.77)

\[ = -\frac{\pi \text{sgn} P_0}{2 \sqrt{P_0^2}} \theta(p^2 - 4m^2) \frac{1}{\sqrt{a_3^2 - b_3^2 + ia_3 0}}, \]

where

\[ a_3 \equiv pq + m^2, \quad b_3 \equiv -\sqrt{N} \sqrt{1 - 4m^2/p^2} \]

(V.78)

and

\[ N \equiv (p \cdot q)^2 - p^2 q^2. \]

(V.79)

Following the same procedure that led to expression (V.77) for \(I\), we can perform the calculation of the integrals

\[ J(p, q) \equiv I_1 + I_2 \]

(V.8)

and

\[ K(p, q) \equiv I_2 - I_1 = -J(q, p) \]

(V.8)

The evaluation of \(J\) can be done in a frame where \(p = 0\), since in this case \(p\) must be time-like in order that the integral not to vanish. Since

\[ |p_0| |q_0| = \sqrt{\rho_0^2 q^2} = \sqrt{\rho_0^2 (q_0^2 - q^2)} = \sqrt{(p\rho_0)^2 - p^2 q^2} = \sqrt{N}, \]

in an arbitrary frame,

\[ J(p, q) = \frac{\text{sgn} P_0}{4\sqrt{p^2}} \theta(p^2 - 4m^2) \int_0^{2\pi} d\varphi \frac{1}{a_1 + b_1 \cos \varphi - i(p^2 - m^2 - 2pq)/0} \]

\[ = \frac{\pi \text{sgn} P_0}{2 \sqrt{p^2}} \theta(p^2 - m^2) \left[ \frac{\theta(p^2 - m^2 - 2pq)}{\sqrt{a_1^2 - b_1^2 - ia_10}} - \frac{\theta(-p^2 + m^2 + 2pq)}{\sqrt{a_1^2 - b_1^2 + ia_10}} \right], \]

(V.8)

with

\[ a_1 \equiv q^2 - m^2 - pq \left(1 - \frac{m^2}{p^2}\right), \quad b_1 \equiv -\sqrt{N} \left(1 - \frac{m^2}{p^2}\right). \]

(V.8)

Using (V.81) and defining

\[ a_2 \equiv p^2 - m^2 - pq \left(1 - \frac{m^2}{q^2}\right), \quad b_2 \equiv -\sqrt{N} \left(1 - \frac{m^2}{q^2}\right), \]

(V.8)

we find an expression for \(K\) and, consequently, the scalar integral can be written as

\[ I(p, q) = \frac{\pi}{2(2\pi)^{1/2}} \left\{ \text{sgn} P_0 \theta(p^2 - 4m^2) \frac{1}{\sqrt{a_3^2 - b_3^2 + ia_3 0}} \right\} \]

\[ \text{sgn} p_0 \theta(p^2 - m^2) \left[ \frac{\theta(p^2 - m^2 - 2pq)}{\sqrt{a_1^2 - b_1^2 - ia_10}} - \frac{\theta(-p^2 + m^2 + 2pq)}{\sqrt{a_1^2 - b_1^2 + ia_10}} \right], \]

(V.8)

\[ \text{sgn} q_0 \theta(q^2 - m^2) \left[ \frac{\theta(q^2 - m^2 - 2pq)}{\sqrt{a_2^2 - b_2^2 - ia_20}} - \frac{\theta(-q^2 + m^2 + 2pq)}{\sqrt{a_2^2 - b_2^2 + ia_20}} \right]. \]

(V.8)

The algebraic calculation of the vector and tensor integrals is lengthy. Each one can be decomposed in terms proportional, respectively, to \(\text{sgn} P_0, \text{sgn} p_0,\) and \(\text{sgn} q_0\), which in turn are constructed from tensor products of the external momenta \(p\) and \(q\).

The vector integral is given by

\[ I^v(p, q) = \frac{I(p, q)}{2N} \{pq(q^2 - m^2) - q^2(p^2 - m^2)\}

\[ + q^2 \{pq(p^2 - m^2) - p^2(q^2 - m^2)\}. \]
Finally, the tensor integral is given by

\[
I^{\mu
u}(p,q) = \frac{I(p,q)}{N} \left\{ -q^\nu q^\nu \left\{ 1 - \frac{1}{N} (q^2 - m^2) + (pq)(\mu^2 - m^2) \right\}^2 
+ \frac{4pq}{N} \theta(p^2 - m^2) \left\{ p^\mu p^\nu \left\{ 3pq - 2q^2 + m^2 
- \frac{2(pq + m^2)(pq - q^2)}{N} \right\} + q^\mu q^\nu \left\{ 3pq - 2p^2 + m^2 
- \frac{2(pq + m^2)(pq - q^2)}{N} \right\} + (p^\nu q^\nu + q^\nu p^\nu) \left\{ pq - p^2 - q^2 - m^2 
- \frac{2(pq + m^2)(pq - q^2)}{N} \right\} - g^\mu p^\nu \left\{ pq + m^2 \right\} \right\}
\]

\[
+ \frac{\pi}{4(2\pi)^{\frac{3}{2}}} \theta(p^2 - 4m^2) \left\{ p^\mu p^\nu \left\{ 3pq - 2q^2 + m^2 
- \frac{2(pq + m^2)(pq - q^2)}{N} \right\} + q^\mu q^\nu \left\{ 3pq - 2p^2 + m^2 
- \frac{2(pq + m^2)(pq - q^2)}{N} \right\} + (p^\nu q^\nu + q^\nu p^\nu) \left\{ pq - p^2 - q^2 - m^2 
- \frac{2(pq + m^2)(pq - q^2)}{N} \right\} - g^\mu p^\nu \left\{ pq + m^2 \right\} \right\}
\]

\[
- (q^2 - m^2) \left\{ |pq|^2 + p^2 q^2 \right\} + (pq + q^2 p^2) \frac{1}{2N} \left\{ 2m(q^2 - m^2) \right\}
- (1 - \frac{m^2}{p^2}) \left\{ |pq|^2 + p^2 q^2 \right\} + q^\mu \frac{p^2}{2} \left\{ pq \left( 1 - \frac{m^2}{p^2} \right) - (q^2 - m^2) \right\}
\]

\[
- \frac{\pi}{4(2\pi)^{\frac{3}{2}}} \theta(p^2 - 4m^2) \left\{ p^\mu p^\nu \left\{ 3pq - 2q^2 + m^2 
- \frac{2(pq + m^2)(pq - q^2)}{N} \right\} + q^\mu q^\nu \left\{ 3pq - 2p^2 + m^2 
- \frac{2(pq + m^2)(pq - q^2)}{N} \right\} + (p^\nu q^\nu + q^\nu p^\nu) \left\{ pq - p^2 - q^2 - m^2 
- \frac{2(pq + m^2)(pq - q^2)}{N} \right\} - g^\mu p^\nu \left\{ pq + m^2 \right\} \right\}
\}

(\text{V.86})
\]

The imaginary parts of the functions

\[
\Sigma^{(1)}(p,q) = \frac{\theta(p^2 - m^2 - 2pq)}{\sqrt{a_1^2 - 4a_2^2}} (p^2 + m^2 + 2pq)
\]

\[
\Sigma^{(2)}(p,q) = \frac{\theta(q^2 - m^2 - 2pq)}{\sqrt{a_3^2 - 4a_4^2}} (q^2 + m^2 + 2pq)
\]

\[
\Sigma^{(3)}(p,q) = \frac{1}{\sqrt{a_5 - 4a_6}}
\]

(V.87)

are given by

\[
3m \Sigma^{(1)} = \frac{1}{\sqrt{b_1^2 - a_2^2}} \theta(q^2 - m^2) \theta(2pq - p^2 - q^2 - m^2) + \frac{q^2 - m^2}{p^2 - m^2}
\]

(V.91)

\[
3m \Sigma^{(2)} = \frac{1}{\sqrt{b_2^2 - a_1^2}} \theta(p^2 - m^2) \theta(2pq - p^2 - q^2 - m^2) + \frac{q^2 - m^2}{p^2 - m^2}
\]

(V.92)

and, therefore, the distributions for the scalar, vector and tensor integrals are real. For the scalar integral we have

\[
3m I_{sc}(p,q) = 3m I_{sc}(p,q)
\]

= \frac{\pi}{4(2\pi)^{\frac{3}{2}}} \theta(p^2 - m^2) \theta(2pq - p^2 - q^2 - m^2)
\]

(V.93)

\[
\times \theta(p^2 - m^2) \theta(q^2 - m^2) \theta(2pq - p^2 - q^2 - m^2) + \frac{q^2 - m^2}{p^2 - m^2}
\]

(V.94)

If \( p \) and \( q \) are the 3-momenta of external electrons, the distribution \( D^\nu(p,q) \) in (V.34) can be simplified by anticommuting \( \hat{p} \) to the left and \( \hat{q} \) to the right, taking into account that it appears between the Dirac spinors \( \psi(p) \) and \( \psi(q) \). Thus, we can write

\[
D^\nu(p,q) = -2m(p^\nu + q^\nu) I + 6pq(p^\nu + q^\nu) \gamma^\nu \eta(p,q)
\]

(V.95)

The scalar integral \( I \) diverges on shell, where only the first divergent term survives for \( \text{sgn} p \neq \text{sgn} q \), while the vector and tensor integrals remain finite in this limit. The first and third terms
in (V.95) have order \( \omega = -2 \) and the remaining have \( \omega = -1 \), between the spinors \( \overline{u}(p) \) and \( u(q) \). However, the last two terms in this expression cancel among themselves.

The retarded distributions retardadas which correspond to each remaining term are obtained through a non-subtracted dispersion relation, setting \( \omega = -1 \) in (IV.52). The splitting of distribution \( D^\mu \) must be done before taking the limit \( p^2 \to m^2, q^2 \to m^2 \), in order to circumvent the infrared divergences just mentioned. With those hypotheses, only four terms contribute to the retarded distribution \( \hat{R}^c \) associated to the vertex function:

\[
\hat{R}^c(p,q) = \frac{i}{2(2\pi)^{3/2}} \frac{\text{sgn} P_0}{\sqrt{p^2}} \sum_{i=1}^{4} \hat{R}^{c(i)}(p,q),
\]

where

\[
\hat{R}^{c(1)}(p,q) = \frac{1}{m} \int_{-\infty}^{\infty} dt \frac{\theta(t^2 P^2 - 4m^2)}{1 + t + i0}.
\]

\[
\hat{R}^{c(2)}(p,q) = -\frac{3}{m^2} \gamma \int_{-\infty}^{\infty} dt \frac{\theta(t^2 P^2 - 4m^2)}{1 + t + i0}.
\]

\[
\hat{R}^{c(3)}(p,q) = -3 g \frac{3m(P^2 + q^2)}{m^2} \int_{-\infty}^{\infty} dt (1 - \text{sgn}(t^2 - 1)) \frac{\theta(t^2 P^2 - 4m^2)}{t^2(1 + t + i0)}.
\]

\[
\hat{R}^{c(4)}(p,q) = -3 g \frac{3m(P^2 + q^2)}{m^2} \int_{-\infty}^{\infty} dt (1 - \text{sgn}(t^2 - 1)) \frac{\theta(t^2 P^2 - 4m^2)}{t^2(1 + t + i0)}.
\]

The integrals (V.99) and (V.100) vanish for \( p^2 \to 4m^2 \). The same is true for the distribution \( \hat{R}^c \), proportional to \( \theta(P^2 - 4m^2) \). In this way the Fourier transform of the three-point function, or the second-order current operator, strictly speaking, is given by

\[
\Lambda^c(p,q) = -i(2\pi)^{3/2} \left[ \hat{R}^c(p,q) - \hat{R}^c(p,q) \right]
\]

\[
= \frac{1}{8\pi} \frac{\text{sgn} P_0}{\sqrt{p^2}} \left[ \log \left( \frac{1 - \sqrt{P^2 + q^2}}{1 + \sqrt{P^2 + q^2}} - \frac{8m^2 P^2}{P^2 - 4m^2} \right) \right]
\]

\[
- \frac{1}{8\pi} \frac{2g}{\sqrt{p^2}} \left[ \log \left( \frac{1 - \sqrt{P^2 + q^2}}{1 + \sqrt{P^2 + q^2}} + \frac{8m^2 P^2}{P^2 - 4m^2} \right) \right].
\]

For \( q \to 0 \) it follows that

\[
\Lambda^c(p,p) = -\frac{3}{4\pi m} \gamma.
\]

Therefore, in a two-dimensional space the vertex correction introduces only an electric form factor \(-3/(4\pi m)\).

### VI. Conclusion and Open Problems

In section II we have seen that the asymptotic behavior of QED3 is singular in the infrared region where an infrared phase factor modifies the evolution operator. This singular phase also appears in the vacuum-vacuum amplitude constructed in section IV, through the causal phase in the S operator which describes the scattering of electrons by an external Coulomb potential

\[
A_s(k) = (q/|k|^2, 0, 0).
\]

For small \( k^2 \) only the coefficient \( A_s(k) \) of the symmetric part of the vacuum polarization tensor contributes:

\[
(\Omega, \Omega') = \exp \left\{ -i\pi \frac{3}{2} \beta \int d^2k \frac{k^2}{2} A_s(k) \right\}
\]

\[
\sim \exp \left\{ i \frac{3}{2} \beta \int d^2k \int_0^\infty \frac{dw}{\omega} \right\}.
\]

The last expression exhibits a collinear logarithmic divergence in the case of forward scattering. Thus, the asymptotic particle states in QED3 are ill-defined in the usual S-matrix formalism.

It is an open problem whether a construction such as the causal one in reference [33], which was used to arrive at a regularized phase in second-order perturbation theory, yields an infrared divergence free phase.

In section III we have shown that, when Pauli-Villars-Raylski regularization is carefully implemented in the standard approach to quantum field theory, the photon mass is not zero but depends on a parameter \( s \), which may be identified with the winding number of homotopical nontrivial gauge transformations. The only way to avoid dependence of the mass on regularization, and thus find a uniquely defined value for the mass, is to make a special choice of parameter \( s \) which is related to restoration of parity as commented in the text.

Also in the Causal theory an indeterminacy arises: there, the assumption of minimal splitting in the causal theory has a stronger physical meaning than restoration of parity in the standard approach: it likewise fixes the value of the magnetic moment of the electron in QED4.

In section IV we have shown that Pauli-Villars-Raylski regularization is carefully implemented in the standard approach to quantum field theory, the photon mass is not zero but depends on a parameter \( s \), which may be identified with the winding number of homotopical nontrivial gauge transformations. The only way to avoid dependence of the mass on regularization, and thus find a uniquely defined value for the mass, is to make a special choice of parameter \( s \) which is related to restoration of parity as commented in the text.
References