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Causal Approach to Quantum Electrodynamics in Three-dimensional Space-time

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Abstract

We study the dinamically generated photon mass, the electron self-energy and the vertex in the causal theory of quantum electrodynamics in three-dimensional space-time (QED_3). We also provide a general discussion of the regularization and gauge (in)dependence of the photon mass. A last section is devoted to challenging open problems.

I. Introduction and Summary

II. The Infrared Problem

III. General Discussion of the Regularization Ambiguity

IV. General Aspects of the Causal Theory: the Phase and Superrenormalizability

V. Results: Dynamical Photon Mass Generation, the Self-energy of the Electron and the Vertex

VI. Conclusion and Open Problems

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I. Introduction and Summary

Quantum electrodynamics in (2+1) space-time dimensions (QED_3) [1] has attracted m interest in recent times. It is a successful model of both the integer and fractional quantum H effects [2]. Some aspects of the model are also of some interest for high-T_c superconductiv insofar as electrons seem to experience an attractive potential, with formation of bound sta ([3],[4]). The greatest interest lies, however, in serving as a laboratory for one of the m important conceptual problems of quantum field theory: the connection between dynami mass generation, gauge invariance and regularizations. The classical Lagrangian density o system of free electrons of mass m and photons of mass μ is

$$\mathcal{L} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4}\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha - \frac{1}{2a}(\partial_\mu A^\mu)^2 . \qquad (I$$

Above, ψ denotes the electron field, $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, A_μ the phot field with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the electromagnetic field tensor, $\not{A}_{ab} = A_\mu\gamma^\mu_{ab}$, $\overline{\psi}_a = \psi^\dagger_b\gamma^0_{ba}$, a $\frac{1}{2a}(\partial_\mu A^\mu)^2$ is a gauge fixing term ($a = 0$ is the Landau gauge and $a = 1$, the Feynman gaug The spinor and γ-matrix indices run from zero to three, with

$$\gamma^0 = \sigma^3 \ , \ \gamma^1 = i\sigma^1 \ , \ \gamma^2 = i\sigma^2 \qquad (I.$$

in terms of the usual Pauli matrices σ^i. In this paper, we adopt a two-dimensional representati for the Fermi fields in d=3 because it is irreducible and from this point of view most natura Since the Pauli matrices form a complete basis, there is no γ^5 and thus chiral symmetry is n defined in the standard fashion. Other discrete symmetries are broken in connection with ma generation. An example of a parity transformation in d=3 is

$$x'^\mu = \Lambda_p{}^\mu{}_\nu x^\nu ,$$

where $x = (x_0, x_1, x_2)$, and

$$\Lambda_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

with $\det\Lambda_p = -1$ as it should be. We might also let the x_2-component change sign, instead the x_1-component, but not both x_1- and x_2-components, since, in two space dimensions, this equivalent to a rotation over 180° with determinant +1. The effect of the parity transformatio $\mathcal{P}$ on the fermion and photon fields is [1]

$$\psi'(x') = \mathcal{P}\psi(x)\mathcal{P}^{-1} = \sigma^1\psi(x')$$
$$A'^\mu(x') = \mathcal{P}A^\mu(x)\mathcal{P}^{-1} = \Lambda_p{}^\mu{}_\nu A^\nu(x)$$

and hence it is easy to see that both mass terms $m\overline{\psi}\psi$ and $\frac{\mu}{4}\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha$ change sign under $\mathcal{P}$

In the term $\frac{\mu}{4}\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha$, μ may be considered to be a bare photon mass. Indeeed, th term is bilinear in A_μ, and the bare photon propagator has a pole at $k^2 = \mu^2$. This term

often called a topological or Chern-Simons mass term, because, for non-Abelian gauge fields, the corresponding mass term is a topologically non-trivial quantity, which must be quantized in terms of winding numbers on manifolds to mantain gauge invariance. Under the gauge transformation

$$\begin{aligned} \psi &\longrightarrow e^{-ie\Lambda}\psi \\ A_\mu &\longrightarrow A_\mu + \partial_\mu \Lambda \end{aligned}$$

this term changes by a total derivative, and hence the Lagrangian density

$$\mathcal{L} \longrightarrow \mathcal{L} + \partial_\nu(\frac{\mu}{4}\epsilon^{\mu\nu\alpha}F_{\mu\nu}\Lambda)\,.$$

If the fields vanish at infinity, i.e., are physical fields, the action $S = \int d^3x\,\mathcal{L}$ is thus gauge-invariant. Hence, physical quantities are gauge-invariant. The Lagrangian density (I.1) describes the so-called Maxwell-Chern-Simons theory, which has been studied extensively [5], most recently in the Coulomb gauge [6], which presents subtle problems. Another interesting direction of recent work was the construction of a pseudoclassical model for Chern-Simons particles ([7],[8]). In this introduction, we should like to explain some relevant issues, and summarize the content of the following sections.

In this paper we do not wish to consider (I.1) with the explicit Chern-Simons mass term, but rather see how this term is generated from the theory with $\mu = 0$. This is the problem of *dynamical mass generation*. The question arises how mass is defined in the quantum theory. We assume a covariant quantization, which generally implies the use of an extended Hilbert space with indefinite metric (see, e.g., [9] for a concise description of this formalism). We may either restrict ourselves to the physical subspace of Fock space (with positive metric) or consider a non-self-adjoint A^0 operator.

Let ψ and A_μ denote the canonically quantized fermion and photon fields, respectively. The corresponding propagators are defined as

$$S_{ab}(x-y) = -i\langle 0|T\psi_a(x)\overline{\psi}_b(y)|0\rangle\,, \tag{I.3}$$

$$D_{\mu\nu}(x-y) = -i\langle 0|TA_\mu(x)A_\nu(y)|0\rangle\,, \tag{I.4}$$

where T is the time-ordering operator and $|0\rangle$ denotes the physical vacuum. The bare propagators $S^0_{ab}(x-y)$ and $D^0_{\mu\nu}(x-y)$ satisfy the equations (for the moment, d is general and denotes the space-time dimension)

$$\begin{aligned} (i\partial\!\!\!/ - m)S^0(x-y) &= \delta^d(x-y)\,, \\ \left[g^{\mu\nu}\Box - \left(1-\frac{1}{a}\right)\partial^\mu\partial^\nu\right]D^0_{\nu\rho}(x-y) &= \delta^\mu{}_\rho\delta^d(x-y)\,, \end{aligned}$$

where $g^{\mu\nu}$ $\left(= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right.$ for $d=3$) is the metric tensor and δ^d is the d-dimensional delta function. In momentum space,

$$S^0(p) = \int d^dx\, e^{ip.x}S^0(x) = \frac{1}{p\!\!\!/ - m}\,, \tag{I.5}$$

$$D^0_{\mu\nu}(k) = -\frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - a\frac{k_\mu k_\nu}{k^4}\,. \tag{I.6}$$

The propagators satisfy the Schwinger-Dyson equations:

$$S^{-1}(p) = S^{0-1}(p) - \Sigma(p)\,, \tag{I.7}$$

$$D^{-1}_{\mu\nu}(k) = D^{0\ -1}_{\mu\nu}(k) - \Pi_{\mu\nu}(k)\,, \tag{I.8}$$

where $\Sigma(p)$ is the fermion self-energy, and $\Pi_{\mu\nu}(k)$ is the vacuum polarization tensor. In lowest order of perturbation theory,

$$\Sigma(p) = ie^2\int\frac{d^dp'}{(2\pi)^d}\gamma^\mu S^0(p')\gamma^\nu D^0_{\mu\nu}(p-p')\,, \tag{I.9}$$

$$\Pi_{\mu\nu}(k) = -ie^2\int\frac{d^dp}{(2\pi)^d}\mathrm{Tr}[\gamma_\mu S^0(p-k)\gamma_\nu S^0(p)]\,. \tag{I.10}$$

Both (I.9) and (I.10) diverge in the ultraviolet, the self-energy for d$\geq$3, and the vacuum polarization for d$\geq$2. Because of this the integrals have to be regularized, and at least three covariant methods are used largely: the Pauli-Villars-Rayski (PVR) regularization [12], based on the replacement

$$\frac{1}{k^2-m^2} \longrightarrow \frac{1}{k^2-m^2} - \frac{1}{k^2-\Lambda^2} \tag{I.11}$$

with Λ a cutoff, the dimensional regularization, due to Bollini, Giambiagi [13] and t'Hooft and Veltman [14], which relies upon an analytic continuation from complex dimensional d, and analytic regularization of propagators (see [15] and references given there).

As is well known, gauge invariance implies

$$k^\mu\Pi_{\mu\nu}(k) = 0\,, \tag{I.12}$$

which is the condition for the vacuum polarization to be transverse. Defining the three ortogonal projectors

$$P^{(1)}_{\mu\nu} = \frac{1}{2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + i\epsilon_{\mu\nu\beta}\frac{k^\beta}{\sqrt{k^2}}\right)\,, \tag{I.13a}$$

$$P^{(2)}_{\mu\nu} = \frac{1}{2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - i\epsilon_{\mu\nu\beta}\frac{k^\beta}{\sqrt{k^2}}\right)\,, \tag{I.13b}$$

$$P^{(3)}_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}\,, \tag{I.13c}$$

it follows that

$$\Pi = \Pi^{(1)}(P^{(1)} + P^{(2)}) + m(P^{(1)} - P^{(2)})\sqrt{k^2}\Pi^{(2)}\,. \tag{I.14}$$

From (I.8), we obtain the modified propagator in the Landau gauge:

$$\mathcal{D}_{\mu\nu}(k) = \frac{-1}{k^2 - \Pi(k^2)}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + im\epsilon_{\mu\nu\alpha}\frac{k^\alpha}{k^2}\frac{\Pi^{(2)}}{1-\Pi^{(1)}/k^2}\right]\,, \tag{I.15}$$

where

$$\Pi(k^2) = \Pi^{(1)}(k^2) + \frac{m^2(\Pi^{(2)}(k^2))^2}{1-\Pi^{(1)}(k^2)/k^2}\,. \tag{I.16}$$

Assuming that

$$\Pi_1(k^2) \stackrel{k^2\to 0}{\longrightarrow} Ck^2 \tag{I.17}$$

with $C > 0$ (see section III), we see from (I.15) that $D_{\mu\nu}(k)$ has a pole at a real positive value of k^2, if we assume that

$$\Pi_2(0) \neq 0 \tag{I.18}$$

and hence a photon mass is *dynamically generated*, under conditions (I.17) and (I.18).

This is a phenomenon of the utmost importance, both in particle physics and in many-body theory. In the latter the mass squared may be identified with the square of the inverse screening length λ^{-2}. Screening is the basis of several phenomena in condensed matter physics, including the stability of matter (see [16] for a clear introduction, and references as well).

The basic property (I.18) shows that dynamical mass generation depends on the infrared structure of the underlying field theory. For QED_3 this was studied in ([17], [4]) and, accordingly, some of the concepts and results are reviewed in section II.

One of the main problems connected with (I.18) is the so-called regularization ambiguity: (I.18) does follow if dimensional regularization [18] or analytic regularization [15] are adopted, but fails if naïve PVR regularization (I.11) is used [1]. This dependence of the renormalized mass on the regularization, as well as on the choice of gauge [1], points to deep problems in the conceptual structure. In section III we revisit this problem, following ref. [19], in the standard approach to quantum field theory. In section IV we introduce the causal theory ([10], [11]), as applied to QED_3, following [20], with a new discussion of general aspects, such as the causal phase, based on [21]. In this theory, no ambiguity due to regularization or to the choice of gauge arises. This is proved in section V, which also completes the treatment of [20], with a new treatment of the vertex based on [22].

Use of the Dyson-Schwinger equation (I.8) in (I.15) shows the *non-perturbative* character of dynamical mass generation: (I.8) may be written in symbolic form

$$D = D^0 \sum_{n=0}^{\infty} (\Pi D^0)^n \tag{I.19}$$

In (I.8), as well as in (I.19), Π is the full vacuum polarization tensor, which satisfies

$$\Pi_{\mu\nu}(k) = -ie^2 \int \frac{d^d p}{(2\pi)^d} \mathrm{Tr}[\gamma_\mu S(p') \Gamma_\nu(p,p') S(p)], \tag{I.20}$$

where Γ_ν is the vertex function, S the full self-energy of the electron, and $k = p' - p$. In our treatment of dynamical mass generation in the causal theory in section V we have *assumed*, however, that the $\Pi_{\mu\nu}$ (in (I.8) and (I.20)) is the *one-loop* vacuum polarization, which is given by (I.10) in lowest order of perturbation theory. We shall return to this, as well as other unsolved questions, in section VI, which is devoted to the conclusion and discussion of open problems.

II. The Infrared Problem

The method for dealing with non-stationary processes as electron scattering by an exter potential with the emission of a finite number of low-energy photons, using the Dyson S mat expansion, leads to the so-called infrared divergences in the probability amplitudes in QED

The program of taking into account the collective effect of low-energy photons was fi accomplished by Bloch and Nordsieck [23], who showed that the probability of emission a finite number of low-energy photons is zero and not infinite, as predicted by perturbati theory, and the transition probability must be extended to all possible final states to end with a finite result.

One can attribute to the infrared divergences an incorrect choice of the asymptotic stat which can be compared with the experimental data, besides the ill definition of the scatteri operator, where the asymptotic dynamics is not taken properly into account [24]. In t 70's Faddeev and Kulish [25] proposed a model where the Fock space representation of t Hilbert space, containing a finite number of low-energy photons, is replaced by a coherent-sta representation for an infinite number of these photons [26], in such a way that matrix elemer of the Dyson S operator between these states are finite and nonzero. These investigations four-dimensional gauge theories can be extended to the three-dimensional case, where a caref analysis of the IR problem is demanded [17].

Consider a quantum system described by a Hamiltonian H constituted by two terms

$$H = H_0 + V(t), \tag{II.}$$

where H_0 is the free Hamiltonian and

$$\lim_{t\to\pm\infty} V(t) = 0. \tag{II.}$$

This implies that asymptotically the scattered particles are supposed to be free. Thus, sin in scattering processes the particles are observed only in asymptotic regions ($t \to \pm\infty$), th asymptotic states are eigenstates of H_0.

Using these ideas, one defines the S operator in perturbation theory assuming that th asymptotic dynamics is given by H_0:

$$S = W_{out}^{\dagger} W_{in}, \tag{II.}$$

where $W_{\substack{in\\out}}$ are the wave operators

$$W_{\substack{in\\out}} = \lim_{t\to\mp\infty} U(t,0)^{\dagger} e^{-iH_0 t} \tag{II.}$$

and

$$U(t,s) = e^{-iH(t-s)}. \tag{II.}$$

Using (II.5), one rewrites the S matrix as

$$S = \lim_{\substack{t\to+\infty\\ s\to-\infty}} e^{iH_0 t} U(t,s) e^{-iH_0 s}. \tag{II.}$$

The transition probability from this state to another final asymptotic state ψ is given by

$$|\langle\psi|\, S\, |\varphi\rangle|^2 , \tag{II.7}$$

where the asymptotic states belong to the Fock space.

Solving the equation of motion for the time evolution operator, the S-matrix can be rewritten as a time-ordered product

$$S = T \exp\left\{-i \int_{-\infty}^{+\infty} V^I(t)\, dt\right\}, \tag{II.8}$$

where $V^I(t)$ is the potential in the interaction picture. If in addition to (II.2) we require

$$\int_{-\infty}^{+\infty} ds \left\| V^I(s) \right\| < \infty, \tag{II.9}$$

then the Dyson series for the S operator (II.8) is absolutely convergent. Thus, the S operator is unitary in Fock space, provided the particles have free dynamics in asymptotic regions.

The above method is not suitable to describing scattering processes when we consider long-range potentials such as the Coulomb potential. In this case, even at regions very far this potential cannot be neglected.

Following [27], let us consider the scattering of a charged particle by a Coulomb potential in two dimensions. The Hamiltonian of the system has the following form:

$$H = \frac{\vec{p}^{\,2}}{2m} + g \ln r = H_0 + V \tag{II.10}$$

where m is the mass of the scattered particle and g is the product of the charges of the particle and the scattering center.

First of all one constructs the potential shape in the asymptotic region in the interaction picture and then one obtains the wave packet which will represent the scattered particle in this region. For this purpose one considers the observables $\vec{x}$ and $\vec{p}$ as the position and momentum operators in the interaction picture. In this representation these operators satisfy the following equations of motion:

$$\frac{dO_I}{dt} = \frac{1}{i}\left[O_I, H_0\right]. \tag{II.11}$$

Using the Hamiltonian (II.10) and the above equation, we see that the momentum of the scattered particle is a constant of motion:

$$\frac{d\,\vec{p}}{dt} = \frac{1}{i}\left[\vec{p}, H_0\right] = 0. \tag{II.12}$$

Similarly, the equation of motion for the coordinates

$$\frac{d\,\vec{x}}{dt} = \frac{1}{i}\left[\vec{x}, H_0\right] = \frac{\vec{p}}{m} \tag{II.13}$$

whose solution

$$\vec{x}\,(t) = \vec{x} + \frac{\vec{p}}{m} t, \tag{II.14}$$

which describes the time evolution of the coordinate operator of the scattered particle, is identical to the classical trajectory of a particle in uniform rectilinear motion. From these results one can obtain the shape of the interaction potential at large distances, assuming that in this region the particles behave as classical particles with well defined trajectories. Thus, for $|t| \to \infty$

$$V(t) = g \ln\left(\frac{|\vec{p}|}{m} t\right). \tag{II.15}$$

This potential, which describes the interaction in the asymptotic region $|t| \to \infty$, is not absolutely convergent and its contribution to the asymptotic dynamics cannot be underestimated. In other words, the asymptotic dynamics is not governed by H_0 but by the operator

$$H_{as}(t) = H_0 + V_{as}(t) = H_0 + g \ln\left(\frac{|\vec{p}|}{m} t\right). \tag{II.16}$$

With this Hamiltonian describing the asymptotic dynamics and taking into account that $V_{as}(t)$ in the interaction picture is the same as in the Schrödinger picture, since

$$\left[V_{as}(t), H_0\right] = 0, \tag{II.17}$$

the wave function which describes the behavior of the particle in this region is obtained by solving the Schrödinger equation for H_{as}

$$i\frac{d}{dt}|\alpha, t\rangle = H_{as}(t)\,|\alpha, t\rangle, \tag{II.18}$$

where $|\alpha, t\rangle$ is the physical state of the scattered particle. In the momentum representation the above equation becomes

$$i\frac{d}{dt}\Psi\left(\vec{p}, t\right) = \left[\frac{\vec{p}^{\,2}}{2m} + g \ln\left(\frac{|\vec{p}|}{m} t\right)\right]\Psi\left(\vec{p}, t\right), \tag{II.19}$$

whose solution is

$$\begin{aligned}
\Psi\left(\vec{x}, t\right) &= \frac{1}{2\pi}\int d\,\vec{p}\;\Psi\left(\vec{p}, t\right) e^{i\vec{p}\cdot\vec{x}} \\
&= \frac{1}{2\pi}\int d\,\vec{p}\; e^{i\vec{p}\cdot\vec{x}} c\left(\vec{p}\right) \\
&\quad \times \exp\left\{-i\frac{\vec{p}^{\,2}}{2m} t - ig\left[t \ln\left(\frac{|\vec{p}|}{m} t - 1\right) - t_0 \ln\left(\frac{|\vec{p}|}{m} t_0 - 1\right)\right]\right\}.
\end{aligned} \tag{II.20}$$

The choice of this solution is due to certain initial conditions for equation (II.19). These are determined by considering that the time variation of the coordinate and momentum distributions for the particle represented by the wave packet (II.20) for $|t| \to \infty$ must be governed by the classical dynamics.

The wave packet (II.20) can be written as

$$\begin{aligned} \Psi(t) &= U_{as}(t)\Psi \\ &= e^{-iH_0 t}\exp\left\{-ig\left[t\ln\left(\frac{|\vec{p}|}{m}t-1\right)-t_0\ln\left(\frac{|\vec{p}|}{m}t_0-1\right)\right]\right\}\Psi. \end{aligned} \tag{II.21}$$

This means that the choice of the Hamiltonian which describes the asymptotic dynamics depends on the physical origin of the problem, in contrast with the usual definitions in the formal theory of scattering, where H_0 is taken as the asymptotic operator in the wave operators.

We then redefine the wave operators in the following way:

$$W_{\substack{in\\out}} = \lim_{t\to\mp\infty} e^{-iHt}U_{as}(t). \tag{II.22}$$

We may derive the asymptotic operator $U_{as}(t)$ in quantum electrodynamics [27]. The interaction operator of the system of photons and charged particles, in the interaction representation, is given by

$$V_I = \int j^{\mu}(x)\cdot A_{\mu}(x)\,d\vec{x}, \tag{II.23}$$

where $j^{\mu}(x)$ is the current operator

$$j^{\mu}(x) = e:\overline{\Psi}(x)\gamma^{\mu}\Psi(x):. \tag{II.24}$$

The decomposition of $\Psi(x)$ into a creation part $\Psi^{(-)}(x)$, and an annihilation part $\Psi^{(+)}(x)$, separates $j^{\mu}(x)$ into four terms

$$j^{\mu++} = e\overline{\Psi}^{(+)}\gamma^{\mu}\Psi^{(+)}, \tag{II.25}$$

$$j^{\mu--} = e\overline{\Psi}^{(-)}\gamma^{\mu}\Psi^{(-)}, \tag{II.26}$$

$$j^{\mu+-} = e:\overline{\Psi}^{(+)}\gamma^{\mu}\Psi^{(-)}:, \tag{II.27}$$

$$j^{\mu-+} = e:\overline{\Psi}^{(-)}\gamma^{\mu}\Psi^{(+)}:. \tag{II.28}$$

We can investigate the asymptotic behavior of this expression for $|t| \to \infty$. In this limit only $j^{\mu+-}$ and $j^{\mu-+}$ survive. In fact, since $V_I(t)$ contains the time t in the form $e^{i\alpha t}$, only those terms for which α approach zero within the range of integration over momentum space will survive. Thus, the terms $j^{\mu--}$ and $j^{\mu++}$ do not contribute in the asymptotic region.

Thus in (2+1) dimensions, the potential which describes the interaction between elect and positrons with the electromagnetic field in the asymptotic region is given by

$$\begin{aligned} V_{as}(t) &= \frac{1}{(2\pi)}\int d\vec{k}\int d\vec{p}\,\frac{1}{\sqrt{2\omega}}\frac{p^{\mu}}{p_0}\rho(\vec{p})\left[a_{\mu}(\vec{k})e^{-i\frac{\vec{k}\cdot\vec{p}}{p^0}t}+a_{\mu}^{\dagger}(\vec{k})e^{i\frac{\vec{k}\cdot\vec{p}}{p^0}t}\right] \\ &= \frac{1}{(2\pi)}\int\frac{d\vec{k}}{\sqrt{2\omega_k}}j_{as}^{\mu}(\vec{k},t)\left[a_{\mu}(\vec{k})e^{-i\omega t}+a_{\mu}^{\dagger}(-\vec{k})e^{i\omega t}\right], \end{aligned} \tag{II}$$

where $\rho(\vec{p})$ is the charge-density operator

$$\rho(\vec{p}) = e\left[b_r^{\dagger}(\vec{p})\,b_r(\vec{p})-d_r^{\dagger}(\vec{p})\,d_r(\vec{p})\right], \tag{II.}$$

and

$$j_{as}^{\mu}(\vec{k},t) = \int p^{\mu}e^{i\frac{\vec{k}\cdot\vec{p}}{p^0}t}\rho(\vec{p})\frac{d\vec{p}}{p^0} \tag{II.}$$

is an operator that has the shape of a current distribution of a particle with charge dens $\rho(\vec{p})$ and uniform velocity $\frac{p^{\mu}}{p^0}$. In fact, the eigenvalues of the operator (II.31), acting in a spa of charged particles, are classical current densities due to the motion of these particles, showi that, asymptotically, the scattered particles behave as classical particles.

As in the non-relativistic case, the Hamiltonian that describes the asymptotic dynamics given by

$$H_{as} = H_0 + V_{as}(t), \tag{II.3}$$

where H_0 is the free Hamiltonian. We can obtain $U_{as}(t)$ by solving the Schrödinger equati for the time evolution operator

$$i\frac{d}{dt}U(t) = H(t)\,U(t), \tag{II.3}$$

where H is the Hamiltonian of the system. In our problem $H = H_{as}$ and, therefore, the abo differential equation must be satisfied by the operator $U_{as}(t)$.

The asymptotic operator consists of two factors that commute. The first involves photo operators. The second is a phase factor which can be written as

$$\phi(t) = \frac{1}{4\pi}\int d\vec{q}\int d\vec{p}\,\frac{p\cdot q}{p^0q^0}\rho(\vec{p})\,\rho(\vec{q})\,t\left[1-\ln\left(2\left|\frac{\vec{p}}{p_0}-\frac{\vec{q}}{q_0}\right|t\right)\right], \tag{II.3}$$

showing that this is a relativistic generalization of the above mentioned Coulomb phase. T spectrum of this operator acting in a space of charged particles derives from the Coulon interaction among all the particles of the system.

The complete asymptotic operator is

$$U_{as}(t) = e^{-iH_0t}e^{R(t)}e^{i\phi(t)} \tag{II.3}$$

where $R(t) = -i\int_{t_0}^{t}V_{as}^{I}(\tau)\,d\tau$, $V_{as}^{I}(t) = e^{iH_0t}V_{as}(t)\,e^{-iH_0t}$ and, following the non-relativist generalization, the operator S is expressed as

$$S = \lim_{\substack{t\to+\infty\\ s\to-\infty}} U_{as}^{\dagger}(t)\,e^{-iH(t-s)}U_{as}(s). \tag{II.3}$$

Thus, the asymptotic dynamics in QED_3 in the infrared region is governed by an evolution operator which contains an infrared phase factor and a factored contribution of the asymptotic electromagnetic field. This leads to definition (II.36) for the S operator, which differs from the Dyson S-matrix (II.4) by the replacement

$$e^{-iH_0t} \to U_{as}(t), \tag{II.37}$$

since the infinite range of the Coulomb potential destroys the behavior of free dynamics in asymptotic regions. According to Faddeev and Kulish, the new S operator maps spaces of coherent states of the electromagnetic field, instead of Fock spaces of initial and final states of free particles into one another. One may also embed the asymptotic dynamics in the S operator by performing an arbitrary splitting of the interaction hamiltonian into H_s+H_h, where H_s carries the contribution of the soft photons [28].

In spite of the conceptual relevance and elegance of the present approach, there remains the long-standing problem of turning the Faddeev-Kulish formalism into a algorithm for the effective computation of infrared-divergence free cross-sections.

III. General Discussion of the Regularization Ambiguity

Gauge theories in (2+1)-dimensional space-time [1], may exhibit inconsistencies at one loop, due to the choice of the regularization method to evaluate ultraviolet divergent amplitudes such as the photon self-energy in spinor QED. In the latter, if we use analytic [15] or dimensional [18] regularization, the photon is induced a topological mass, in contrast with the result obtained through the Pauli-Villars-Rayski regularization, where the photon remains massless when we remove the regulators. Recently, alternative treatments have been given to the subject and, by using dispersion relations [29], it was shown that the photon indeed dynamically acquires a topological mass.

In order to get a new insight, let us reexamine the ordinary PVR prescription and analyse the conditions that must be imposed on the masses and coupling constants of the auxiliary fields such that a regularized closed fermion loop in 2+1 dimensions is rendered finite. Consider the integral corresponding to a fermion loop containing n vertices with n external photon lines atached, with momenta k_i (i=1,2,...,n). This integral is proportional to

$$\int d^3p \frac{Tr[\gamma_{\mu_1}(m+\not p)\gamma_{\mu_2}(m+\not p+\not k_1)...\gamma_{\mu_n}(m+\not p+...+\not k_{n-1})]}{(m^2-p^2+i\epsilon)[m^2-(p+k_1)^2+i\epsilon]...[m^2-(p+...+k_{n-1})^2+i\epsilon]} \tag{III.1}$$

so, for large p, its integrand behaves like p^{-n} whereas for $n<4$ the integral diverges as

$$\int_0^\infty \frac{p^2\,dp}{p^n} \sim \int_0^\infty \frac{dp}{p^{n-2}} .$$

The integrand $\mathcal{I}$ in equation (III.1) behaves like

$$\mathcal{I} \sim \sum_k m^k a_{-(n+k)}(p) , \tag{III.2}$$

where

$$a_{-(n+k)}(p) \sim p^{-(n+k)} . \tag{III.3}$$

Therefore, in making the substitution

$$\mathcal{I}(m) \to \sum_i^{n_f} c_i \mathcal{I}(M_i) ,$$

where n_f is the number of auxiliary fermion fields, we must impose in the vacuum polarization case ($n=2$) the constraints

$$\sum_i^{n_f} c_i = 0 , \tag{III.4}$$

$$\sum_i^{n_f} c_i M_i = 0 , \tag{III.5}$$

in order to get rid of the linear and logarithmic divergences, respectively.

Having settled down the basis for the Pauli-Villars-Rayski regularization method, we calculate the vacuum polarization tensor in spinor QED_3. In the standard notation, the regularized expression for the vacuum polarization tensor reads

$$\Pi^M_{\mu\nu}(k) = \frac{ie^2}{(2\pi)^3}\sum_{i=0}^{n_f} c_i \int d^3p \frac{P(M_i)}{({M_i}^2-{p_1}^2)({M_i}^2-{p_2}^2)} , \tag{III.6}$$

where

$$c_0 = 1 \quad , \quad M_0 = m \quad , \quad M_i = m\lambda_i \ (i=1,...,n_f) \tag{III.7}$$

and

$$P(M_i) = Tr\{\gamma_\mu(\not p_1 + M_i)\gamma_\nu(\not p_2 + M_i)\} . \tag{III.8}$$

We choose both the electron mass and that of the auxiliary field M_1 to be positive quantities; the coefficients λ_i ultimately go to infinity to recover the original theory. Using the Feynman parametrization

$$\frac{1}{({M_i}^2-{p_1}^2)({M_i}^2-{p_2}^2)} = \int_0^1 d\xi \frac{1}{[{M_i}^2-{p_1}^2-({p_2}^2-{p_1}^2)\xi]^2} , \tag{III.9}$$

we obtain

$$\Pi^M_{\mu\nu}(k) = (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\Pi^M_1(k^2) + im\epsilon_{\mu\nu\alpha}k^\alpha \Pi^M_2(k^2) , \tag{III.10}$$

where

$$\Pi^M_1(k^2) \equiv 4ie^2k^2\sum_{i=0}^{n_f} c_i \int_0^1 d\xi\, \xi(1-\xi)\int \frac{d^3p}{(2\pi)^3}\frac{1}{({Q_i}^2-p^2)^2} , \tag{III.11}$$

$$\Pi^M_2(k^2) \equiv -\frac{2ie^2}{m}\sum_{i=0}^{n_f} c_i M_i \int_0^1 d\xi \int \frac{d^3p}{(2\pi)^3}\frac{1}{({Q_i}^2-p^2)^2} , \tag{III.12}$$

with

$$Q_i^2 \equiv M_i^2 - \xi(1-\xi)k^2 \ . \tag{III.13}$$

The vacuum polarization tensor results to be transversal as expected by gauge invariance. From the subsidiary conditions (III.4) and (III.5), we realize that the number of regulators must be at least two, otherwise we can't get the coefficients λ_i cannot become arbitrarily large.

So, let us take

$$c_1 = \alpha - 1 \ , \quad c_2 = -\alpha \ , \quad c_j = 0 \ ; \ j > 2 \ , \tag{III.14}$$

where the parameter α can assume any real value except zero and the unity, so that condition (III.4) is satisfied. For $|\lambda_1|, |\lambda_2| \to \infty$, $\Pi^{(1)}(0) = 0$, whereas

$$\Pi^{(2)}(0) = \frac{\alpha e^2}{4\pi m}(1-s) \ , \tag{III.15}$$

$$s \equiv sign(1-\alpha^{-1}) \ . \tag{III.16}$$

For $0 < \alpha < 1$, which corresponds to $s = -1$, $\Pi^{(2)}(0) \neq 0$; in this case the photon acquires a topological mass, proportional to $\Pi^{(2)}(0)$, coming from proper insertions of the antisymmetric sector of the vacuum polarization tensor in the free photon propagator. If we assume that α is outside this range, $s = 1$, c_1 and c_2 have opposite signs and $\Pi^{(2)}(0)$ vanishes. We then conclude that this arbitrariness in the choice of the parameter α reflects itself in different values for the photon mass. The new parameter s may be identified with the winding number of homotopically nontrivial gauge transformations and also appears in lattice regularization [30].

Now we must look for the value of α leads to the correct photon mass. From equation (III.12) and we see that $\Pi^{(2)}(k^2)$ is ultraviolet finite. We remind that a closed fermion loop must be regularized as a whole (III.7),(III.8) so to preserve gauge invariance. However, having done this, we have affected the finite antisymmetric sector of the vacuum polarization tensor and, consequently, the photon mass. The same reasoning applies when, using PVR regularization, we calculate the anomalous magnetic moment of the electron; again, if care is not taken, we obtain an incorrect physical result.

In order to solve this problem we should take the value of α which cancels the contribution coming from the regulator fields. This occurs for $\alpha = 1/2$. Hence

$$\Pi^{(2)}(0) = \frac{e^2}{4\pi m} \ , \tag{III.17}$$

in agreement with the other approaches already mentioned. We should remember that PVR regularization violates parity symmetry in 2+1 dimensions. Nevertheless, for this particular choice of α, this symmetry is restored as the regulator masses get larger and larger. The result quoted above suggests that the ordinary parity-breaking PVR regularization, if carefully implemented, does not introduce any residual contribution to the photon topological mass.

IV. General Aspects of The Causal Theory: the Pha and Superrenormalizability

Although the physics of the quantized electron-positron field in interaction with a class electromagnetic field is well understood, some mathematical aspects of the theory have not established themselves as standard textbook material (an exception is [11]), in spite of tl fundamental importance. We therefore review briefly one of these aspects, which we neec this section: the scattering operator **S** in Fock space for quantum electrodynamics in (2-dimensional space-time, in an external time-dependent electromagnetic field A.

We start from the one-particle Hamiltonian (II.1) with

$$V(t) = e(V(t,\vec{x}) - \vec{\alpha}.\vec{A}(t,\vec{x})) \ . \tag{IV}$$

The potentials are assumed to vanish for $t \longrightarrow \pm\infty$ in such a way that the wave operat exist, together with a unitary S-matrix (II.3). The second-quantized free Dirac field is giv on Fock space by

$$\psi(f) = b(P_+^0 f) + d(P_-^0 f)^\dagger \ . \tag{IV}$$

Here $P_\pm^0$ are the projection operators on the positive and negative spectral subspaces of t one-particle free Dirac Hamiltonian H_0, respectively.

The second quantized S-matrix in Fock space is now defined by

$$\psi(S^\dagger f) = \mathbf{S}^{-1}\psi(f)\mathbf{S} \ , \tag{IV.}$$

$$\psi(S^\dagger f)^\dagger = \mathbf{S}^{-1}\psi(f)^\dagger\mathbf{S} \ , \ \forall f \in \mathcal{H}_1 \ , \tag{IV.}$$

if it exists, where $\mathcal{H}_1$ denotes the one-particle Hilbert space. We have taken the adjoint $S^\dagger$ the test functions since $\psi(f)$ is antilinear in f. It follows from the above definitions that **S** unitary and uniquely determined up to a phase. However, this phase $\lambda[A]$ is physical becau it depends on the external potential $A_\mu(x)$. As we shall see this phase will be fixed by tl requirement of causality of **S**.

The S-matrix **S** in Fock space exists, if and only if P_+SP_- is a Hilbert-Schmidt operato In this case it is given by

$$\mathbf{S} = C\, e^{S_{+-}S_{--}^{-1}b^\dagger d^\dagger} : e^{(S_{++}^{\dagger -1}-1)b^\dagger b} :: e^{(1-S_{--}^{-1})dd^\dagger} : e^{S_{--}^{-1}S_{-+}db} \ , \tag{IV.}$$

where : : denotes Wick ordering,

$$S_{ij} = P_i S P_j \ , \ i,j = +,- \tag{IV.}$$

and

$$|C|^2 = \det(1 - S_{+-}S_{-+}^\dagger) \ . \tag{IV.}$$

The first factor in (IV.5) describes electron-positron pair creation, the second one electro scattering, the third one positron scattering and the last one pair annihilation.

In the one-particle theory the condition that a change in the interaction law in any space-time region can influence the evolution of the system only at subsequent times can be translated into the factorization of the S-matrix [31]

$$S[A] = S_2 S_1 \ , \ \ S_j \stackrel{def}{=} S[A_j] \ , \tag{IV.8}$$

where we have written the electromagnetic potential as

$$A^\mu(x) = A_1^\mu(x) + A_2^\mu(x) \ , \tag{IV.9}$$

which is the sum of two parts with disjoint supports in time

$$\text{supp}\, A_1 \subset (-\infty, r] \ , \ \ \text{supp}\, A_2 \subset [r, +\infty) \ . \tag{IV.10}$$

A similar factorization should hold for the S-operator $\mathbf{S}$ in Fock space,

$$(\Omega, \mathbf{S}\Omega) = (\Omega, \mathbf{S_2 S_1}\Omega) \ . \tag{IV.11}$$

We call (IV.11) global causality condition for the Fock space S-operator in contrast to the differential condition

$$\frac{\delta}{\delta A_\mu(y)} \left(\Omega, \mathbf{S}^\dagger \frac{\delta \mathbf{S}}{\delta A_\nu(x)} \Omega \right) = 0, \ \text{ for } \ x^0 < y^0 \ . \tag{IV.12}$$

We mentioned that the S-matrix in Fock space can be uniquely determined up to a phase,

$$\mathbf{S} = e^{i\varphi} \tilde{\mathbf{S}} \ , \tag{IV.13}$$

where $\tilde{\mathbf{S}}$ is unitary, and given by expression (IV.5). Inserting (IV.13) into (IV.12) we obtain

$$\frac{\delta}{\delta A_\mu(y)} \left(\mathbf{S}\Omega, \frac{\delta \mathbf{S}}{\delta A_\nu(x)} \Omega \right) =$$

$$i \frac{\delta^2 \varphi}{\delta A_\mu(y) \delta A_\nu(x)} + \frac{\delta}{\delta A_\mu(y)} \left(\tilde{\mathbf{S}}\Omega, \frac{\delta \tilde{\mathbf{S}}}{\delta A_\nu(x)} \Omega \right) \ . \tag{IV.14}$$

It can be shown from the unitarity of $\tilde{\mathbf{S}}$ that the last term in (IV.14) is purely imaginary. Consequently, the real part of the causality condition (IV.12) is automatically satisfied while for the imaginary part we may choose φ conveniently such that (IV.12) holds.

We now turn to the determination of the causal phase in lowest order of perturbation theory. From (IV.5) we have

$$\tilde{\mathbf{S}}\Omega = C(\Omega + \sum_{mn} (S_{+-})_{mn} b_m^\dagger d_n^\dagger \Omega + \ldots) \ , \tag{IV.15}$$

where we have put S_{--}^{-1} equal to the unity in lowest order. Taking the functional derivative of (IV.15) with respect to $A_\nu(x)$ and keeping only terms of order $O(A)$ in the resulting expression, we arrive at

$$\left(\tilde{\mathbf{S}}\Omega, \frac{\delta \tilde{\mathbf{S}}}{\delta A_\nu(x)} \Omega \right) = i C^2 \Im m \text{Tr} \left(S_{-+}^\dagger \frac{\delta S_{+-}}{\delta A_\nu(x)} \right) \ . \tag{IV.16}$$

In lowest order we may set $C^2 = 1$.

The local causality condition (IV.12) together with expressions (IV.14) and (IV.16) yield

$$F(x,y) \stackrel{def}{=} \frac{\delta^2 \varphi}{\delta A_\mu(y) \delta A_\nu(x)} + \Im m \frac{\delta}{\delta A_\mu(y)} \text{Tr} \left((S_{+-})^\dagger \frac{\delta S_{+-}}{\delta A_\nu(x)} \right) = 0 \tag{IV.17}$$

for $x^0 < y^0$.

Next we calculate the second term in (IV.17). In lowest order of perturbation theory, we have

$$S_{+-}^{(1)} = -i(2\pi)^{-1} P_+(\mathbf{p}) \gamma^0 e \not{A}(p+q) P_-(-\mathbf{q}) \ . \tag{IV.18}$$

Using the following representation (I.2) for the Dirac matrices, we obtain from (IV.18)

$$\text{Tr} \frac{\delta}{\delta A_\mu(y)} (S_{+-})^\dagger \frac{\delta S_{+-}}{\delta A_\nu(x)} =$$

$$e^2 (2\pi)^{-5} \int d^2p \int d^2q \, e^{i(p+q)(x-y)} \text{tr}[P_-(-\mathbf{q}) \gamma^0 \gamma^\mu P_+(\mathbf{p}) \gamma^0 \gamma^\nu P_-(-\mathbf{q})]$$

$$= -\frac{1}{(2\pi)^2} \int d^3k \, e^{ik(x-y)} \hat{P}^{\mu\nu}(k) \ , \tag{IV.19}$$

$\hat{P}^{\mu\nu}(k)$ is related to the tensor of pair creation in (2+1) dimensions.

Since the symmetric and the antisymmetric parts of $\hat{P}^{\mu\nu}(k)$ are, respectively, real and imaginary, we can write

$$F(x,y) = \frac{\delta^2 \varphi}{\delta A_\mu(y) \delta A_\nu(x)} +$$

$$- \frac{1}{(2\pi)^2} \left[\int_{k_0 > 0} d^3k \, \sin k(x-y) \hat{P}_S^{\mu\nu}(k) - i \int_{k_0 > 0} d^3k \, \cos k(x-y) \hat{P}_A^{\mu\nu}(k) \right] \ . \tag{IV.20}$$

In order to write the last term in (IV.20) as a complex Fourier transform we must continue $\hat{P}_S^{\mu\nu}(k)$ and $\hat{P}_A^{\mu\nu}(k)$ antisymmetrically to $k_0 < 0$

$$F(x,y) = \frac{\delta^2 \varphi}{\delta A_\mu(y) \delta A_\nu(x)} - \frac{1}{(2\pi)^2} \frac{i}{2} \int d^3k \, e^{-ik(x-y)} [d_S^{\mu\nu}(k) - d_A^{\mu\nu}(k)] \ , \tag{IV.21}$$

where

$$d_S^{\mu\nu}(k) = (k^\mu k^\nu - k^2 g^{\mu\nu}) B(k^2) \ , \tag{IV.22}$$

$$d_A^{\mu\nu}(k) = i m \varepsilon^{\mu\nu\alpha} k_\alpha \Pi^{(2)}(k^2) \ . \tag{IV.23}$$

and

$$B(k^2) = \frac{-e^2}{2(4\pi)^2} \frac{k^2 + 4m^2}{k^2} \Theta(k^2 - 4m^2) \frac{\text{sgn}(k_0)}{\sqrt{k^2}} \ , \tag{IV.24}$$

$$\Pi^{(2)}(k^2) = \frac{-e^2}{2(2\pi)^2} \Theta(k^2 - 4m^2) \frac{\text{sgn}(k_0)}{\sqrt{k^2}} \ . \tag{IV.25}$$

According to a theorem by Titchmarsh, the Fourier transform of a causal function vanishing for $x^0 - y^0 = t < 0$ satisfies a dispersion relation. Since $d_S^{\mu\nu}(k)$ and $d_A^{\mu\nu}(k)$ are real and purely imaginary, respectively, they cannot be the Fourier transform of a causal function. The lacking imaginary part of $d_S^{\mu\nu}(k)$ and the lacking real part of $d_A^{\mu\nu}(k)$ must de supplied by the first term containing the phase $\varphi[A]$,

$$\frac{\delta^2\varphi}{\delta A_\mu(y)\delta A_\nu(x)} = -\frac{1}{(2\pi)^2}\frac{i}{2}\int d^3k\, e^{-ik(x-y)}[iq_S^{\mu\nu}(k) - q_A^{\mu\nu}(k)] \,, \tag{IV.26}$$

where

$$q_S^{\mu\nu}(k) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{+\infty} dt\,\frac{d_S^{\mu\nu}(kt)}{t^2(1-t)}$$

$$= \frac{\alpha}{\pi}(k^\mu k^\nu - k^2 g^{\mu\nu})\left[\frac{1}{\sqrt{k^2}}\left(1+\frac{4m^2}{k^2}\right)\log\left(\frac{1-\sqrt{\frac{4m^2}{k^2}}}{1+\sqrt{\frac{4m^2}{k^2}}}\right) + \frac{4m}{k^2}\right]\mathrm{sgn}(k_0) \,, \tag{IV.27}$$

and

$$q_A^{\mu\nu}(k) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{+\infty} dt\,\frac{-id_A^{\mu\nu}(kt)}{t(1-t)}$$

$$= -\frac{\beta}{\pi}m\varepsilon^{\mu\nu\alpha}\frac{k_\alpha}{\sqrt{k^2}}\log\left(\frac{1-\sqrt{\frac{4m^2}{k^2}}}{1+\sqrt{\frac{4m^2}{k^2}}}\right)\mathrm{sgn}(k_0) \,, \tag{IV.28}$$

with $\alpha \equiv -e^2/[2(4\pi)^2]$ and $\beta \equiv -e^2/[2(2\pi)^2]$. In the above dispersion relations, $\mathcal{P}$ denotes the principal-value of the respective integral.

The causal phase is obtained by two integrations

$$\varphi[A] = \frac{1}{2}\int d^3x \int d^3y\,\frac{\delta^2\varphi}{\delta A_\mu(y)\delta A_\nu(x)}A_\mu(y)A_\nu(x) + O(A^4)$$

$$= \frac{\pi}{2}\int d^3k\left[\left(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu}\right)\Pi_1^{(1)}(k) + im\varepsilon^{\mu\nu\alpha}k_\alpha\Pi_1^{(2)}(k)\right]A_\mu(k)A_\nu^*(k) \,, \tag{IV.29}$$

where

$$\Pi_1^{(1)}(k) = \frac{\alpha}{\pi}\left[\sqrt{k^2}\left(1+\frac{4m^2}{k^2}\right)\log\left(\frac{1-\sqrt{\frac{4m^2}{k^2}}}{1+\sqrt{\frac{4m^2}{k^2}}}\right) + 4m\right]\mathrm{sgn}(k_0) \,, \tag{IV.30}$$

$$\Pi_1^{(2)}(k) = -\frac{\beta}{\pi\sqrt{k^2}}\log\left(\frac{1-\sqrt{\frac{4m^2}{k^2}}}{1+\sqrt{\frac{4m^2}{k^2}}}\right)\mathrm{sgn}(k_0) \,. \tag{IV.31}$$

If we decompose the electromagnetic fields which appear in the integrand of (IV.29) into the respective real and imaginary parts we see that $\varphi[A]$ is indeed real. The S-operator in Fock space $\mathbf{S}[A]$ is then completely determined.

The vacuum-vacuum amplitude is

$$(\Omega, \mathbf{S}\Omega) = Ce^{i\varphi}(\Omega, e^{S_+ - S_-^{-1}b^\dagger d^\dagger}\Omega) = Ce^{i\varphi} \,. \tag{IV.}$$

The absolute square

$$|(\Omega, \mathbf{S}\Omega)|^2 = C^2 = 1 - P \tag{IV.}$$

must be equal to one minus the total probability P of pair creation,

$$P = -2\pi\int d^3k\,\hat{P}^{\mu\nu}(k)A_\mu(k)A_\nu^*(k) \,, \tag{IV.}$$

since the external field can change the vacuum state only into pair states.

The resulting expression for the vacuum-vacuum amplitude reads

$$(\Omega, \mathbf{S}\Omega) = \exp\{i\frac{\pi}{2}\int d^3k\left[\left(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu}\right)\Pi^{(1)}(k^2) + im\varepsilon^{\mu\nu\alpha}k_\alpha\Pi^{(2)}(k^2)\right]$$

$$\times A_\mu(k)A_\nu^*(k)\} \,, \tag{IV.3}$$

where

$$\Pi^{(1)}(k^2) = \Pi_1^{(1)}(k^2) - i\Pi_2^{(1)}(k^2) \,,$$
$$\Pi^{(2)}(k^2) = \Pi_1^{(2)}(k^2) - i\Pi_2^{(2)}(k^2) \,, \tag{IV.3}$$

and

$$\Pi_2^{(1)}(k^2) = \alpha\sqrt{k^2}\left(1+\frac{4m^2}{k^2}\right)\Theta(k^2 - 4m^2) \,, \tag{IV.37}$$

$$\Pi_2^{(2)}(k^2) = \frac{\beta\Theta(k^2 - 4m^2)}{\sqrt{k^2}} \,. \tag{IV.38}$$

For $k^2 < 4m^2$ the expression between square-brackets in (IV.35) coincides up to a mult plicative factor with the complex conjugate of the two-point function that corresponds to th vacuum polarization tensor. The vacuum-vacuum amplitude is ultraviolet finite and exhibit an additional contribution from the antisymmetric part of the vacuum polarization tensor i (2+1)-dimensional space-time, which emerges from the topological structure of the theory. W should like to emphasize that our construction of the phase above is entirely perturbative.

The program of constructing the S-matrix by means of causality in quantum field theor goes back to Stueckelberg and Bogoliubov (see [32]). The S-matrix is completely determine by causality and translation invariance, supposed the coupling T_1 is given:

$$S(g) = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^3x_1\ldots d^3x_n\, T\{T_1(x_1)\ldots T_1(x_n)\}g(x_1)\ldots g(x_n) \,, \tag{IV.39}$$

where $g(x) \in \mathcal{S}(\Re^3)$ is a C-number test function, assumed to be in the Schwartz space. W consider that the limit $g \to 1$ exists for the right physically measurable quantities. Howeve (IV.39) contains ultraviolet divergences for $n > 1$, and, consequently, must be renormalized.

In the early 70's, Epstein and Glaser [10] proposed an axiomatic construct where ultraviolet divergences do not appear, leading directly to the renormalized perturbation series. They have shown that in the causal theory no UV problem arises if the causal distributions are correctly split (distribution splitting) [10].

In the causal theory the S-matrix is viewed as an operator-valued distribution and has the following form

$$S(g) = 1 + \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3x_1 \ldots d^3x_n \, T_n(x_1 \ldots x_n) g(x_1) \ldots g(x_n) \; . \tag{IV.40}$$

The n-point operator-valued distributions T_n are the basic objects of the theory. They can be constructed inductively from T_1 through a number of physical requirements, the most essential one being causality. Let the operator-valued distributions $\tilde{T}_n$ be defined by

$$S(g)^{-1} = 1 + \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3x_1 \ldots d^3x_n \, \tilde{T}_n(x_1 \ldots x_n) g(x_1) \ldots g(x_n) \; . \tag{IV.41}$$

Then one defines, for arbitrary sets of points X, Y in the (2+1)-dimensional Minkowski space, the following distributions

$$A'_n(x_1 \ldots x_n) = \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n) \; , \tag{IV.42}$$

$$R'_n(x_1 \ldots x_n) = \sum_{P_2} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X) \; , \tag{IV.43}$$

where the sums run over all partitions

$$P_2 : \; \{x_1, \ldots, x_{n-1}\} = X \cup Y \; , \;\; X \neq \emptyset$$

into disjoint subsets with $|X| = n_1$, $|Y| \leq n-2$. We also introduce

$$D_n(x_1 \ldots x_n) = R'_n - A'_n . \tag{IV.44}$$

If the sums are extended over all partitions P_2^0, including the empty set $X = \emptyset$, we obtain the distributions

$$A_n(x_1 \ldots x_n) = A'_n + T_n(x_1 \ldots x_n) \; , \tag{IV.45}$$

$$R_n(x_1 \ldots x_n) = R'_n + T_n(x_1 \ldots x_n) \tag{IV.46}$$

These distributions are not known by the induction assumption because they contain the unknown T_n. Only the difference

$$D_n = R'_n - A'_n = R_n - A_n \tag{IV.47}$$

is known. We can determine R_n or A_n separately by investigating the support properties of the various distributions. It turns out that R_n is a retarded and A_n an advanced distribution. Hence by causal distribution splitting of (IV.47) one gets R_n (and A_n), and T_n then folows from (IV.45) (or (IV.46)).

In QED$_3$, D_n is of the form

$$D_n(x_1 \ldots x_n) = \sum_k : \prod_j \bar{\psi}(x_j) d_n^k(x_1 \ldots x_n) \prod_l \psi(x_l) :: \prod_m A(x_m) : \; , \tag{IV.48}$$

where $\psi(\bar{\psi})$ are *free* fermion field operators and A the *free* radiation field operators. The double dots denote the usual normal ordering. In the above expression, d_n^k are tempered numerical distributions, which have causal support. They must be split as follows:

$$d_n^k = r_n(x) - a_n(x) \; , \tag{IV.49}$$

where r_n and a_n have support in the forward and backward light-cone, respectively. The simplest way of splitting would be

$$r_n(x) = \chi_n(x) d_n^k \; , \tag{IV.50}$$

with

$$\chi_n(x) = \prod_{j=1}^{n-1} \Theta(x_j^0 - x_n^0) \; . \tag{IV.51}$$

This would lead to the usual UV divergent expression (IV.39). Since d_n^k are distributions they cannot, in general, be multiplied by discontinuous functions.

The causal splitting can directly be done in momentum space by means of the following dispersion formula

$$\hat{r}(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \, \frac{\hat{d}(tp)}{(t - i0)^{\omega+1}(1 - t + i0)} \; . \tag{IV.52}$$

where ω is the order of singularity of $\hat{d}$. This last expression is called the symmetric splitting solution. We refer to [11] for further details on the splitting mechanism. In particular, the theory of quasi-asymptotics [32] has been used in [11] to simplify the analysis of distribution splitting given in [10].

The singular order of a generic graph in QED$_3$ is given by

$$\omega = 3 - f - \frac{1}{2}b - \frac{1}{2}n \; , \tag{IV.53}$$

where f (b) is the number of external fermions (bosons), and n the order of perturbation theory. We see that the only singular graphs are the vacuum polarization ($n = 2$, $\omega = 1$), the electron self-energy ($n = 2$, $\omega = 0$) and the vacuum polarization in fourth order ($n = 4$, $\omega = 0$). Therefore, we conclude that QED in (2+1) dimensions is *superrenormalizable*. See [20] for further details.

V. Results: Dynamical Photon Mass Generation, the Self-energy of the Electron and the Vertex

The first order term T_1 of QED is given by

$$T_1(x) = ie : \bar{\psi}(x)\gamma^\mu\psi(x) : A_\mu(x) = -\tilde{T}_1(x) \tag{V.1}$$

On going from $n = 1$ to $n = 2$ the inductive method proceeds by forming

$$A_2'(x_1, x_2) = \tilde{T}_1(x_1)T_1(x_2) = -T_1(x_1)T_1(x_2) \ , \tag{V.2}$$
$$R_2'(x_1, x_2) = T_1(x_2)\tilde{T}_1(x_1) = -T_1(x_2)T_1(x_1) \ , \tag{V.3}$$

and

$$D_2(x_1, x_2) = R_2' - A_2' = T_1(x_1)T_1(x_2) - T_1(x_2)T_1(x_1) \tag{V.4}$$

By using Wick's theorem, the term due to vacuum polarization is obtained by two fermionic contractions

$$D_2^{vp}(x_1, x_2) = d^{\mu\nu}(x_1, x_2) : A_\mu(x_1)A_\nu(x_2) : \ , \tag{V.5}$$

where

$$d^{\mu\nu}(x_1, x_2) \equiv P^{\mu\nu}(x_1 - x_2) - P^{\nu\mu}(x_2 - x_1) \ , \tag{V.6}$$

with

$$P^{\mu\nu}(x_1 - x_2) \equiv e^2 Tr\{\gamma^\mu S^+(x_1 - x_2)\gamma^\nu S^-(x_2 - x_1)\} \ . \tag{V.7}$$

In momentum space we find [20]

$$d^{\mu\nu}(k) = d_S^{\mu\nu}(k) + d_A^{\mu\nu}(k) \ , \tag{V.8}$$

where

$$d_S^{\mu\nu}(k) = (k^\mu k^\nu - k^2 g^{\mu\nu})B(k^2) \ , \tag{V.9}$$
$$d_A^{\mu\nu}(k) = im\epsilon^{\mu\nu\alpha}k_\alpha \tilde{B}(k^2) \ . \tag{V.10}$$

In Eqs.(V.9) and (V.10), $B(k^2)$ and $\tilde{B}(k^2)$ are distributions such that $d_S^{\mu\nu}(k)$ and $d_A^{\mu\nu}(k)$ are of order $\omega = 1$ and $\omega = 0$, respectively.

From (IV.52) and the explicit form of $\tilde{B}(k^2)$ we obtain the antisymmetric splitting solution

$$r_A^{\mu\nu}(k) = im\epsilon^{\mu\nu\alpha}k_\alpha \Pi^{(2)}(k^2) \ , \tag{V.11}$$

where

$$\Pi^{(2)}(k^2) = \frac{me^2}{2(2\pi)\sqrt{k^2}} \log \frac{1 - \sqrt{k^2/4m^2}}{1 + \sqrt{k^2/4m^2}} \ . \tag{V.12}$$

The vacuum polarization tensor can be written in the form

$$\Pi_{\mu\nu}(k) = (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\Pi^{(1)}(k^2) + im\epsilon_{\mu\nu\alpha}k^\alpha \Pi^{(2)}(k^2) \ . \tag{V.13}$$

From (IV.52) and the explicit form of $B(k^2)$ we also obtain the symmetric splitting solution $r_S^{\mu\nu}(k)$. Then, in the limit $k^2 \to 0$ it follows that

$$\Pi^{(1)}(0) = 0 \ , \tag{V.14}$$

and, from (V.12),

$$\Pi^{(2)}(0) = \frac{e^2}{4\pi m} \ . \tag{V.15}$$

The photon propagator modified by the proper vacuum polarization insertions is given

$$\mathcal{D}_{\mu\nu}(k) = \frac{-1}{k^2 - \Pi(k^2)}[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - im\epsilon_{\mu\nu\alpha}\frac{k^\alpha}{k^2}\frac{\Pi^{(2)}}{1 - \Pi^{(1)}/k^2}] \ , \quad \text{(V.}$$

with

$$\Pi(k^2) = \Pi^{(1)}(k^2) + \frac{m^2(\Pi^{(2)}(k^2))^2}{1 - \Pi^{(1)}/k^2} \ . \quad \text{(V.}$$

By (V.16) and (V.17), we see that the modified propagator have a pole dislocated from t origin and, thus, we conclude that the photon acquires a "dinamically generated" mass order e^2, in agreement with the usual procedure from (IV.39), using a gauge invariant analy regularization.

We now discuss in more detail other second order diagram of QED_3 with non-negative the electron self-energy. The corresponding c-number distribution is

$$\begin{aligned} d(y) &= -e^2\gamma^\mu[S^{(-)}(y)D_0^{(+)}(-y) + S^{(+)}(y)D_0^{(+)}(y)]\gamma_\mu \\ &= -e^2\gamma^\mu[d_-(y) + d_+(y)]\gamma_\mu \ , \end{aligned} \quad \text{(V.1}$$

where

$$\begin{aligned} d_-(y) &= S^{(-)}(y)D_0^{(+)}(-y) \\ d_+(y) &= S^{(+)}(y)D_0^{(+)}(y) \ . \end{aligned} \quad \text{(V.1}$$

The Fourier transform of d_- is

$$\begin{aligned} \hat{d}_-(p) &= -(2\pi)^{3/2}\int d^3q \hat{D}_0^{(-)}(p-q)(\not{q} + m)\hat{D}_m^{(-)}(q) \\ &= (2\pi)^3(I_1 + I_2) \ , \end{aligned} \quad \text{(V.20}$$

where

$$I_1 \equiv \int d^3q\, \theta(q^0 - p^0)\delta[(p-q)^2]m\theta(-q^0)\delta(q^2 - m^2) \quad \text{(V.2}$$

and

$$I_2 \equiv \int d^3q\, \theta(q^0 - p^0)\delta[(p-q)^2]\not{q}\theta(-q^0)\delta(q^2 - m^2) \ . \quad \text{(V.2}$$

Taking time-like p in the form $p = (p_0, \mathbf{0})$ yields

$$\begin{aligned} I_1 &= m\int d^3q\, \theta(q^0 - p^0)\theta(-q^0)\delta(p_0^2 - 2p_0q_0 + m^2)\delta(q^2 - m^2) \\ &= m\int \frac{d^2q}{2E_q}\delta(p_0^2 + 2p_0E_q + m^2)\theta(-E_q - p_0) \ . \end{aligned} \quad \text{(V.2}$$

It follows from (V.23) that $p_0 < 0$ and

$$E_q = \frac{m^2 + p_0^2}{-2p_0} = \sqrt{m^2 + \mathbf{q}^2} \ , \quad \text{(V.2}$$

so that

$$|\mathbf{q}| = \frac{m^2 - p_0^2}{2p_0} \text{ e } m^2 - p_0^2 < 0 \,. \tag{V.25}$$

Therefore,

$$\begin{aligned} I_1 &= \frac{2\pi}{2} m\theta(p_0^2 - m^2)\theta(-p_0) \int_0^\infty d|\mathbf{q}| \frac{|\mathbf{q}|}{E_q} \frac{1}{2|p_0|} \delta\left(E_q - \frac{p_0^2 + m^2}{2|p_0|}\right) \\ &= \pi m\theta(p_0^2 - m^2)\theta(-p_0) \int_m^\infty dE \frac{E}{E\, 2|p_0|} \delta\left(E - \frac{p_0^2 + m^2}{2|p_0|}\right) \\ &= \frac{\pi}{2}\theta(p_0^2 - m^2)\theta(-p_0) \frac{m}{\sqrt{p_0^2}} \,. \end{aligned} \tag{V.26}$$

We may write the integral (V.22) in the form $I_2 = I_2^\nu \gamma_\nu$. For $\nu \neq 0$, I_2^ν vanishes by symmetry. For I_2^0 we have

$$\begin{aligned} I_2^0 &= \int d^3q\, q^0 \theta(q^0 - p^0)\delta(p_0^2 - 2p_0 q_0 + m^2) \frac{\delta(q^0 + E_q)}{2E_q} \\ &= \frac{\pi}{2}\theta(p_0^2 - m^2)\theta(-p_0) \frac{p_0^2 + m^2}{2p_0^2} \frac{p^0}{\sqrt{p_0^2}} \,, \end{aligned} \tag{V.27}$$

in the same way above for I_1. For generic p, we must replace p_0^2 by p^2 in I_1 and I_2 and the linear term $p^0\gamma_0$ in $I_2^0\gamma_0$, with I_2^0 given by (V.27), by $\not{p}$. We then obtain

$$\hat{d}_-(p) = (2\pi)^{-3}\frac{\pi}{2}\theta(p^2 - m^2)\theta(-p_0)\frac{1}{\sqrt{p^2}}\left[m + \frac{\not{p}}{2}\left(1 + \frac{m^2}{p^2}\right)\right] \,, \tag{V.28}$$

which coincides with the distribution $\hat{r}'(p)$. The distribution $\hat{d}_+(p)$, with support in the advanced light-cone, is obtained in an analogous manner from (V.19), or simply substituting $\theta(p_0)$ for $-\theta(-p_0)$ in (V.28).

It follows from (V.18), (V.28) that

$$\begin{aligned} \hat{d}(p) &= -e^2\gamma^\mu[\hat{d}_-(p) + \hat{d}_+(p)]\gamma_\mu \\ &= e^2(2\pi)^{-3}\frac{\pi}{2}\theta(p^2 - m^2)\frac{\operatorname{sgn} p_0}{\sqrt{p^2}}\left[3m - \frac{\not{p}}{2}\left(1 + \frac{m^2}{p^2}\right)\right] \,. \end{aligned} \tag{V.29}$$

From (V.29) we conclude that the singular order of $\hat{d}$ is $\omega = 0$. Therefore the symmetric solution of the splitting problem is

$$\begin{aligned} \hat{r}(p) = e^2(2\pi)^{-3}\frac{\pi}{2}\frac{i}{2\pi}\int_{-\infty}^{\infty} &\frac{dt}{(t - i0)(1 - t + i0)}\theta(t^2p^2 - m^2) \\ &\times \frac{\operatorname{sgn} t p_0}{|t|\sqrt{p^2}}\left[3m - \frac{t\not{p}}{2}\left(1 + \frac{m^2}{t^2p^2}\right)\right] \,. \end{aligned} \tag{V.30}$$

Thus,

$$\begin{aligned} \hat{r}(p) = e^2(2\pi)^{-4}\frac{i\pi}{2}\frac{\operatorname{sgn} p_0}{\sqrt{p^2}} &\left\{\left[3m - \frac{\not{p}}{2}\left(1 + \frac{m^2}{p^2}\right)\right]\log\left(\frac{1 - \sqrt{p^2/m^2}}{1 + \sqrt{p^2/m^2}}\right)\right. \\ &\left. + \sqrt{\frac{p^2}{m^2}}\left(6m - \not{p}\frac{m^2}{p^2}\right)\right\} + e^2(2\pi)^{-3}\frac{\pi}{4}\theta(p^2 - m^2)\frac{\operatorname{sgn} p_0}{\sqrt{p^2}}\left[3m - \frac{\not{p}}{2}\left(1 + \frac{m^2}{p^2}\right)\right] \,, \end{aligned} \tag{V.31}$$

which, from the tensor structure of $\hat{r}$, is the general solution.

The electron self-energy is defined as

$$\begin{aligned} \Sigma(p) &= -i(2\pi)^2[\hat{r}(p) - \hat{r}'(p)] \\ &= \frac{e^2}{8\pi}\frac{1}{\sqrt{p^2}}\left\{\left[3m - \frac{\not{p}}{2}\left(1 + \frac{m^2}{p^2}\right)\right]\log\left(\frac{1 - \sqrt{p^2/m^2}}{1 + \sqrt{p^2/m^2}}\right)\right. \\ &\left. + \sqrt{\frac{p^2}{m^2}}\left(6m - \not{p}\frac{m^2}{p^2}\right)\right\} - \frac{ie^2}{16\pi}\theta(p^2 - m^2)\frac{1}{\sqrt{p^2}}\left[3m - \frac{\not{p}}{2}\left(1 + \frac{m^2}{p^2}\right)\right] \,, \end{aligned} \tag{V.32}$$

when p is in the advanced light-cone. The logarithmic on shell singularity is more severe than in four dimensions. Anyway, $S(g)$ given by remains well defined in perturbation theory even in three dimensions, although in this case the adiabatic limit $g \to 1$ is more delicate.

The proper two-point insertions lead to the corrected second-order propagators. There remains another basic insertion in the theory, known as the vertex function, which, as the electron self-energy, also exhibits a singular behaviour in the infrared region. We construct the three-point function that corresponds to the vertex function from the retarded and advanced distributions

$$\begin{aligned} R_3'(x_1, x_2, x_3) &= \sum_{P_2} T(Y, x_3)\tilde{T}(X) \\ &= \underbrace{T_2(x_1, x_3)\tilde{T}_1(x_2)}_{R_{31}'} + \underbrace{T_2(x_2, x_3)\tilde{T}_1(x_1)}_{R_{32}'} + \underbrace{T_1(x_3)\tilde{T}_2(x_1, x_2)}_{R_{33}'} \end{aligned} \tag{V.33}$$

and

$$\begin{aligned} A_3'(x_1, x_2, x_3) &= \sum_{P_2} \tilde{T}(X)T(Y, x_3) \\ &= \underbrace{\tilde{T}_1(x_2)T_2(x_1, x_3)}_{A_{31}'} + \underbrace{\tilde{T}_1(x_1)T_2(x_2, x_3)}_{A_{32}'} + \underbrace{\tilde{T}_2(x_1, x_2)T_1(x_3)}_{A_{33}'} \,. \end{aligned} \tag{V.34}$$

Each term in (V.33) and (V.34) corresponds to a certain decomposition of the vertex diagram in a vertex part ($T_1 = -\tilde{T}_1$) and a second-order diagram ($T_2, \tilde{T}_2$). The last may be a two-point function for Möller scattering,

$$T_2^{(1)}(x_1, x_2) = -ie^2 : \overline{\psi}(x_1)\gamma^\mu\psi(x_1)\overline{\psi}(x_2)\gamma_\mu\psi(x_2) : D_0^F(x_1 - x_2) \,, \tag{V.35}$$

or that for Compton scattering,

$$T_2^{(2)}(x_1,x_3) = -ie^2 : \overline{\psi}(x_1)\gamma^\mu S_F(x_1-x_3)\gamma^\nu\psi(x_3) :: A_\mu(x_1)A_\nu(x_3) :$$
$$= -ie^2 : \overline{\psi}(x_3)\gamma^\nu S_F(x_3-x_1)\gamma^\mu\psi(x_1) :: A_\mu(x_1)A_\nu(x_3) : \ . \qquad (V.36)$$

The inverse two-point functions are given by

$$\hat{T}_2(x_1,x_2) = -T_2(x_1,x_2) + T_1(x_1)T_1(x_2) + T_1(x_2)T_1(x_1) \ . \qquad (V.37)$$

For the first term in (V.33) we get

$$R'_{31} \equiv -T_2^{(2)}(x_1,x_3)T_1(x_2)$$
$$= ie^2 : \overline{\psi}(x_1)\gamma^\mu S_F(x_1-x_3)\gamma^\nu S^{(+)}(x_3-x_2)ie\gamma^\lambda\psi(x_2) :$$
$$\times g_{\mu\lambda}D_0^{(+)}(x_1-x_2)A_\nu(x_3) \ , \qquad (V.38)$$

such that to x_1 is associated an outgoing electron, to x_2 an incoming electron and to x_3 an external photon. Analogously,

$$R'_{32} \equiv -T_2^{(2)}(x_2,x_3)T_1(x_1)$$
$$= -ie^2 : \overline{\psi}(x_1)\gamma_\mu S^{(-)}(x_1-x_3)\gamma^\nu S_F(x_3-x_2)ie\gamma^\mu\psi(x_2) :$$
$$\times D_0^{(+)}(x_2-x_1)A_\nu(x_3) \ . \qquad (V.39)$$

The last term in (V.33) can be written as

$$R'_{33} \equiv T_1(x_3)\hat{T}_2^{(1)}(x_1,x_2)$$
$$= \underbrace{-T_1(x_3)T_2^{(1)}(x_1,x_2)}_{R'_1} + \underbrace{T_1(x_3)T_1(x_1)T_1(x_2)}_{R'_{21}} + \underbrace{T_1(x_3)T_1(x_2)T_1(x_1)}_{R'_{22}} \ , \qquad (V.40)$$

where

$$R'_1 = e^3 : \overline{\psi}(x_1)\gamma^\mu\frac{1}{i}S^{(-)}(x_1-x_3)\gamma^\nu\frac{1}{i}S^{(+)}(x_3-x_2)\gamma_\mu\psi(x_2) :$$
$$\times D_0^F(x_1-x_2)A_\nu(x_3) \ , \qquad (V.41)$$
$$R'_{21} = e^3 : \overline{\psi}(x_1)\gamma^\mu S^{(-)}(x_1-x_3)\gamma^\nu S^{(+)}(x_3-x_2)\gamma^\lambda\psi(x_2) :$$
$$\times g_{\mu\lambda}D_0^{(+)}(x_1-x_2)A_\nu(x_3) \ , \qquad (V.42)$$
$$R'_{22} = e^3 : \overline{\psi}(x_1)\gamma^\lambda S^{(-)}(x_1-x_3)\gamma^\nu S^{(+)}(x_3-x_2)\gamma^\mu\psi(x_2) :$$
$$\times g_{\mu\lambda}D_0^{(+)}(x_2-x_1)A_\nu(x_3) \ . \qquad (V.43)$$

The advanced distribution is obtained in a similar way,

$$A'_{31} = -T_1(x_2)T_2^{(2)}(x_1,x_3)$$

$$= e^3 : \overline{\psi}(x_1)\gamma^\mu S_F(x_1-x_3)\gamma^\nu S^{(-)}(x_3-x_2) : \gamma^\lambda\psi(x_2) :$$
$$\times g_{\lambda\mu}D_0^{(+)}(x_2-x_1)A_\nu(x_3) \ , \qquad (V.4$$

$$A'_{32} = -T_1(x_1)T_2^{(2)}(x_2,x_3)$$
$$= -e^3 : \overline{\psi}(x_1)\gamma^\lambda S^{(+)}(x_1-x_3)\gamma^\nu S_F(x_3-x_2)\gamma^\mu\psi(x_2) :$$
$$\times g_{\lambda\mu}D_0^{(+)}(x_1-x_2)A_\nu(x_3) \ . \qquad (V.4$$

The last term in (V.34) split into three pieces

$$A'_{33} = \hat{T}_2(x_1,x_2)T_1(x_3)$$
$$= \underbrace{-T_2^{(1)}(x_1,x_2)T_1(x_3)}_{A'_1} + \underbrace{T_1(x_1)T_1(x_2)T_1(x_3)}_{A'_{21}} + \underbrace{T_1(x_2)T_1(x_1)T_1(x_3)}_{A'_{22}} \ , \qquad (V.4$$

with the following results:

$$A'_1 = -e^3 : \overline{\psi}(x_1)\gamma^\mu S^{(+)}(x_1-x_3)\gamma^\nu S^{(-)}(x_3-x_2)\gamma_\mu\psi(x_2) :$$
$$\times D_0^F(x_1-x_2)A_\nu(x_3) \ , \qquad (V.4$$
$$A'_{21} = e^3 : \overline{\psi}(x_1)\gamma^\mu S^{(+)}(x_1-x_3)\gamma^\nu S^{(-)}(x_3-x_2)\gamma^\lambda\psi(x_2) :$$
$$\times g_{\mu\lambda}D_0^{(+)}(x_1-x_2)A_\nu(x_3) \ , \qquad (V.4$$
$$A'_{22} = e^3 : \overline{\psi}(x_1)\gamma^\lambda S^{(+)}(x_1-x_3)\gamma^\nu S^{(-)}(x_3-x_2)\gamma^\mu\psi(x_2) :$$
$$\times g_{\mu\lambda}D_0^{(+)}(x_2-x_1)A_\nu(x_3) \ . \qquad (V.4$$

We may now write the difference

$$D = R' - A' \stackrel{def}{=} -e^3 : \overline{\psi}(x_1)D^\nu(x_1,x_2,x_3)\psi(x_2) : A_\nu(x_3) \ , \qquad (V.5$$

where all the 10 terms which contribute for D^ν have the same matrix structure. The followi pairs of terms combine: R'_{31} with R'_{21}, R'_{32} with R'_{22}, $-A'_{31}$ with $-A'_{22}$ and $-A'_{32}$ with $-A'_{21}$. A a consequence all the Feynman fermion propagators are converted into retarded and advance propagators.

In momentum space we have

$$\hat{D}^\nu(p,q) = (2\pi)^{-3}\int dy_1dy_2D^\nu(x_1,x_2,x_3)e^{ipy_1+iqy_2} = \hat{R}' - \hat{A}' \ , \qquad (V.5$$

where

$$y_1 = x_1 - x_3 \ , \quad y_2 = x_3 - x_2 \ .$$

The resulting retarded and advanced distributions resultantes are given, respectively, by

$$\hat{R}'(p,q) = (2\pi)^{-3/2}\int d^3k\left[\gamma^\mu\hat{S}^{(-)}(p-k)\gamma^\nu\hat{S}^{(+)}(q-k)\gamma_\mu\hat{D}_0^F(k)\right.$$

$$+\gamma^\mu \hat{S}^{(-)}(p+k)\gamma^\nu \hat{S}^{av}(q+k)\gamma_\mu \hat{D}_0^{(+)}(k)$$
$$-\gamma^\mu \hat{S}^{ret}(p-k)\gamma^\nu \hat{S}^{(+)}(q-k)\gamma_\mu \hat{D}_0^{(+)}(k)\Big] \ , \tag{V.52}$$

$$\hat{A}'(p,q) = -(2\pi)^{-3/2}\int d^3k \Big[-\gamma^\mu \hat{S}^{(+)}(p-k)\gamma^\nu \hat{S}^{(-)}(q-k)\gamma_\mu \hat{D}_0^F(k)$$

$$+\gamma^\mu \hat{S}^{(+)}(p-k)\gamma^\nu \hat{S}^{ret}(q-k)\gamma_\mu \hat{D}_0^{(+)}(k)$$
$$-\gamma^\mu \hat{S}^{av}(p+k)\gamma^\nu \hat{S}^{(-)}(q+k)\gamma_\mu \hat{D}_0^{(+)}(k)\Big] \ . \tag{V.53}$$

After simplifying the matrix part, D^ν can be written as

$$D^\nu(p,q) = -3\not{q}\gamma^\nu \not{p}\mathcal{I} + 3\not{q}\gamma^\nu\gamma^\mu \mathcal{I}_\mu + 3\mathcal{I}_\mu\gamma^\mu\gamma^\nu\not{p} + 3\mathcal{I}^\mu{}_\mu\gamma^\nu$$
$$-6\mathcal{I}^\nu{}_\mu\gamma^\mu + 4m(p^\nu + q^\nu)\mathcal{I} - 6m\mathcal{I}^\nu - m(\not{p}\gamma^\nu + \gamma^\nu\not{q})\mathcal{I} - m^2\gamma^\nu\mathcal{I} \ , \tag{V.54}$$

where $\mathcal{I}$, $\mathcal{I}^\mu$ and $\mathcal{I}^{\mu\nu}$ are the following scalar, vector and tensor integrals

$$\mathcal{I}^{\cdots}(p,q) \stackrel{def}{=} (2\pi)^{-3/2}\int \frac{d^3k}{(2\pi)^3}\{1, k^\mu, k^{\mu\nu}\}$$

$$\times\Big[-D^{(+)}(k-p)D^{(-)}(k-q)D_0^F(k) + D^{(-)}(k-p)D^{(+)}(k-q)D_0^F(k)$$
$$-D^{(+)}(k-p)D^{ret}(k-q)D_0^{(-)}(k) + D^{(-)}(k-p)D^{av}(k-q)D_0^{(+)}(k)$$
$$-D^{av}(k-p)D^{(-)}(k-q)D_0^{(+)}(k) + D^{ret}(k-p)D^{(+)}(k-q)D_0^{(-)}(k)\Big] \ . \tag{V.55}$$

and

$$D^{(\pm)}(k) = \theta(\pm k_0)\delta(k^2 - m^2) \ , \ D^F(k) = \frac{1}{k^2 - m^2 + i0} \ , \tag{V.56}$$
$$D^{av}(k) = \frac{1}{k^2 - m^2 - ik_0 0} \ , \ D^{ret} = \frac{1}{k^2 - m^2 + ik_0 0} \ . \tag{V.57}$$

The quantities with index 0 correspond to $m = 0$.

We calculate the scalar integral

$$\mathcal{I}(p,q) = \frac{1}{(2\pi)^{9/2}}\sum_{j=1}^{6}\mathcal{I}_j \ , \tag{V.58}$$

with

$$\mathcal{I}_1 \equiv \int d^3k\,\theta(p^0 - k^0)\delta[(k-p)^2 - m^2]\frac{1}{(k-q)^2 - m^2 - i0(k^0 - q^0)}$$
$$\times\theta(k^0)\delta(k^2) \ , \tag{V.59}$$
$$\mathcal{I}_2 \equiv \int d^3k\frac{1}{(k-p)^2 - m^2 + i0(k^0 - p^0)}\theta(k^0 - q^0)\delta[(k-q)^2 - m^2]$$
$$\times\theta(-k^0)\delta(k^2) \ , \tag{V.60}$$

$$\mathcal{I}_3 \equiv \int d^3k\,\theta(p^0 - k^0)\delta[(k-p)^2 - m^2]\theta(k^0 - q^0)\delta[(k-q)^2 - m^2]$$
$$\times\frac{1}{k^2 + i0} \tag{V.61}$$
$$\mathcal{I}_4 \equiv -\int d^3k\frac{1}{(k-p)^2 - m^2 - i0(k^0 - p^0)}\theta(q^0 - k^0)\delta[(k-q)^2 - m^2]$$
$$\times\theta(k^0)\delta(k^2) \ , \tag{V.62}$$
$$\mathcal{I}_5 \equiv -\int d^3k\,\theta(k^0 - p^0)\delta[(k-p)^2 - m^2]\frac{1}{(k-q)^2 - m^2 + i0(k^0 - q^0)}$$
$$\times\theta(-k^0)\delta(k^2) \ , \tag{V.63}$$
$$\mathcal{I}_6 \equiv -\int d^3k\,\theta(k^0 - p^0)\delta[(k-p)^2 - m^2]\theta(q^0 - k^0)\delta[(k-q)^2 - m^2]$$
$$\times\frac{1}{k^2 + i0} \ . \tag{V.64}$$

Defining

$$I(p,q) \equiv -(\mathcal{I}_3 + \mathcal{I}_6) \tag{V.65}$$

and performing the shift $k \to k + p$ in the integration variables, we obtain

$$I = I_a + I_b \ , \tag{V.66}$$

where

$$I_a = \int \frac{d^3k}{(k+p)^2 + i0}\theta(k_0)\delta(k^2 - m^2)\theta(-P_0 - k_0)\delta[(P+k)^2 - m^2] \ , \tag{V.67}$$
$$I_b = -\int \frac{d^3k}{(k+p)^2 + i0}\theta(-k_0)\delta(k^2 - m^2)\theta(P_0 + k_0)\delta[(P+k)^2 - m^2] \ , \tag{V.68}$$

with

$$P \equiv p - q \ . \tag{V.69}$$

The product of the first θ and δ functions in the integrand of (V.67) gives $\delta(k_0 - E_{\mathbf{k}})/(2E_{\mathbf{k}})$, which implies

$$k_0 = E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2} \ , \ P_0 < -k_0 < -m \ . \tag{V.70}$$

For space-like P there exists a Lorentz frame with $P_0 = 0$, such that I_a vanishes. Consequently P must be time-like. Then, there is a frame with $\mathbf{P} = 0 = \mathbf{p} - \mathbf{q}$ (center-of-mass system). The last function δ in (V.67) implies

$$P_0^2 = -2E_{\mathbf{k}}P_0 \ , \ E_{\mathbf{k}} = -\frac{P_0}{2} = -\frac{1}{2}(p_0 - q_0) \ , \ k_0 = -\frac{1}{2}P_0 \ . \tag{V.71}$$

Analogously, in (V.68) we have

$$k_0 = -E_{\mathbf{k}} = -\sqrt{\mathbf{k}^2 + m^2} \ , \ P_0 > -k_0 > m \ . \tag{V.72}$$

and

$$P_0^2 = -2E_{\mathbf{k}}P_0 \ , \ E_{\mathbf{k}} = \frac{P_0}{2} = \frac{1}{2}(p_0 - q_0) \ , \ k_0 = -\frac{1}{2}P_0 \ . \tag{V.73}$$

Thus,

$$I_a = \int \frac{d^2\mathbf{k}}{2E_{\mathbf{k}}} \frac{1}{(E_{\mathbf{k}} + p_0)^2 - (\mathbf{k}+\mathbf{p})^2 + i0} \frac{\delta(E_{\mathbf{k}} + P_0/2)}{2|P_0|}$$

$$\text{for } -P_0 - E_{\mathbf{k}} > 0 \ , \tag{V.74}$$

$$I_b = -\int \frac{d^2\mathbf{k}}{2E_{\mathbf{k}}} \frac{1}{(E_{\mathbf{k}} - p_0)^2 - (\mathbf{k}+\mathbf{p})^2 + i0} \frac{\delta(E_{\mathbf{k}} - P_0/2)}{2|P_0|}$$

$$\text{for } P_0 - E_{\mathbf{k}} > 0 \ . \tag{V.75}$$

If we use $E = E_{\mathbf{k}}$ as integration variable

$$\int d^2\mathbf{k} = \int_0^{2\pi} d\varphi \int_0^{\infty} |\mathbf{k}| d|\mathbf{k}| = \int_0^{2\pi} \int_m^{\infty} E\, dE \ ,$$

the integration in E can be easily performed due to the δ function in (V.74) and (V.75). So, we obtain

$$I = -\mathrm{sgn}P_0 \frac{\theta(P_0^2 - 4m^2)}{4|P_0|} \times \int_0^{2\pi} d\varphi \frac{1}{p_0q_0 - \mathbf{p}^2 + m^2 - \sqrt{P_0^2 - 4m^2}|\mathbf{p}|\cos\varphi + i0} \ . \tag{V.76}$$

We can rewrite the above integral in a covariant form if we notice that

$$p_0q_0 - \mathbf{p}^2 = p_0q_0 - \mathbf{p}.\mathbf{q} = p.q$$

and

$$|P_0||\mathbf{p}| = \sqrt{(p_0 - q_0)^2\mathbf{p}^2} = \sqrt{p_0^2\mathbf{q}^2 + q_0^2\mathbf{p}^2 - 2p_0q_0\mathbf{p}.\mathbf{q}}$$
$$= \sqrt{(p_0q_0 - \mathbf{p}.\mathbf{q})^2 - (p_0^2 - \mathbf{p}^2)(q_0^2 - \mathbf{q}^2)} = \sqrt{(p.q)^2 - p^2q^2} \ .$$

In an arbitrary frame,

$$I = -\frac{\mathrm{sgn}P_0}{4\sqrt{P^2}}\theta(P^2 - 4m^2)\int_0^{2\pi} d\varphi \frac{1}{a_3 + b_3\cos\varphi + i0}$$
$$= \frac{\pi}{2}\frac{\mathrm{sgn}P_0}{\sqrt{P^2}}\theta(P^2 - 4m^2)\frac{1}{\sqrt{a_3^2 - b_3^2 + ia_30}} \ , \tag{V.77}$$

where

$$a_3 \equiv pq + m^2 \ , \ b_3 \equiv -\sqrt{N}\sqrt{1 - \frac{4m^2}{P^2}} \tag{V.78}$$

and

$$N \equiv (p.q)^2 - p^2q^2 \ . \tag{V.79}$$

Following the same procedure that led to expression (V.77) for I, we can perform t calculation of the integrals

$$J(p,q) \equiv \mathcal{I}_1 + \mathcal{I}_5 \tag{V.8}$$

and

$$K(p,q) \equiv \mathcal{I}_2 + \mathcal{I}_4 = -J(q,p) \ . \tag{V.8}$$

The evaluation of J can be done in a frame where $\mathbf{p} = 0$, since in this case p must be time-li in order that the integral not to vanish. Since

$$|p_0||\mathbf{q}| = \sqrt{p_0^2\mathbf{q}^2} = \sqrt{p_0^2(q_0^2 - q^2)} = \sqrt{(p_0q_0)^2 - p_0^2q^2} = \sqrt{N} \ ,$$

in an arbitrary frame,

$$J(p,q) = \frac{\mathrm{sgn}p_0}{4\sqrt{p^2}}\theta(p^2 - m^2)\int_0^{2\pi} d\varphi \frac{1}{a_1 + b_1\cos\varphi - i(p^2 - m^2 - 2pq)0}$$
$$= \frac{\pi}{2}\frac{\mathrm{sgn}p_0}{\sqrt{p^2}}\theta(p^2 - m^2)\left[\frac{\theta(p^2 - m^2 - 2pq)}{\sqrt{a_1^2 - b_1^2 - ia_10}} - \frac{\theta(-p^2 + m^2 + 2pq)}{\sqrt{a_1^2 - b_1^2 + ia_10}}\right] \ , \tag{V.8}$$

with

$$a_1 \equiv q^2 - m^2 - pq\left(1 - \frac{m^2}{p^2}\right) \ , \ b_1 \equiv -\sqrt{N}\left(1 - \frac{m^2}{p^2}\right) \ . \tag{V.83}$$

Using (V.81) and defining

$$a_2 \equiv p^2 - m^2 - pq\left(1 - \frac{m^2}{q^2}\right) \ , \ b_2 \equiv -\sqrt{N}\left(1 - \frac{m^2}{q^2}\right) \ , \tag{V.84}$$

we find an expression for K and, consequently, the scalar integral can be written as

$$\mathcal{I}(p,q) = \frac{\pi}{2(2\pi)^{9/2}}\left\{-\frac{\mathrm{sgn}P_0}{\sqrt{P^2}}\theta(P^2 - 4m^2)\frac{1}{\sqrt{a_3^2 - b_3^2 + ia_30}}\right.$$
$$\frac{\mathrm{sgn}p_0}{\sqrt{p^2}}\theta(p^2 - m^2)\left[\frac{\theta(p^2 - m^2 - 2pq)}{\sqrt{a_1^2 - b_1^2 - ia_10}} - \frac{\theta(-p^2 + m^2 + 2pq)}{\sqrt{a_1^2 - b_1^2 + ia_10}}\right]$$
$$\left.-\frac{\mathrm{sgn}q_0}{\sqrt{q^2}}\theta(q^2 - m^2)\left[\frac{\theta(q^2 - m^2 - 2pq)}{\sqrt{a_2^2 - b_2^2 - ia_20}} - \frac{\theta(-q^2 + m^2 + 2pq)}{\sqrt{a_2^2 - b_2^2 + ia_20}}\right]\right\} \ . \tag{V.85}$$

The algebraic calculation of the vector and tensor integrals is lenghty. Each one can b decomposed in terms proportional, respectively, to $\mathrm{sgn}P_0$, $\mathrm{sgn}p_0$ and $\mathrm{sgn}q_0$, which in turn a constructed from tensor products of the external momenta p and q.

The vector integral is given by

$$\mathcal{I}^\mu(p,q) = \frac{\mathcal{I}(p,q)}{2N}\left\{p^\mu\left[pq(q^2 - m^2) - q^2(p^2 - m^2)\right]\right.$$
$$+q^\mu\left[pq(p^2 - m^2) - p^2(q^2 - m^2)\right]$$

$$+\frac{\pi}{4(2\pi)^{9/2}N}\left\{\frac{\mathrm{sgn}P_0}{\sqrt{P^2}}\theta(P^2-4m^2)\left[p^\mu(pq-q^2)+q^\mu(pq-p^2)\right]\right.$$

$$\left.+\frac{\mathrm{sgn}p_0}{\sqrt{p^2}}\theta(p^2-m^2)\left[q^\mu p^2-p^\mu(pq)\right]-\frac{\mathrm{sgn}q_0}{\sqrt{q^2}}\theta(q^2-m^2)\left[p^\mu q^2-q^\mu(pq)\right]\right\}\ . \tag{V.86}$$

Finally, the tensor integral is given by

$$\mathcal{I}^{\mu\nu}(p,q)=\frac{\mathcal{I}(p,q)}{N}\left\{-q^\mu q^\nu\left\{-\frac{1}{N}\left[(q^2-m^2)p^2-(pq)(p^2-m^2)\right]^2\right.\right.$$

$$\left.+\frac{1}{4}(p^2-m^2)^2\right\}-(p^\mu q^\nu+q^\mu p^\nu)\frac{1}{8}\left\{2(p^2-m^2)(q^2-m^2)\right.$$

$$\left.+\frac{4pq}{N}\left[(q^2-m^2)^2p^2-2(q^2-m^2)(p^2-m^2)pq+(p^2-m^2)^2q^2\right]\right\}$$

$$-p^\mu p^\nu\left\{-\frac{1}{2N}\left[(p^2-m^2)q^2-(pq)(q^2-m^2)\right]^2+\frac{1}{4}(q^2-m^2)^2\right\}$$

$$\left.+g^{\mu\nu}\frac{1}{4}\left[p^2(q^2-m^2)^2-2pq(q^2-m^2)(p^2-m^2)+q^2(p^2-m^2)^2\right]\right\}$$

$$+\frac{\pi}{4(2\pi)^{9/2}N}\left\{\frac{\mathrm{sgn}P_0}{2\sqrt{P^2}}\theta(P^2-4m^2)\left\{p^\mu p^\nu\left[3pq-2q^2+m^2\right.\right.\right.$$

$$\left.-\frac{2(pq+m^2)(pq-q^2)^2}{N}\right]+q^\mu q^\nu\left[3pq-2p^2+m^2\right.$$

$$\left.-2\frac{(pq+m^2)(pq-p^2)^2}{N}\right]+(p^\mu q^\nu+p^\nu q^\mu)\left[pq-p^2-q^2-m^2\right.$$

$$\left.\left.-2\frac{(pq+m^2)(pq-q^2)(pq-p^2)}{N}\right]-g^{\mu\nu}P^2(pq+m^2)\right\}$$

$$+\frac{\mathrm{sgn}p_0}{\sqrt{p^2}}\theta(p^2-m^2)\left\{q^\mu q^\nu\frac{p^2}{N}\left[pq(p^2-m^2)-p^2(q^2-m^2)\right]\right.$$

$$+p^\mu p^\nu\frac{1}{2N}\left\{pq\left(1-\frac{m^2}{p^2}\right)\left[3p^2q^2-(pq)^2\right]\right.$$

$$\left.-(q^2-m^2)\left[(pq)^2+p^2q^2\right]\right\}+(p^\mu q^\nu+p^\nu q^\mu)\frac{p^2}{2N}\left\{2pq(q^2-m^2)\right.$$

$$\left.\left.-(1-\frac{m^2}{p^2})\left[(pq)^2+p^2q^2\right]\right\}+g^{\mu\nu}\frac{p^2}{2}\left[pq\left(1-\frac{m^2}{p^2}\right)-(q^2-m^2)\right]\right\}$$

$$\left.-\frac{\mathrm{sgn}q_0}{\sqrt{q^2}}\theta(q^2-m^2)\times\{p\leftrightarrow q\}\right\}\ . \tag{V.87}$$

The distributions R' (A') for the scalar, vector and tensor integrals are obtained replacing $\mathrm{sgn}p_0$ by $-\theta(-p_0)$ ($-\theta(p_0)$), $\mathrm{sgn}q_0$ by $\theta(q_0)$ ($\theta(-q_0)$) and $\mathrm{sgn}P_0$ by $-\theta(-P_0)$ ($-\theta(P_0)$) in (V.85),

(V.86) and (V.87). The imaginary parts of the functions

$$\Sigma^{(1)}(p,q)\equiv\frac{\theta(p^2-m^2-2pq)}{\sqrt{a_1^2-b_1^2-ia_10}}-\frac{\theta(-p^2+m^2+2pq)}{\sqrt{a_1^2-b_1^2+ia_10}}\ , \tag{V.88}$$

$$\Sigma^{(2)}(p,q)\equiv\frac{\theta(q^2-m^2-2pq)}{\sqrt{a_2^2-b_2^2-ia_20}}-\frac{\theta(-q^2+m^2+2pq)}{\sqrt{a_2^2-b_2^2+ia_20}}\ , \tag{V.89}$$

$$\Sigma^{(3)}(p,q)\equiv\frac{1}{\sqrt{a_3^2-b_3^2+ia_30}} \tag{V.90}$$

are given by

$$\Im m\,\Sigma^{(1)}=-\frac{1}{\sqrt{b_1^2-a_1^2}}\theta(q^2-m^2)\theta(2pq-\frac{p^2-m^2}{q^2-m^2}q^2-\frac{q^2-m^2}{p^2-m^2}p^2)\ , \tag{V.91}$$

$$\Im m\,\Sigma^{(2)}=-\frac{1}{\sqrt{b_2^2-a_2^2}}\theta(p^2-m^2)\theta(2pq-\frac{p^2-m^2}{q^2-m^2}q^2-\frac{q^2-m^2}{p^2-m^2}p^2)\ , \tag{V.92}$$

$$\Im m\,\Sigma^{(3)}=0\ . \tag{V.93}$$

and, therefore, the distributions for the scalar, vector and tensor integrals are real. For the scalar integral we have

$$\Im m\,\mathcal{I}_{R'}(p,q)=\Im m\,\mathcal{I}_{A'}(p,q)$$

$$=\frac{\pi}{2(2\pi)^{9/2}}\frac{1}{\sqrt{2pq(p^2-m^2)(q^2-m^2)-p^2(q^2-m^2)^2-q^2(p^2-m^2)^2}}$$

$$\times\theta(p^2-m^2)\theta(q^2-m^2)\theta(2pq-p^2\frac{q^2-m^2}{p^2-m^2}-q^2\frac{p^2-m^2}{q^2-m^2})$$

$$=\frac{\pi}{2(2\pi)^{9/2}}\frac{\theta(p^2-m^2}{\sqrt{p^2-m^2}}\frac{\theta(q^2-m^2)}{\sqrt{q^2-m^2}}$$

$$\times\frac{\theta(2pq-p^2(q^2-m^2)/(p^2-m^2)-q^2(p^2-m^2)/(q^2-m^2)}{\sqrt{2pq-p^2(q^2-m^2)/(p^2-m^2)-q^2(p^2-m^2)/(q^2-m^2)}}\ . \tag{V.94}$$

If p and q are the 3-momenta of external electrons, the distribution $D^\nu(p,q)$ in (V.54) can be simplified by anticommuting $\not p$ to the left and $\not q$ to the right, taking into account that it appears between the Dirac spinors $\overline{u}(\mathbf{p})$ and $u(\mathbf{q})$. Thus, we can write

$$D^\nu(p,q)=-2m(p^\nu+q^\nu)\mathcal{I}+6pq\gamma^\nu\mathcal{I}+6(p^\nu+q^\nu)\gamma^\mu\mathcal{I}_\mu$$
$$-6\gamma^\nu(p^\mu+q^\mu)\mathcal{I}_\mu+3\gamma^\nu\mathcal{I}^\mu{}_\mu-6\mathcal{I}^\nu{}_\mu\gamma^\mu\ . \tag{V.95}$$

The scalar integral $\mathcal{I}$ diverges on shell, where only the first divergent term survives for $\mathrm{sgn}p_0=\mathrm{sgn}q_0$, while the vector and tensor integrals remain finite in this limit. The first and third terms

in (V.95) have order $\omega = -2$ and the remaining have $\omega = -1$, between the spinors $\overline{u}(\mathbf{p})$ and $u(\mathbf{q})$. However, the last two terms in this expression cancel among themselves.

The retarded distributions retardadas which correspond to each remaining term are obtained through a non-subtracted dispersion relation, setting $\omega = -1$ in (IV.52). The splitting of distribution D^ν must be done before taking the limit $p^2 \to m^2$, $q^2 \to m^2$, in order to circumvent the infrared divergences just mentioned. With those hipotheses, only four terms contribute to the retarded distribution $\hat{R}^\nu$ associated to the vertex function:

$$\hat{R}^\nu(p,q) = \frac{i}{2(2\pi)^{9/2}} \frac{\mathrm{sgn}\, P_0}{\sqrt{P^2}} \sum_{i=1}^{4} \hat{R}^{\nu(i)}(p,q) \ , \tag{V.96}$$

where

$$\hat{R}^{\nu(1)}(p,q) = \frac{p^\nu + q^\nu}{m} \int_{-\infty}^{\infty} dt \frac{\theta(t^2P^2 - 4m^2)}{(1-t+i0)|t^2-1|} \ , \tag{V.97}$$

$$\hat{R}^{\nu(2)}(p,q) = -\frac{3pq}{m^2}\gamma^\nu \int_{-\infty}^{\infty} dt\, t \frac{\theta(t^2P^2 - 4m^2)}{(1-t+i0)|t^2-1|} \ , \tag{V.98}$$

$$\hat{R}^{\nu(3)}(p,q) = \frac{3m(p^\nu + q^\nu)}{pq + m^2} \int_{-\infty}^{\infty} dt\, [1 - \mathrm{sgn}(t^2-1)] \frac{\theta(t^2P^2 - 4m^2)}{t^2(1-t+i0)} \ , \tag{V.99}$$

$$\hat{R}^{\nu(4)}(p,q) = -3\gamma^\nu \int_{-\infty}^{\infty} dt\, [1 - \mathrm{sgn}(t^2-1)] \frac{\theta(t^2P^2 - 4m^2)}{t(1-t+i0)} \ . \tag{V.100}$$

The integrals (V.99) and (V.100) vanish for $P^2 < 4m^2$. The same is true for the distribution $\hat{R}'^\nu$, proportional to $\theta(P^2 - 4m^2)$. In this way the Fourier transform of the three-point function that corresponds to the vertex function, or the second-order current operator, strictly speaking, is given by

$$\Lambda^\nu(p,q) = -i(2\pi)^{7/2}[\hat{R}^\nu(p,q) - \hat{R}'^\nu(p,q)]$$

$$= -\frac{1}{8\pi}\frac{\mathrm{sgn}\, P_0}{\sqrt{P^2}}\frac{(p^\nu + q^\nu)}{m}\left[\log\left(\frac{1-\sqrt{\frac{P^2}{4m^2}}}{1+\sqrt{\frac{P^2}{4m^2}}}\right) - \frac{8m^2}{P^2-4m^2}\sqrt{\frac{P^2}{4m^2}}\right]$$

$$-\frac{3}{8\pi}\frac{\mathrm{sgn}\, P_0}{\sqrt{P^2}}\frac{pq}{m^2}\gamma^\nu\left[\log\left(\frac{1-\sqrt{\frac{P^2}{4m^2}}}{1+\sqrt{\frac{P^2}{4m^2}}}\right) + \frac{8m^2}{P^2-4m^2}\sqrt{\frac{P^2}{4m^2}}\right] \ . \tag{V.101}$$

For $q \to p$ it follows that

$$\Lambda^\nu(p,p) = -\frac{3}{4\pi m}\gamma^\nu \ . \tag{V.102}$$

Therefore, in a two-dimensional space the vertex correction introduces only an electric form factor $-3/(4\pi m)$.

VI. Conclusion and Open Problems

In section II we have seen that the asymptotic bahavior of QED_3 is singular in the infrared region where an infrared phase factor modifies the evolution operator. This singular phase also

appears in the vacuum-vacuum amplitude constructed in section IV, through the causal pha in the S operator which describes the scattering of electrons by an external Coulomb potenti

$$A_\mu(k) = (q/|\mathbf{k}|^2, 0, 0) \ .$$

For small k^2 only the coefficient $\Pi_1^{(1)}$ of the symmetric part of the vacuum polarization tens contributes:

$$\begin{aligned}(\Omega, \mathbf{S}\Omega) &= \exp\left\{-i\pi^2\beta \int d^3k \frac{\mathbf{k}^2}{k^2}\Pi_1^{(1)}(k^2)A_0(k)^2\right\} \\ &\sim \exp\left\{iq^2\frac{8\pi^3}{3}\beta \int dk^0 \int_0^\infty \frac{d\omega}{\omega}\right\} \ .\end{aligned}$$

The last expression exhibits a collinear logarithmic divergence in the case of forward scatterin Thus, the asymptotic particle states in QED_3 are ill-defined in the usual S-matrix formalism. is an open problem whether a construction such as the causal one in reference [33], which w used to arrive at a regularized phase in second-order perturbation theory, yields an infrare divergence free phase.

In section III we have shown that, when Pauli-Villars-Rayski regularization is carefully ir plemented in the standard approach to quantum field theory, the photon mass is not zero b depends on a parameter s, which may be identified with the winding number of homotopical nontrivial gauge transformations. The only way to avoid dependence of the mass on regula ization, and thus find a uniquely defined value for the mass, is to make a special choice of which is related to restoration of parity as commented in the text.

Also in the Causal theory an indeterminacy arises: there, the assumption of minimal spli ting of the causal distribution related to the vacuum polarization tensor fixes the value of t mass, which is the same as in the standard approach for the particular value of s mention above. The assumption of minimal splitting in the causal theory has a stronger physical m tivation than restoration of parity in the standard theory: it likewise fixes the value of t magnetic moment of the electron in QED_4.

As remarked in section I, our proof in section V of dynamical mass generation in the caus approach relies on (I.20), with Π replaced by the one-loop polarization. In condensed matt physics, this is the RPA (random phase approximation) [16]. At present the only justificati of this procedure is by taking the $N \to \infty$ limit of $QED_3(N)$ [34]. Improvement of this result possible by studying various Ansätze which allow a solution of the Schwinger-Dyson equatio (see [34] and references given there). The results of various Ansätze have been compared a numerically tested, and consistency was found, suggesting that dynamical mass generation do occur [34], but no rigorous proof exists, such as the one which was given for the Gross-Nev model in three space-time dimensions [35].

Perhaps the most challenging open problem is a conclusive treatment of the nonrelativis limit of the theory ([3],[4]). One possible approach is through the non-perturbative Ansät [34], but several conceptual problems remain to be solved.

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