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First-order perturbed KdV solitons

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Abstract

We consider the Korteweg-de Vries equation with a perturbation arising naturally in many physical situations. Although being asymptotically integrable, we show that the corresponding perturbed solitons do not have the usual scattering properties. Specifically, we show that there is a solution, correct up to $\mathcal{O}(\epsilon)$, where ϵ is the perturbative parameter, consisting, at $t \to -\infty$, of two superposed deformed solitons characterized by wave-numbers k_1 and k_2, which give rise, for $t \to +\infty$ to the same but phase-shifted superposed solitons, plus a coupling term depending on k_1 and k_2. We also find the condition on the original equation for which this coupling vanishes.

03.40.Kf; 02.30.Jr

Typeset using REVTEX

The importance of certain *integrable partial differential equations* for the description of physical systems comes from the fact that they may describe asymptotic limits of equations supposed to govern real systems. The property of integrability, although being rare in general, in the realm of asymptotic equations is ubiquitous [1]. A most notorious example is that of the Korteweg-de Vries equation (KdV). This equation describes the long-wave limit of spatially extended systems which are conservative and dispersive . The Nonlinear Schrödinger equation (NLS) also enjoys this universality character, describing the slow nonlinear modulation of wave-amplitudes in dispersive systems. The physical systems concerned form a wide range of examples, including, to quote a few [2], water-waves, gas dynamics, plasma physics and waves in ferrites. The concept of soliton is directly associated to equations integrable by the inverse scattering method [3]. Indeed, such equations display N-soliton solutions, and a general solution for sufficiently well-behaved initial conditions evolves asymptotically in time to a superposition of both solitons and radiative terms.

Predictions derived from equations like KdV and NLS, which are obtained from perturbation theory, must pass the test of being stable against higher-order perturbations. To this date, a plethora of results exists concerning the behaviour of solutions of perturbed equations, most frequently obtained through the use of the perturbed inverse scattering transform method [4]. To come to our point, let us first specialize the problem. We will be considering perturbations to KdV equation, that is, equations reading

$$u_t = 6uu_x - u_{xxx} + \epsilon P(u) \quad , \tag{1}$$

where u is a function of (x,t), $P(u)$ is a function of u and its derivatives in x, $\epsilon \ll 1$ is a perturbative parameter, and subscripts denote partial differentiation. We will not be concerned with an arbitrary $P(u)$. Rather, we will be concerned with what could be called *natural* perturbation, namely those arising in perturbative expansions from basic equations that are conservative and dispersive, with a dispersion relation admitting an expansion of the form $\omega(k) = a_1 k + a_3 k^3 + a_5 k^5 + \ldots$. We further assume that no constants scale with ϵ. Under such conditions, a simple scaling analysis shows that the possible form of $P(u)$ is :

$$P(u) = \alpha_0 u_{xxxxx} - 10\alpha_1 u u_{xxx} - 20\alpha_2 u_x u_{xx} + 30\alpha_3 u^2 u_x \quad , \tag{2}$$

where α_i are arbitrary constants. If $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$, then the above equation is integrable and displays N-soliton solutions. For instance, the 1-soliton solution reads simply

$$u = -2k \operatorname{sech}^2 \left\{ k \left[x - \left(4k^2 - 16\alpha_0 \epsilon k^4 \right) t \right] \right\} \quad . \tag{3}$$

Let us now put the central question: given Eq.(1), with the perturbation (2), can we still define the concept of soliton? Explicitly, do we still have a solution which, for $t \to \pm\infty$, is the superposition of functions $\phi_i(x, t, k_i)$, each of them depending only on one k_i, and such that they scatter elastically, differing from $t \to -\infty$ to $t \to +\infty$ by a phase-shift only? It should be remarked that we are working with an equation coming from a perturbative expansion, truncated at $\mathcal{O}(\epsilon)$. Thus, we do not look for exact solutions, but for solutions correct up to $\mathcal{O}(\epsilon)$. Being so, we come to the formalism of asymptotic integrability, as introduced in refs. [5–7].

Suppose we make a ϵ-dependent transformation $u \to w$ of the form

$$u = w + \epsilon \phi(w) \quad , \tag{4}$$

known under the name of near-identity transformation. What has been shown in refs. [5,7] is that Eq.(1), with $P(u)$ given by Eq.(2), can always be transformed to

$$w_t = 6ww_x - w_{xxx} + \alpha_0 \epsilon \left(w_{xxxxx} - 10ww_{xxx} - 20w_x w_{xx} + 30w^2 w_x \right) + \mathcal{O}(\epsilon^2) \quad , \tag{5}$$

which is just the sum of the KdV equation and the first higher-order equation in the KdV hierarchy [3], and its solutions are the solutions of the KdV equation with a renormalized velocity, just like in (3). Thus, in order to find solutions to the original problem, we have just to insert solutions of Eq.(5) into (4). Before giving the explicit form of the near-identity transformation we should stress that it not possible, in general, to extend this procedure to arbitrarily higher orders. This comes from the existence of obstacles to asymptotic integrability as discussed in ref. [6] for the NLS equation, but equally applicable to KdV.

Furthermore, it is possible to find a transformation which maps our original equation (1) to KdV itself, but which is x-dependent [7]. This is, however, not of practical use here.

The correct form of $\phi(w)$ turns out to be

$$\phi(w) = \alpha w^2 + \beta w_{xx} + \gamma w_x \partial^{-1} w \quad , \tag{6}$$

where ∂^{-1}denotes integration with respect to x, and α, β and γ are given in terms of α_i by:

$$\alpha = \frac{1}{3}(10\alpha_0 + 5\alpha_1 - 15\alpha_3) \quad , \tag{7a}$$

$$\beta = \frac{1}{6}(15\alpha_3 - 10\alpha_2 - 5\alpha_0) \quad , \tag{7b}$$

$$\gamma = \frac{10}{3}(\alpha_0 - \alpha_1) \quad . \tag{7c}$$

We can now look for the u coming from the 1-soliton and 2-soliton solutions, for example. This is instructive for the 1-soliton case, but the expression for the 2-soliton case is not very illuminating. For this reason, we are going to look at the asymptotics for $t \to \pm\infty$. This is most easily accomplished by using well known techniques of soliton theory. First, transform from w to F by

$$w = -2(\log F)_{xx} \quad . \tag{8}$$

getting a transformed $\phi(F)$:

$$\begin{aligned}\phi = \frac{4}{F^4}\Bigg[&\left(\alpha + \frac{3\beta}{2}\right) F^2 F_{xx}^2 - (2\alpha + 6\beta + 3\gamma) F F_x^2 F_{xx} \\ &+ (\alpha + 3\beta + 2\gamma) F_x^4 + (2\beta + \gamma) F^2 F_x F_{xx} - \left(\frac{\beta}{2}\right) F^3 F_{xxxx}\Bigg] \quad .\end{aligned} \tag{9}$$

As is well known, if we define $\eta_k = k\left[x - (k^2 - \epsilon\alpha_0 k^4)\, t\right]$, then for a 1-soliton solution we have

$$F = 1 + \exp(\eta_k) \quad , \tag{10}$$

and for a 2-soliton solution.

$$F = 1 + \exp(\eta_{k_1}) + \exp(\eta_{k_2}) + \exp(\eta_{k_1} + \eta_{k_2} + A_{12}) \quad . \tag{11}$$

with $\exp A_{12} = ((k_1 - k_2)/(k_1 + k_2))^2$. Inserting Eq. (10) into (9), and going through the algebra gives us the perturbation to the 1-soliton solution:

$$\begin{aligned}\frac{\phi}{k^4} = &-\left(\frac{\beta+\gamma}{2}\right)\operatorname{sech}^2(\eta_k/2) + \left(\frac{\alpha+3\beta+2\gamma}{4}\right)\operatorname{sech}^4(\eta_k/2) \\ &-\left(\frac{\gamma}{2}\right)\operatorname{sech}^2(\eta_k/2)\tanh(\eta_k/2) \quad .\end{aligned} \tag{12}$$

The first term represents a perturbation to the amplitude of the solitary wave. The second and the third represent deformations of the soliton form. However. in this case the term $\operatorname{sech}^2(\eta_k/2)\tanh(\eta_k/2)$ can always be eliminated. Indeed, in Eq. (4) a term Aw_x can always be added, A being an arbitrary constant which can be adjusted as to eliminate the third term in Eq.(12). This is related to the fact that, if we had tried to solve Eq.(1) by a perturbative series, at $\mathcal{O}(\epsilon)$ we could always sum the solution of the linearized KdV equation. Without this third term, we have the result of ref. [8]. Lets us now go to the 2-soliton case, but let us take into account the existence of an additional term Aw_x from the beginning. Instead of writing out the solution explicitly, we give the asymptotic behaviour for $t \to \pm\infty$, taking the limits by using standard techniques [3]. We obtain the following results. First, define

$$v(\eta_k, k) = k^4\left[-\left(\frac{\beta+\gamma}{2}\right)\operatorname{sech}^2(\eta_k/2) + \left(\frac{\alpha+3\beta+2\gamma}{4}\right)\operatorname{sech}^4(\eta_k/2)\right] \quad . \tag{13}$$

With this definition, we have. for $t \to +\infty$:

$$\begin{aligned}0 = &\, v(\eta_{k_1}, k_1) + v(\eta_{k_2} + A_{12}, k_2) + \left(\frac{Ak_1^3\gamma}{2} - \frac{k_1^4}{2}\gamma\right)\operatorname{sech}^2(\eta_{k_1}/2)\tanh(\eta_{k_1}/2) \\ &+ \left(\frac{\gamma A k_2^3}{2} - \gamma k_1 k_2^3 + \gamma\frac{k_2^4}{2}\right)\operatorname{sech}^2[(\eta_{k_2} + A_{12})/2]\tanh[(\eta_{k_2} + A_{12})/2] \quad ;\end{aligned} \tag{14}$$

and for $t \to -\infty$

$$\begin{aligned}0 = &\, v(\eta_{k_1} + A_{12}, k_1) + v(\eta_{k_2}, k_2) + \left(\frac{Ak_2^3\gamma}{2} - \frac{k_2^4}{2}\gamma\right)\operatorname{sech}^2(\eta_{k_2}/2)\tanh(\eta_{k_2}/2) \\ &+ \left(\frac{\gamma A k_1^3}{2} - \gamma k_2 k_1^3 + \gamma\frac{k_1^4}{2}\right)\operatorname{sech}^2[(\eta_{k_1} + A_{12})/2]\tanh[(\eta_{k_1} + A_{12})/2] \quad .\end{aligned} \tag{15}$$

If we now impose that for $t \to -\infty$ we have a sum of functions, each one depending only on either k_1 or k_2, we must choose $A = 2k_2$. But with this choice, for $t \to +\infty$ we will get a coupling term involving k_1 and k_2. Indeed, we can see that there is no constant A such that for both limits $t \to \pm\infty$ we can have a superposition of functions, none of which depending on the mixed (k_1, k_2)-coupling terms. We see thus that we do not have a process of scattering of two solitons that are uncorrelated asymptotically in time. Even for a solution consisting of the superposition of two 1-solitons at $t \to -\infty$, a linkage appears for $t \to +\infty$ through a (k_1, k_2) coupling. This means that we cannot define two separate particle-like objects that scatter and emerge maintaining their identity. Note however that if $\gamma = 0$ then this linkage disappears and the usual picture of elastic scattering of solitons, although deformed, shows up.

Let us now summarize our results. We have a perturbed KdV equation, where the perturbation is, generically, non-integrable. Instead of looking for an exact solution, we search an expression valid up to $\mathcal{O}(\epsilon)$, as the original equation is supposed to be valid only to this same order. In this sense, we have found the general effects of the perturbation given by Eq.(2), which are summarized in Eqs.(14) and (15). Qualitatively we can say the following: in the general case, the solitons are asymptotically deformed and do not preserve the usual particle-like properties because of the appearance of a linking term involving k_1 and k_2. However, an interesting case emerges when $\alpha_0 = \alpha_1$ in Eq.(2), in which there are particle-like structures, deformed solitons, that have asymptotically, up to $\mathcal{O}(\epsilon)$, the same collisional properties as solitons in integrable systems.

ACKNOWLEDGEMENT

The author would like to thank CNPq and FAPESP (Brazil), for partial support. Fruitfull discussions with J. G. Pereira are also acknowledged.

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