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temperature \***

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# **$Q$ -DEFORMED FERMIONIC LIPKIN MODEL AT FINITE TEMPERATURE**

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## **Abstract**

The interplay between temperature and  $q$ -deformation in the phase transition properties of many-body systems is studied in the particular framework of the collective  $q$ -deformed fermionic Lipkin model. It is shown that in phase transitions occurring in many-fermion systems described by  $su(2)_q$ -like models

are strongly influenced by the  $q$ -deformation.

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## I. INTRODUCTION

$Q$ -deformed algebras turned out to be a fertile area of research in the last few years. Many applications of  $q$ -deformation ideas can be found in the literature, in areas as different as optics or particle physics. A natural scenario for seeking a physical interpretation of the  $q$ -deformation parameter is the many-body physics. The multiple correlations among the constituents of the system may provide us with a framework where the physical effects of such a deformation may show up in an amplified way.

The many-body problem in all its complexity calls for the use of approximate methods or the development of simple solvable models which should entail most of the relevant physics combined with a technically simple treatment [1]. A long heritage of such models is available in the nuclear physics literature, among which the Lipkin model [2] has been extensively used as a laboratory to test approximate methods and to point out the main features of the many-body systems.

From the point of view of  $q$ -deformed algebra applications to physical systems it is important to understand how the basic characteristics and the general behavior of many-body systems are modified when the underlying algebra is deformed. The use of  $q$ -deformed algebra in the description of some many-body systems has led to the appearance of new features when compared to the non-deformed case. In this connection we mention some examples: a) in the  $q$ -oscillator many-body problem [3] it was shown that, when promoting the symmetries of the standard oscillator system to  $q$ -symmetries, the spectrum of the system is found to exhibit interactions between the levels of the individual oscillators, b) the revivals phenomenon present in the Jaynes-Cummings model [4] disappears when the original  $su(2)$  symmetry is deformed, c) a good agreement with the experimental data was obtained through a  $\kappa$ -deformed Poincaré phenomenological fit to the rotational and radial excitations of mesons [5], d) a purely  $su(2)_q$ -based mass formula for quarks and leptons was developed using an inequivalent representation [6], e) phonons in the superfluid  ${}^4\text{He}$  were shown to satisfy the Heisenberg  $q$ -deformed algebra [7], f) the chemical potential  $\mu(T)$  has

a linear dependence on  $T$  in addition to the usual behavior for a free  $q$ -deformed fermionic system [8].

Recently, the quasi-spin version of the Lipkin model has been treated by  $q$ -deforming the  $su(2)$  algebra and the ground state phase transition was discussed at  $T = 0$  by directly diagonalizing the energy matrix constructed out of the representation states  $|jm\rangle$  [9]. It was then found a suppression of that phase transition for  $q > 1$  [10]. Afterwards, a different treatment of the Lipkin model has been developed in which the  $q$ -deformation was performed already at the fermionic level. In this approach a new  $q$ -deformed collective Lipkin Hamiltonian was obtained which differs from that of the previous one by the presence of a  $q$ -deformed mean field term. The reason behind the care in treating the fermionic mean field of the many-body Hamiltonian is the necessity of preserving the symmetry of the original problem when the algebra is deformed. The  $T = 0$  ground state phase transition was studied again by using a variational energy expression constructed out of  $q$ -deformed coherent states. As an outcome of the careful treatment of the fermionic degrees of freedom it emerged that for some values of  $q$ , the phase transition is not suppressed anymore [11]. In the present paper and still in the same spirit of a variational approach, we have included an explicit temperature dependence in the fermionic  $q$ -deformed Lipkin model, in order to study the interplay between temperature and deformation of the algebra. In this context, Gilmore and Feng [12] have treated the same problem for the  $q = 1$  case.

The present paper is organized as follows. In section II we introduce the temperature dependence in the  $q$ -deformed Lipkin many-fermion model. Through the analysis of the free energy associated to this system, using  $q$ -deformed coherent states, we discuss the existence of phase transitions at finite temperature and its dependence with the  $q$ -deformation of the underlying algebra. In section III we present our conclusions.

## II. Q-DEFORMED LIPKIN MODEL WITH TEMPERATURE

The Lipkin model is a valuable tool to test specific methods to treat fermionic many-body systems. Besides, due to its  $su(2)$  structure, it has been widely used in attempts to obtain a physical meaning to the  $q$ -deformation concept as applied to many particle systems [10,11,13]. In Ref. [11] in contrast with [10,13] a careful treatment of the fermionic mean field was performed, which embodies  $q$ -deformation effects. There it has been shown that the  $q$ -deformed fermionic Lipkin Hamiltonian

$$H = \frac{\epsilon}{4 \sinh\left(\frac{\gamma}{2}\right)} \sinh(2\gamma S_0) + \frac{V}{2[N]_q} (S_+^2 + S_-^2). \quad (2.1)$$

written in terms of  $su_q(2)$  operators obeying the commutation relations

$$[S_0, S_{\pm}] = \pm S_{\pm} \quad (2.2)$$

$$[S_+, S_-] = [2S_0]_q \quad (2.3)$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{e^{\gamma x} - e^{-\gamma x}}{e^{\gamma} - e^{-\gamma}}, \quad (2.4)$$

undergoes a second order phase transition, characterizing the spherical symmetry breaking in quasi-spin space.  $q$ -coherent states were used to define  $\theta$  and  $\varphi$  as collective variables in terms of which the phase transition was analyzed through the behavior of the variationally obtained ground state energy. As a result of that analysis the phase transitions depend not only on the strength of the interaction,  $V$ , but also on the deformation of the algebra and on the number of particles through the product  $\gamma(N-1)$ . The  $q$ -dependent critical value of the strength parameter  $\chi$  characterizing the phase transition is

$$\chi_c = 1 + 2 \sinh^2 \left[ \frac{\gamma}{2} (N-1) \right], \quad (2.5)$$

meaning that a universal character can no longer be assigned to  $\chi$  as a system-independent indicator of the phase transition in a  $q$ -deformed system.

In many-body systems undergoing phase transitions, temperature plays a role since it can restore the symmetry. The  $q$ -deformation also acts as a symmetry restoring parameter [11]. It is therefore important to investigate the interplay of both effects in the  $q$ -deformed Lipkin model.

Let us consider again the  $q$ -deformed Lipkin Hamiltonian in order to calculate the free energy and study the phase transitions of the model. In this sense we must also select some test states so as to have a variational expression for the free energy which will be determined by a variational method. In this connection we will use the set of  $q$ -deformed coherent states already discussed in the literature [14,15] and defined by

$$\begin{aligned} |j, z\rangle &= c_q^{z^* j} |j, -j\rangle \\ &= \sum_j \binom{2j}{j+m}_q^{\frac{1}{2}} z^{*(j+m)} |j, m\rangle, \end{aligned} \quad (2.6)$$

to write the free energy inequality

$$F \leq \text{Tr}(\hat{\rho}_{tr} \hat{H}) + \beta^{-1} \text{Tr}(\hat{\rho}_{tr} \ln \hat{\rho}_{tr}), \quad (2.7)$$

where  $\hat{\rho}_{tr}$  is the trial density operator given by

$$\hat{\rho}_{tr} = Y^{-1}(N, j) \sum_{i=1}^{Y(N, j)} |\widetilde{j, z}_i\rangle \langle \widetilde{j, z}_i| \quad (2.8)$$

$Y(N, j)$  is the multiplicity of the degenerate  $j$ -multiplet

$$Y(N, j) = \frac{N! (2j+1)}{\left(\frac{N}{2} + j + 1\right)! \left(\frac{N}{2} - j\right)!}, \quad (2.9)$$

which is the same as in the standard Lipkin model because the  $q$ -deformation procedure does not change the number and labelling of the states [9]. The same reason guarantees that the  $q$ -deformed Lipkin Hamiltonian is block diagonal, being each block associated with a given  $j$ . Using the variational principle we obtain

$$F \leq \min_{j, \theta, \phi} \left( \frac{\langle j, \theta, \phi | \hat{H} | j, \theta, \phi \rangle}{\langle j, \theta, \phi | j, \theta, \phi \rangle} - \beta^{-1} \ln Y(N, j) \right) \quad (2.10)$$

which, by substituting the  $q$ -deformed Lipkin Hamiltonian, Eq. (2.1), gives rise to

$$F \leq \min_{j,\theta,\phi} \left( \epsilon_q E_q(j, \theta, \phi) - \beta^{-1} \ln Y(N, j) \right), \quad (2.11)$$

where

$$\epsilon_q = \frac{\epsilon}{2[1/2]_q}, \quad (2.12)$$

is a  $q$ -dependent single particle spacing, and

$$E_q(j, \theta, \phi) = \frac{[2j]_q}{2} \left( -\frac{\cos \theta}{D(\gamma, \theta, j)} + \frac{\chi(j) \sin^2 \theta \cos 2\phi}{2 D(\gamma, \theta, j)} \right). \quad (2.13)$$

In the above expression the  $q$ -dependent strength parameter now is

$$\chi(j) = \frac{|V|}{\epsilon_q [N]_q} [2j-1]_q \quad (2.14)$$

and

$$D(\gamma, \theta, j) = 1 + \sinh^2 \left[ \frac{\gamma}{2} (2j-1) \right] \sin^2 \theta. \quad (2.15)$$

Minimization on the parameters  $\theta$  and  $\phi$  gives rise to the extrema,

$$\phi = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \quad (2.16)$$

$$\theta = 0 \text{ or } \pi,$$

in the  $V \geq 0$  case. New physics appears due to the interplay between the parameters  $\theta, \gamma, N$  and  $j$ ; the remaining analysis of the behavior of  $E_q(j, \theta, \phi)$  around the extrema goes similarly as in Ref. [11]. However, there we were restricted to the analysis of the ground state behavior, in which case  $j$  is fixed to  $N/2$ . Due to the temperature dependence of the second term of the free energy,  $j$  will not be fixed anymore to the ground state multiplet, being therefore an additional parameter in the present extrema analysis.  $\theta = \pi$  is still a maximum, but the condition at  $\theta = 0$  is now  $j$  dependent

$$-1 + \chi(j) \cos \theta - 2C \cos \theta \frac{(\cos \theta - \frac{\chi(j)}{2} \sin^2 \theta)}{1 + C \sin^2 \theta} = 0 \quad (2.17)$$

where

$$C = \sinh^2 \left[ \frac{\gamma}{2} (2j-1) \right], \quad (2.18)$$

being the solution a maximum or a minimum according to the inequality

$$\text{maximum: } 2C - \chi(j) + 1 < 0 \quad (2.19)$$

$$\text{minimum: } 2C - \chi(j) + 1 > 0.$$

The solutions to the equation 2.17 can be written as

$$\cos \theta = \frac{\chi(j)}{2C} \left( 1 \pm \sqrt{1 - G^2(\gamma)} \right), \quad (2.20)$$

where

$$G^2(\gamma) = \frac{4C(C+1)}{\chi^2(j)}. \quad (2.21)$$

These solutions must still obey the following conditions

$$G(\gamma) \leq 1 \quad (2.22)$$

$$0 \leq |\cos \theta| \leq 1.$$

The first condition is related to the determination of  $\gamma_{\max}$  given by the inequality

$$\epsilon \sinh(\gamma N) \cosh\left(\frac{\gamma}{2}\right) - |V| \leq 0. \quad (2.23)$$

whereas the second one leads to the determination of critical  $j$ ,

$$j_c = \frac{1}{2} + \gamma^{-1} \operatorname{arccoth} \left( \frac{1 + \sqrt{1 - G^2(\gamma)}}{G(\gamma)} \right). \quad (2.24)$$

Once one is given  $j_c$ , one can readily obtain the corresponding  $\chi_c$  by a direct substitution in Eq. (2.14),

$$\chi_c = \frac{\sinh[\gamma(2j_c-1)]}{G(\gamma)}. \quad (2.25)$$

To summarize,  $\gamma < \gamma_{\max}$  and  $j > j_c$  are the necessary and sufficient conditions to the existence of the extremum  $\theta_m$  given by,

$$\cos \theta_m = \frac{\coth \gamma (j - \frac{1}{2})}{G(\gamma)} \left( 1 - \sqrt{1 - G^2(\gamma)} \right), \quad (2.26)$$

leading to  $\left. \frac{\partial^2 E_q}{\partial \theta^2} \right|_{\theta=\theta_m} > 0$ , indicating that  $\theta_m$  is a minimum.

Table I below presents the behavior of the energy extrema with respect to  $j_c$ .

Figure 1 shows the free energy  $F(\beta)$  as a function of  $kT' = 1/\beta$ . As in the non-deformed case, the free energy is given by the lower frontier of the family of straight lines, the main difference lying on the crossing points between the curves, which in the present case are shifted towards higher temperatures. The range of allowed values of  $j$  is given by  $\sqrt{N}/2 \leq j \leq N/2$ , as in the non-deformed case.

When  $j_c$  exists, the crossing of the above mentioned straight lines determines the phase transition temperature given by

$$\beta_c^{-1} = \frac{\min_{\theta, \phi} \epsilon_q E_q(j_c, \theta, \phi) - \min_{\theta, \phi} \epsilon_q E_q(j_c - 1, \theta, \phi)}{\ln Y(N, j_c) - \ln Y(N, j_c - 1)} \quad (2.27)$$

Upon a substitution of the explicit expression of  $E_q(j, \theta, \phi)$  we get

$$\beta_c^{-1} = \frac{\epsilon_q \cosh(\gamma(2j_c - 1))}{\ln Y(N, j_c) - \ln Y(N, j_c - 1)}. \quad (2.28)$$

Figure 2 shows the behavior of  $\beta_c^{-1}$  as a function of  $q = \exp(\gamma)$  in the same physical situation as figure 1.

### III. CONCLUSIONS

It has already been seen previously [11], that the critical value of the coupling constant  $\chi_c$  is a number and  $q$ -deformation dependent at  $T = 0$ . With the introduction of the temperature, as performed in the present work, we could think that no new effect would appear, since the only temperature dependent contribution to the free energy,  $\beta^{-1} \ln Y(N, j)$ , does not contain any  $q$  dependence, being  $Y(N, j)$  just a level degeneracy counter. However, the temperature dependent effects are embodied through  $j_c$ , the temperature dependent critical  $j$  given by Eq.2.24. Similarly, the interplay between temperature and  $q$ -deformation has

also appeared in a  $q$ -bosonic description of the Bose condensation, where the critical temperature increases with the deformation parameter [16].

As shown in Fig. 2, the critical temperature presents strong dependence on the deformation of the algebra. The critical temperature decreases with increasing of the  $q$ -deformation. This means that phase transitions occurring in interacting many-fermion systems described by  $su_q(2)$ -like models are in fact strongly influenced by the  $q$ -deformation, indicating that any physical conclusions regarding the phase transition behavior must be taken with care if the  $q$ -deformed concept has indeed any physical meaning.

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FIGURE CAPTIONS

**Figure 1.** This figure shows the behavior of a family of straight lines for  $j$  values running from  $j = 6$  to  $j = 0$  as a function of  $\beta^{-1}$ , for  $N = 12$ ,  $V = 2$ ,  $\epsilon = 1$  and  $q = 1.116$ . The free energy is the envelope formed by the lower frontier of these straight lines.

**Figure 2.** The critical temperature  $\beta_c^{-1}$  is shown as a function of  $q$ , the deformation parameter of the algebra, for  $N = 12$ ,  $V = 2$ ,  $\epsilon = 1$ .

TABLES

$j$	Minimum	Maximum
$j < j_c$	$\theta = 0$	$\theta = \pm\pi$
$j > j_c$	$\theta = \pm\theta_m$	$\theta = 0, \pm\pi$

TABLE I. The  $\theta$  values at the extrema of the energy functional are shown for different values of  $j$ .

