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Can BCS and BEC be synthesized? *

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Abstract

The superconducting phase is widely believed to be a kind of Bose condensate. Transition temperatures that are substantially higher than BCS T_c values for a many-fermion system interacting pairwise via the familiar BCS interaction model indeed follow from Bose-Einstein condensation (BEC) in either a pure gas of "bosonic" Cooper pairs or in a mixture of the latter plus background fermions when pair-breaking effects are explicitly allowed. Such effects can arise when the pairs either move too fast or are thermally excited. Cooper pairs of definite center-of-mass momentum (CMM) but not definite relative momentum are "bosonic" even though they do not obey the usual Bose commutation relations, since they exhibit indefinite maximum occupation in a given state and thus obey the Bose-Einstein distribution. This transition occurs even in weak coupling, and more significantly even in two dimensions where ordinary BEC is prohibited -- due to the novel ingredient of an (almost) linear, as opposed to quadratic, dispersion relation for the Cooper pair binding energy as function of its CMM. Nonzero-CMM pairs, neglected in the BCS many-fermion theory, are indeed found to be vanishingly small in number, particularly for small coupling and/or for small Debye-to-Fermi-temperature ratio, but play a crucial role. For example, BEC in the pure "bosonic" gas model is possible for all space dimensions > 1, thus allowing *all* known superconductors — for quasi-1D organics through quasi-2D organics or cuprates to fully-3D materials — to be Bose condensates in principle.

1. Introduction

The phenomenon of superconductivity has now been observed in all four broad classes of materials: *metals* (in 1911, by Onnes), *semiconductors* (in 1964 [1]), *polymers* (in 1975 [2]) and *ceramics* (in 1986 [3]). A recent review [4] of some Russian experimental work in certain polymers even suggests the possibility of room-temperature superconductivity.

It seems to be universally believed that, though not yet proved, superconductivity is just another example of Bose-Einstein condensation, as is superfluidity in liquid helium-4 [5] or less directly in liquid helium-3 [6]. The notion of superconductivity as a Bose-Einstein (BE)-like transition of an assembly of bosonic objects is not new, going back at least to Ogg [7] in the forties and to Ginzburg [8] and to Schafroth [9] in the fifties. The idea has resurfaced [10] more recently with the discovery [3] of short-coherence-length cuprate superconductors. Anderson [11] envisages crystalline electrons (or holes) as pair-clusters of excitations that are fermionic, chargeless "spinons" and bosonic, charged "holons"—the latter susceptible [12] to a kind of Bose-Einstein condensation (BEC). Schrieffer and co-workers [13] deal with a bosonic "spin-bag" which is shared by two holes. Friedberg, Lee and Ren [14] achieve fits to the cuprate data of Uemura *et al.* [15] with a BE-like condensation in $2+\epsilon$ dimensions [16] by assuming an effective bosonic pair mass in the direction perpendicular to the copper-oxide planes approaching the pronounced uniaxial anisotropy of 10^5 reported experimentally [17], e.g., in TlBaCaCuO. Indeed, the value of ϵ itself can be determined analytically in idealized situations [12], and is estimated [18] to be about 0.03 in cuprate superconductors—its small but nonzero value being a measure of the coupling between copper-oxide planes. Alexandrov and Mott [19], as well as Ranninger and co-workers [20], concentrate on a bipolaronic picture and note an amazing similarity between at least two cuprate superconductors (with $T_c \simeq 92$K and 107K) and liquid ^{4}He ($T_c \simeq 2.2$K) as regards their empirical specific heat singularities across T_c. Fujita and co-workers [21-24] have generalized the Bardeen-Cooper-Schrieffer (BCS) formalism to include hole-hole (as well as particle-particle) Cooper pairs, stressing in addition that either type of pair (called "pairons") propagate not with a quadratic but with a *linear* dispersion relation, and are bosons which may BE-condense in 2D as well as in 3D according to very specific, unique T_c formulae which differ markedly from the familiar BEC T_c formula associated with quadratic-dispersion-relation bosons. The linear dispersion behavior of Cooper pairs was noted at least as early as 1964 in Schrieffer's monograph [25] on superconductivity.

Indeed, a BE *paradigm* in superconductivity is suggested by the recent T_c *vs* T_F or T_{BE} data extended beyond the cuprates by Uemura *et al.* [26] to virtually *all* exotic superconductors, whether 1D-like, 2D-like or 3D-like, where T_F is the Fermi temperature and the 3D BEC temperature $T_{BE} = 0.218T_F$ if *all* fermions in the original fermion gas are imagined paired. The exotic superconducting samples studied by Uemura *et al.* [26] have T_c values spanning almost three orders of magnitude and reveal an intriguing universal behavior roughly parallel to but shifted *down* from the straight line designating T_{BE} in the "Uemura plot" of T_c *vs* T_F or T_{BE}, thus suggesting a BE mechanism somehow implicit in an appropriately generalized BCS formalism.

Based on earlier work by Eagles [27] and by Leggett [28], Miyake [29] and later Randeria, Duan and Shieh [30] formulated the 2D many-fermion problem at zero absolute temperature ($T=0$) within a BCS formalism whereby *both* the gap equation and the number equation are solved self-consistently [31] but without explicit

reference to the underlying (possibly singular) two-fermion interaction potential which is replaced by a scattering t-matrix. The latter in turn is then related, at low-scattering energies, to the s-wave "scattering length". This self-consistent formulation leads to the usual BCS theory in the limit of weak coupling, and to an ideal gas of tightly-bound, well-separated bosons in the opposite, strong-coupling, limit. The so-called "BCS-Bose crossover" [32] formulation in 2D has been extended to finite T by van der Marel [33], as well as by Drechsler and Zwerger [34] who used an elegant functional integral approach which in lowest-order gives a Ginzburg-Landau theory. Following Ref. [30], a generalized coherence length (or, more precisely, a root-mean-square pair radius) was formulated within the BCS-Bose crossover picture in 1D, 2D and 3D by Casas *et al.* [35], and the 2D case compared with cuprate superconductor data. Their results suggested that these latter materials, among other 3D-like superconductors, might be moderately well described, at least in lowest order, as *weakly-coupled* within the BCS-Bose crossover formalism.

The 3D BCS-Bose crossover problem was incisively analyzed by Nozières and Schmitt-Rink [36]—in fact shortly before the 1986 discovery [3] of high-T_c cuprate superconductivity. Its definitive formulation in two transparent papers [37,38] by Haussmann stressed the vital importance of *triple* self-consistency (*viz.*, in the gap, number *and* single-particle-energy equations; cf. also Ref. [39]). Haussmann employs the Thouless criterion [40] whereby the divergence of the real part of the temperature-dependent t-matrix evaluated at zero momentum and zero frequency leads to a (mean-field) superfluid transition temperature T_c that increases monotonically and smoothly from the weak-coupling (BCS) to the strong-coupling (Bose) extreme, the resulting T_c exactly reproducing, at the two limits, respectively, the BCS T_c formula (given in terms of the s-wave scattering length) as well as the familiar Bose-Einstein condensation temperature formula. Coherence lengths in 3D at $T=0$ have also been calculated [41] over the entire range of coupling/density within the BCS-Bose picture.

In this paper we derive explicit T_c-formulae for BEC in d (>0) dimensions for an ideal gas of identical bosons having a quadratic (Section 2) or a linear dispersion relation (Section 4); Cooper-pair dispersion relations, *viz.*, binding-energy *vs.* center-of-mass-momenta curves are obtained numerically in 2D and in 3D in Section 3 by assuming Coulomb plus electron-phonon interactions mimicked via the familiar BCS interaction model; in Section 5 Cooper pairs are clearly distinguished from familiar elementary excitations such as zero-sound phonons or plasmons; Section 6 elaborates on the Davydov interpretation of the BCS ground state as an ideal mixture of fermion and boson ideal gases; Section 7 sketches a four-fluid statistical model of such a mixture that again leads to substantially higher critical transition temperatures than the BCS theory; and Section 8 gives conclusions.

2. BEC of quadratic-dispersion-relation bosons in any dimension

According to Ref. [16] on an ideal quantum gas of permanent (i.e., number-conserving) bosons in d dimensions, there exists a non-zero absolute temperature T_c below which a macroscopic occupation emerges for a *single* (of an infinitely many) quantum state only if $d > 2$. (The $d = 2$ case, in fact, displays the *same* [42] smooth, singularity-free temperature-dependent specific heat for either bosons or fermions.) The Bose-Einstein distribution summed over all states yields the total number of bosons N_B, each of mass m, of which, say $N_{B,0}$, are in the lowest state $\varepsilon_k = \hbar^2 k^2/2m$ (= 0 in the thermodynamic limit). Explicitly,

$$N_B = N_{B,0} + \sum_{\mathbf{k}>0} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} \tag{1}$$

where $\beta \equiv 1/k_B T$ and $\mu \leq 0$ is the chemical potential. For $T > T_c$, $N_{B,0}$ is negligible compared with N_B; while for $T < T_c$, $N_{B,0}$ is a sizeable fraction of N_B. At precisely $T = T_c$, $N_{B,0} \simeq 0$ and $\mu \simeq 0$, while at $T = 0$, $N_B = N_{B,0}$ (*viz.*, absence of any exclusion principle). To find T_c, the sum in (1) can be converted to an integral over *positive* $k \equiv |\mathbf{k}|$, where $\mathbf{k}$ is a d-dimensional vector. The volume $V_d(R)$ of a hypersphere of radius R in $d \geq 0$ dimensions is given [43] by

$$V_d(R) = \frac{\pi^{d/2} R^d}{\Gamma(1 + d/2)} \tag{2}$$

For $d=3$, this is just $4\pi R^3/3$; for $d=2$ it is the area πR^2 of a circle of radius R; for $d=1$ it is just the "diameter" $2R$ of a line of "radius" R; and for $d=0$ it is unity. Using (2) for $d > 0$, the summation in (1) over our d-dimensional vector $\mathbf{k}$ becomes, in the thermodynamic limit,

$$\sum_{\mathbf{k}} \longrightarrow \left[\frac{2\pi^{d/2}}{\Gamma(d/2)}\right] \left(\frac{L}{2\pi}\right)^d \int dk k^{d-1} \tag{3}$$

with the prefactor in square brackets reducing as it should to 2, 2π and 4π for d = 1, 2 and 3, respectively. Defining the number density in d dimensions through $n_B \equiv N_B/L^d$, (1) with $T = T_c$, $N_{B,0} \simeq 0$ and $\mu \simeq 0$ becomes an elementary integral easily evaluated in terms of the so-called Bose integrals [43] (with $z \equiv e^{\mu/k_B T}$ the so-called fugacity)

$$g_\sigma(z) \equiv \frac{1}{\Gamma(\sigma)} \int_0^\infty dx \frac{x^{\sigma-1}}{z^{-1}e^x - 1} = \sum_{l=1}^{\infty} \frac{z^l}{l^\sigma} \xrightarrow[z\to 1]{} \zeta(\sigma), \tag{4}$$

where $\zeta(\sigma)$ is the Riemann zeta function of order σ. Solving (1) for T_c then gives

$$T_c = \frac{2\pi\hbar^2}{mk_B} \left[\frac{n}{\zeta(d/2)}\right]^{2/d}. \tag{5}$$

This result is formally valid for all $d > 0$. Note, however, that for $0 < d \leq 2$, $T_c = 0$ since $\zeta(\sigma) = \infty$ for $\sigma \leq 1$, the case $d = 2$ dimensions giving the celebrated harmonic series $\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ which diverges. Clearly, for $d = 1$, the series $\zeta(1/2) = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots$ diverges even more severely, etc. All this is consistent with the well-known fact that BEC does *not* occur for quadratic-dispersion-relation bosons for $d \leq 2$ dimensions. For $d = 3$ dimensions (5) becomes

$$T_c = \frac{2\pi\hbar^2 n_B^{2/3}}{mk_B[\zeta(3/2)]^{2/3}} \simeq \frac{3.31\hbar^2 n_B^{2/3}}{mk_B}, \tag{6}$$

since $\zeta(3/2) \simeq 2.612$. This is the familiar T_c-formula for BEC in 3D, a phenomenon finally observed experimentally [44] in ultra-cold alkali-atom gas clouds only recently.

3. Cooper-pair dispersion relations

Let fermions with kinetic energies $\epsilon_k \equiv \hbar^2 k^2/2m^*$ and $\epsilon_{k'} \equiv \hbar^2 k'^2/2m^*$ interact pairwise via the *BCS model interaction*

$$V_{kk'} = \begin{cases} -V & \text{if } E_F - \hbar\omega_D < \epsilon_k, \ \epsilon_{k'} < E_F + \hbar\omega_D \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

with $V > 0$ and $\hbar\omega_D$ the maximum energy of a vibrating-ionic-lattice phonon, where $V_{kk'}$ is the double Fourier transforms of the interaction, and m^* the fermion effective mass.

The total energy (besides the rest masses of the constituents) E_T eigenvalue equation for a (Cooper) pair of fermions interacting via the BCS interaction model and immersed in a background of $N-2$ inert spectator fermions in a spherical Fermi surface (in $\mathbf{k}$-space) of radius k_F is given [45] by

$$1 = V\sum_{\mathbf{k}}{}'[2\epsilon_k - (E_T - \hbar^2K^2/4m^*)]^{-1} \tag{8}$$

where $\hbar\mathbf{K} = \hbar(\mathbf{k}_1 + \mathbf{k}_2)$ is the center-of-mass momentum of the pair, while $\hbar\mathbf{k} = \hbar(\mathbf{k}_1 - \mathbf{k}_2)$ is its relative momentum. The prime on the summation sign denotes the conditions

$$k_F < k_1 \equiv |\mathbf{k} + \frac{1}{2}\mathbf{K}| < (k_F^2 + k_D^2)^{1/2}, \quad k_F < k_2 \equiv |\mathbf{k} - \frac{1}{2}\mathbf{K}| < (k_F^2 + k_D^2)^{1/2} \tag{9}$$

where $\hbar^2k_D^2/2m^* \equiv \hbar\omega_D$ the Debye energy. Setting $E_T \equiv 2E_F - \Delta_K$, the pair is bound if $\Delta_K > 0$ and (8) becomes an eigenvalue equation for the pair (positive) binding energy Δ_K. For $K = 0$ (8) is just

$$1 = V\sum_{\mathbf{k}}{}'[2\epsilon_k - 2E_F + \Delta_0]^{-1} = V\int_{E_F}^{E_F+\hbar\omega_D}\frac{g(\epsilon)d\epsilon}{2\epsilon - 2E_F + \Delta_0}, \tag{10}$$

from which one immediately obtains for the $K = 0$ pair binding energy, *exact* in 2D [as well as in 1D or 3D provided that $\hbar\omega_D \ll E_F$ so that the density of (fermionic) states per spin $g(\epsilon) \simeq g(E_F)$, a constant that can be taken outside the integral], the familiar result

$$\Delta_0 = \frac{2\hbar\omega_D}{e^{2/\lambda} - 1} \xrightarrow[\lambda\to 1]{} 2\hbar\omega_D e^{-2/\lambda}, \tag{11}$$

where $\lambda \equiv g(E_F)V$ is a dimensionless coupling constant. Finite-temperature BCS theory, on the other hand, gives the T_c formula

$$T_c = 1.13\Theta_D e^{-1/\lambda} \tag{12}$$

where $\lambda \leq 1/2$. Since $\Theta_D \simeq 300K$, the critical temperature (12) is at most about 46K. This has been dubbed the "phonon barrier". Since actual superconductors are now known to have $T_c \leq 164K$, the BCS "phonon barrier" of 46K has prompted many workers to search for non-phonon mechanisms such as excitons, plasmons, magnons, etc., that can substitute sizably larger values for the temperature scale Θ_D in (12), and thus lead to higher T_c's. Note that since (11)

yields only $\Delta_0 \leq 11$K (for $\lambda \leq 1/2$ and $\Theta_D \simeq 300$K) compared with the total rest mass of two electrons which is $\simeq 10^{10}$K, a Cooper pair is very weakly bound indeed when compared, say, with the deuteron for which these two energies are, respectively, about 2 MeV and 2,000 MeV, or with the pi-meson as made up of a down and an up quark (each assumed to have a mass ~ 350 Mev/c^2) where the two energies are respectively 560 MeV and 700 MeV..

For $K \geq 0$ and $d = 2$, eq. (8) reduces to

$$1 = \frac{4\lambda}{\pi}\int_0^{\pi/2}d\phi\int_{[1-\kappa^2(1+\nu)sin^2\phi]^{1/2}+\kappa(1+\nu)^{1/2}cos\phi}^{[1+\nu-\kappa^2(1+\nu)sin^2\phi]^{1/2}-\kappa(1+\nu)^{1/2}cos\phi} d\xi\xi[\tilde{\Delta}_\kappa + 2(1+\nu)\kappa^2 - 2 + 2\xi^2]^{-1} \tag{13}$$

where $g(E_F) \equiv L^2m^*/2\pi\hbar^2$ is the 2D density of states; $\xi \equiv k/k_F$; $\kappa \equiv K/2(k_F^2 + k_D^2)^{1/2}$; $\tilde{\Delta}_\kappa \equiv \Delta_K/E_F$; $\nu \equiv \hbar\omega_D/E_F = k_D^2/k_F^2$. For small K, one obtains from (13)

$$\Delta_K \xrightarrow[K\to 0]{} \Delta_0 - \frac{2}{\pi}\frac{[(1+\nu)^{1/2} + e^{2/\lambda}]}{e^{2/\lambda} - 1}\hbar v_F K + O(K^2) \tag{14}$$

which for weak coupling $\lambda \to 0$ reduces to

$$\Delta_K \xrightarrow[\substack{K\to 0\\ \lambda\to 0}]{} \Delta_0 - \frac{2}{\pi}\hbar v_F K + O(K^2). \tag{15}$$

Figure 1 compares the linear approximation (14) to the exact dispersion relation obtained numerically from (13), for the specified values of λ and ν. Indeed, the linear approximation is very good for moderately small λ and ν over the entire range of K values for which $\Delta_K \geq 0$.

For $d = 3$, assuming $\nu \ll 1$ and the 3D density of states $g(E_F) = (L^3/\pi^2\hbar^3)\sqrt{m^{*3}E_F/2}$ eq. (8) becomes

$$1 = 2\lambda\int_0^{\pi/2}d\phi sin\phi\int_{[1-\kappa^2(1+\nu)sin^2\phi]^{1/2}+\kappa(1+\nu)^{1/2}cos\phi}^{[1+\nu-\kappa^2(1+\nu)sin^2\phi]^{1/2}-\kappa(1+\nu)^{1/2}cos\phi} d\xi\xi^2[\tilde{\Delta}_\kappa + 2(1+\nu)\kappa^2 - 2 + 2\xi^2]^{-1} \tag{16}$$

which for small K, using the weak-coupling expression $\tilde{\Delta}_0 \simeq 2\nu e^{-2/\lambda}$, gives

$$\Delta_K \xrightarrow[K\to 0]{} \Delta_0 - e^{4/\lambda}\frac{\sqrt{1-\nu e^{-2/\lambda}}1(2e^{-2/\lambda} + \nu e^{-2/\lambda} + 1)}{e^{2/\lambda}\nu(lnA + lnB) + \nu ln(AB) + 2e^{2/\lambda}\sqrt{1-\nu e^{-2/\lambda}}}$$

$$\frac{1}{(e^{2/\lambda} + 1 - \sqrt{1+\nu})}\hbar v_F K + O(K^2) \tag{17}$$

where $A \equiv \frac{\sqrt{1+\nu} - \sqrt{1-\nu e^{-2/\lambda}}}{\sqrt{1+\nu} + \sqrt{1-\nu e^{-2/\lambda}}}$, and $B \equiv \frac{1+\sqrt{1-\nu e^{-2/\lambda}}}{1-\sqrt{1-\nu e^{-2/\lambda}}}$. For $\nu << 1$, $\lambda \to 0$ and $K \to 0$ eq. (17) reduces to the result cited without proof in [25], namely

$$\Delta_K \xrightarrow[\lambda \to 0]{K \to 0} \Delta_0 - \frac{1}{2}\hbar v_F K + O(K^2). \tag{18}$$

Results in 3D are qualitatively similar to those in 2D illustrated in Fig. 1.

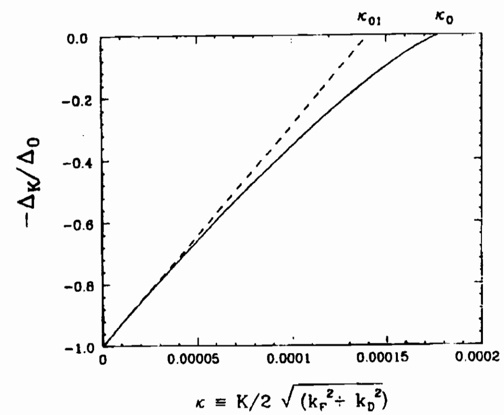

Figure 1: Exact 2D Cooper-pair dispersion relation calculated numerically from (13) for $\lambda = 0.5$ and $\nu = 10^{-2}$, compared with its linear approximation (14).

4. BEC of linear-dispersion-relation bosons in any dimension

Fujita and co-workers [24] (cf. Ref. [43], p. 211) have shown that BEC *is* possible in 2D for bosons with a linear, instead of the usual quadratic, dispersion relation. Photons and phonons are examples of such bosons but, however, are non-number-conserving. Cooper pairs do *not* obey the standard Bose commutation relations and being characterized by definite $\mathbf{k}_1$and $\mathbf{k}_2$ have occupation number 0 or 1, which is the same occupation number for fermions. Following Ref. [21], we shall refer to a *pairon* as a Cooper pair of definite CMM $\mathbf{K}$ only, but we distinguish

it from an ordinary Cooper pair of definite $\mathbf{K}$ *and* $\mathbf{k}$. Pairons do not obey BE commutation relations either, but *do obey the BE distribution* and are hence considered as "quasi-bosons". A detailed proof of this is found in [21], chapter 9, but is also clear from the following. If n_{k_1} is the occupation number, 0 or 1, of a fermion in state $\mathbf{k}_1$, the occupation number for a singlet Cooper pair will be $n_{\mathbf{k}_1\uparrow}n_{\mathbf{k}_2\downarrow}$ and continues to be 0 or 1. Alternately, a given singlet pair can be characterized by $\mathbf{k}$ and $\mathbf{K}$ defined below (8), instead of by $\mathbf{k}_1$ and $\mathbf{k}_2$, in which case the occupation number is say, $\mathcal{N}_{\mathbf{k},\mathbf{K}} = 0$ or 1. Finally, the occupation number of a pairon with specific CMM $\mathbf{K}$ is then $N_{B,\mathbf{K}} \equiv \sum_{\mathbf{k}} \mathcal{N}_{\mathbf{k},\mathbf{K}} = 0, 1, 2, \ldots, QED$.

A Cooper pair is thus a pair of fermions bound just outside the momentum-space Fermi surface enclosing $N-2$ background, inert, spectator fermions of the N-fermion system. It has partner wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ which may or may not add up to a zero center-of-mass wave number $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$. From (15) and (18), a pairon has an excitation energy which is *linear* [25] in K for small K (long wavelength limit), namely,

$$\varepsilon_K \equiv \Delta_0 - \Delta_K \xrightarrow[K \to 0]{} \frac{1}{2} v_F \hbar K, \tag{19}$$

which is valid provided the coupling is small, where Δ_K is the (positive) binding energy of a Cooper pair with net center-of-mass momentum $\hbar K$, and v_F is the Fermi velocity defined by $E_F \equiv \hbar^2 k_F^2/2m^* = \frac{1}{2}m^* v_F^2$.

Although it has been traditionally argued, correctly, in the literature since 1957 that $K = 0$ Cooper pairs are overwhelmingly more tightly bound that $K > 0$ pairs which are ignored in the BCS theory, Fujita and co-workers [24] conjecture that it is precisely the latter pairs that pre-exist at $T > T_c$ and that *drive* BEC at $T = T_c$. Indeed, the number of Cooper pairs with a specific $K \geq 0$ is proportional to a number which is somewhat *less than* (because of finite-temperature smearing effects at the Fermi surface) the overlap volume in $\mathbf{k}$-space swept out by all possible vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ joined head-to-tail as in Figure 2, with both head and tail *within* the energy-shell $\hbar\omega_D$, to give a specific CMM $\mathbf{K}$. This latter overlap volume V_K is just

$$\begin{aligned} V_K = & \int d^3k\, \theta(|\mathbf{K}/2 + \mathbf{k}| - k_F)\theta(|\mathbf{K}/2 - \mathbf{k}| - k_F) \times \\ & \theta(\sqrt{k_F^2 + k_D^2} - |\mathbf{K}/2 + \mathbf{k}|)\theta(|\mathbf{K}/2 - \mathbf{k}| - \sqrt{k_F^2 + k_D^2}) \end{aligned} \tag{20}$$

The integral is exact, though tedious [46], and comprises four distinct regions in the interval $0 < K < 2\sqrt{k_F^2 + k_D^2}$, see below. For $K = 0$ it becomes the volume of the spherical shell, namely

$$V_0 = (4\pi/3)k_F^3[(1+\nu)^{\frac{3}{2}} - 1] \tag{21}$$

The fractional number of pairons with a specific value of K, to those with $K = 0$, will then be somewhat less than

$$\begin{aligned}
\frac{V_K}{V_0} &= \frac{(1+\nu)^{\frac{3}{2}}}{[(1+\nu)^{\frac{3}{2}}-1]}\Big[1-(1+\nu)^{-3/2}+\kappa^3 \\
&\quad -(3/2)\kappa(2+\nu)/(1+\nu)\Big] && \text{if } 0<\kappa<(1-1/\sqrt{1+\nu})/2 \\
&= \frac{(1+\nu)}{\frac{2}{3}[(1+\nu)^{\frac{3}{2}}-1]}[3\nu^2/16\kappa(1+\nu)^{\frac{3}{2}}] && \text{if } (1-1/\sqrt{1+\nu})/2 < \kappa < 1/\sqrt{1+\nu} \\
&= \frac{1}{[(1+\nu)^{\frac{3}{2}}-1]}\Big[-1+(3/16)\nu^2/(\kappa\sqrt{1+\nu}) \\
&\quad +(3/2)\kappa\sqrt{1+\nu}-(1/2)(1+\nu)^{\frac{3}{2}}\kappa^3\Big] && \text{if } 1/\sqrt{1+\nu} < \kappa < (1+1/\sqrt{1+\nu})/2 \\
&= \frac{(1+\nu)^{\frac{3}{2}}}{[(1+\nu)^{\frac{3}{2}}-1]}[1-(3/2)\kappa+(1/2)\kappa^3] && \text{if } (1+1/\sqrt{1+\nu})/2 < \kappa < 1.
\end{aligned} \tag{22}$$

where κ and ν are defined just before (14). Finally, it is clear from Fig. 2 that

$$V_K/V_0 = 0 \text{ if } \kappa \geq 1 \tag{23}$$

These upper bounds are exhibited in Figs. 3 and 4 for different values of ν including the value of $\nu = 3060$ appropriate for the low-carrier concentration ($\simeq 10^{15} cm^{-3}$) superconducting semiconductor $SrTi0_3$ doped with Zr [48]. For $\nu = \infty$ the problem reduces to that of the overlap volume of two solid spheres [49], p. 28, namely $V_K/V_0 = 1 - \frac{3}{2}\kappa + \frac{1}{2}\kappa^3$, to which the last expression of (22) reduces when $\nu >> 1$. In general, note the (small but nonzero) fraction of $K > 0$ pairons to $K = 0$ Cooper pairs, particularly for small ν. Nonetheless, the premise of Refs.[22-24] is that, *without abandoning the phonon mechanism* modeled by (7), superconductivity is really a BEC in either 2D or 3D, of excited ($K > 0$) pairons pre-existing *above* T_c. At $T = 0$ *all* pairons are at rest ($K = 0$), while a mixture of both kinds ($K = 0$ and $K > 0$) is present for $0 < T < T_c$, a $\mathbf{K} = 0$ pairon being an ordinary Cooper pair. For d dimensions we have the general "excitation energy"

$$\varepsilon_K \equiv \Delta_0 - \Delta_K \xrightarrow[K\to 0]{} a(d)v_F\hbar K \tag{24}$$

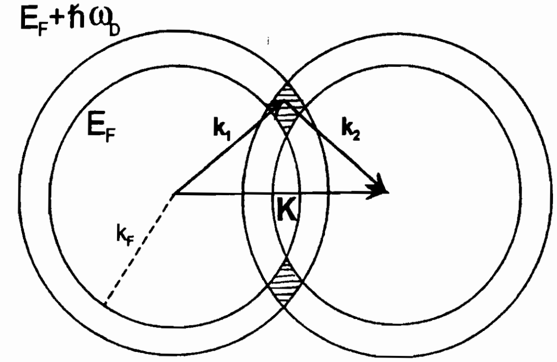

Figure 2: Cross section of overlap volume (cross-hatched) giving an upper bound as explained in text to the number of Cooper pairs with a definite center-of-mass momentum $\hbar\mathbf{K}$, if the pair partners interact via the BCS model interaction (7).

where [21] $a(1) = 1$, $a(2) = 2/\pi$ and $a(3) = 1/2$. Using (24) instead of $\varepsilon_k = \hbar^2k^2/2m$ in (1) and performing the integral implied by (3) gives [47] (for $N_0 \simeq 0$, $\mu \simeq 0$) the weak-coupling T_c-formula in d space dimensions

$$T_c = \frac{a(d)v_F\hbar}{k_B}\left[\frac{\pi^{\frac{d+1}{2}}n_B}{\Gamma(\frac{d+1}{2})\zeta(d)}\right]^{1/d} \tag{25}$$

Since $\zeta(2) = \pi^2/6 \simeq 1.64493$ and $\zeta(3) \simeq 1.20206$, this reduces to the T_c formulae of Refs. [22,23], for 2D and 3D respectively, namely $T_c = 1.244\hbar k_B^{-1}v_Fn_B^{1/2}$ in 2D and $T_c = 1.008\hbar k_Bv_Fn_B^{1/3}$ in 3D [note that the coefficient 1.244 in 2D should replace the coefficient 0.977 of Ref. [22] since $a(2)$ equals $2/\pi$ in 2D instead of the 1/2 mistakenly assumed there]. Note from (25) that $T_c > 0$ for $d > 1$, a result that might conceivably be relevant in understanding quasi-1D organic superconductors [50]. Organic superconductors comprise $(1+\epsilon)$D materials such as the Bechgaard salts, $(2+\epsilon)$D materials like the ET salts and fully-3D materials such as the alkali- and alkaline-earth-doped fullerene crystals called "fulleride" superconductors [51]. The $(1+\epsilon)$D and $(2+\epsilon)$D compounds consist of *coupled* parallel chains and planes, respectively, of atoms.

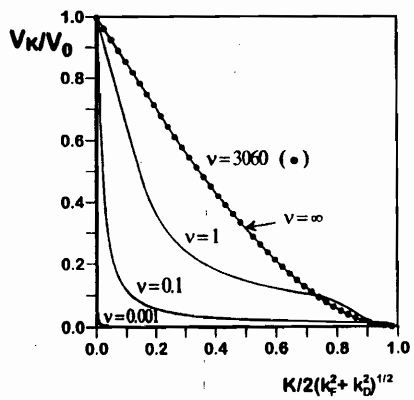

Figure 3: Overlap volume (20) to (22) of two spherical shells of Fig. 2 as function of K, relative to volume when $K = 0$, for various values of $\nu \equiv \Theta_D/T_F \equiv k_D^2/k_F^2$: $\nu = 10^{-3}$ applies to *conventional* superconductors, $0.03 \leq \nu \leq 0.07$ to cuprates; $\nu = 3060$ to Zr-doped $SrTiO_3$ [46]; and the limit $\nu \to \infty$ refers to overlap volume of two solid spheres [49], p. 28.

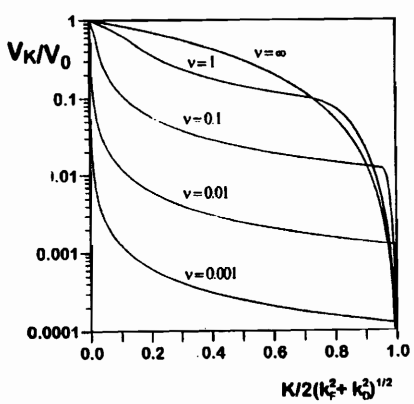

Figure 4: Same as Fig. 3 but on a semilog plot.

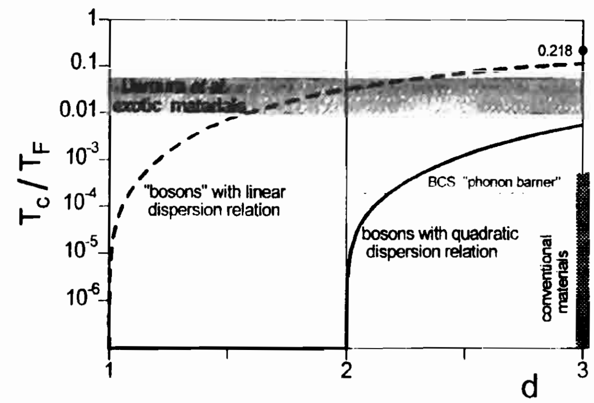

Figure 5: Full curve refers to BEC in d-dimensions according to (5), while dashed curve refers to BEC according to (25) if $a(d) = 1$ is used, for $\Theta_D/T_F = 10^{-3}$ and $n_B/n = d\nu/4$, $m = 2m^*$ as explained in text. The dot at $d = 3$ refers to (5) with $n_B/n = 1/2$ and $m = 2m^*$, namely *all* fermions paired. Light and dark crosshatchings comprise Uemura plot [26] data for *exotic* and *conventional* superconductors, respectively. The thin horizontal line marked BCS "phonon barrier" corresponds to Eq. (12) with $\lambda \leq 1/2$, namely $T_c/T_F \leq (1.13e^{-2})\Theta_D/T_F \simeq 0.153\Theta_D/T_F$ for the case $\Theta_D/T_F = 10^{-3}$.

The large dot in Fig. 5 on the $d = 3$ ordinate denotes the previously mentioned BEC value of T_c, in units of T_F, for a 3-dimensional fermion gas in which we imagine *all* the fermions paired into quadratic-dispersion-relation bosons, i.e., (6) with $n_B = n/2$ and $m = 2m^*$, with $n = k_F^3/3\pi^2$ the 3D fermion-number density. In d-dimension, using (3) for the number of fermions $N = 2\sum_{\mathbf{k}} \theta(k_F - k)$ one obtains $n = k_F^d/2^{d-2}\pi^{d/2}d\Gamma(d/2)$. On the other hand, the number of bosons $N_{B,0}(0)$ actually formed under interaction (7) is $g(E_F)\hbar\omega_D$, where $g(\epsilon) \equiv (L/2\pi)^d d^dk/d\epsilon = (m^*/2\pi\hbar^2)^{d/2}L^d\epsilon^{d/2-1}/\Gamma(d/2)$. Thus, instead of the maximum possible $n_B/n = 1/2$ used before, one really has only $n_B/n = d\hbar\omega_D/4E_F \equiv d\nu/4 \ll 1/2$, which allows rewriting (25) as

$$\frac{T_c}{T_F} = 2a(d)\left[\frac{\nu}{2\Gamma(d)\zeta(d)}\right]^{1/d} \qquad \text{(pure pairon gas).} \tag{26}$$

If the pairons are breakable, i.e., $\Delta_K < 0$ for all $K > K_{01} = \Delta_0 a(d)\hbar v_F$, it is easy to show from (1) and (24) that for $d > 1$

$$\frac{T_c}{T_F} \simeq \nu a(d)^d (d-1)\left(\frac{e^{2/\lambda}}{\upsilon}\right)^{d-1} \xrightarrow[\lambda\to 0]{} \infty. \tag{27}$$

This is expected since for vanishingly small coupling, K_{01} also vanishes so that *all* the pairos are $K = 0$ bosons and the system remains BE-condensed at all finite T. We have plotted (5) in units of T_F and (26) [assuming $a(d) = 1$] *vs.* d in Fig. 5 for the special case $\lambda = 1/2$ and $\nu = 0.05$. This value of υ is intermediate to the range $0.03 \leq \upsilon \leq 0.07$ appropriate for cuprates.

Finally, the average separation $R_0 \sim n_B^{-1/d}$ between pairon centers relative to that between fermions in the normal state $r_0 \sim n^{-1/d}$, is

$$\begin{aligned}\frac{R_0}{r_0} &= (n/n_B)^{1/d} = (4T_F/d\Theta_D)^{1/d} \equiv (4/d\upsilon)^{1/d}\\ &\simeq 6 \qquad \text{(in 2D with } \upsilon = 0.05)\\ &\simeq 37 \quad\ \text{(in 3D with } \upsilon = 10^{-3})\end{aligned} \tag{28}$$

where $n_B/n = d\Theta_D/4T_F$ was used. Since Coulomb repulsions between electrons (or holes) are screened to zero beyond distances of order 1 Å, which is of order r_0, *interactions between pairons of charge 2e are negligible.*

5. Zero-sound phonons, plasmons and pairons contrasted

Pairons are *entities distinct* from zero-sound phonons or plasmons since the former: a) are bounded in number, and b) carry mass $(2m^* - \Delta_0/c^2)$ and fixed charge (2e), while phonons or plasmons *do not* share either property.

Fig.6 compares and contrasts them in the longwavelength limit ($K \to 0$). The dashed quadratic curve is the plasmon dispersion relation [49], p. 180,

$$\omega_K = \omega_P[1 + \frac{9}{10}(K/K_{TF})^2 + \ldots] \tag{29}$$

in the "ring (RPA) approximation" valid for $r_s \equiv r_0/a_0 \equiv (\frac{4\pi}{3}n_F)^{-1/3}/(\hbar^2/me^2) << 1$, where r_0 is an average electron spacing, $n_F \equiv k_F^3/3\pi^2$ being the electron number-density, a_0 the first Bohr radius $\hbar^2/me^2$ with m the electron mass, while the plasmon frequency is $\omega_P \equiv \sqrt{4\pi n_F e^2/m}$ and the "Thomas-Fermi inverse screening length" is $K_{TF} \equiv \sqrt{6\pi n_F e^2/E_F}$ with $E_F \equiv \hbar^2 k_F^2/2m$ as before. The

dot-dashed curve is the weak-coupling zero-sound phonon dispersion curve for repulsive interactions between fermions at $T = 0$, and is given by [49] p.183,

$$\omega_K \simeq [1 + 2e^{-(2\pi^2\hbar^2/mk_F\nu(0) + 2)}]v_F K \tag{30}$$

for $\nu(0) << \hbar^2/mk_F$, where $\nu(\mathbf{q}) \equiv \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}}V(\mathbf{r})$ and $V(\mathbf{r})$ the (repulsive) interparticle interaction potential. The slope of this straight line *rises* as coupling is increased, and assumes the form

$$\omega_K \simeq [\nu(0)/3\pi^2(\hbar^2/mk_F)]^{1/2} v_F K \tag{31}$$

for $\nu(0) >> \hbar^2/mk_F$. Note that (31) can be rewritten as

$$\omega_K^2 \simeq \frac{\nu(0)}{3\pi^2(\hbar^2/mk_F)} v_F^2 K^2 \tag{32}$$

and becomes the plasmon frequency squared ω_P^2 if $\nu(K)$ is taken as the Fourier integral of the Coulomb interaction, $4\pi e^2/K^2$.

On the other hand, for attractive interactions $V < 0$ between the fermions one has the so-called "Anderson mode" [52],

$$\omega_K \simeq [1 - 4g(E_F)|V|]\frac{1}{\sqrt{3}}v_F K \tag{33}$$

in the weak-coupling limit, which is shown as the dotted curve in Fig. 6. Finally, the weak-coupling Cooper pair dispersion relation (19) is represented by the full curve. In 2D, the Anderson mode (33) carries [53] a factor $\frac{1}{\sqrt{2}}$ instead of the $\frac{1}{\sqrt{3}}$ of 3D; it thus also lies higher than the pairon dispersion relation (15) since $1/\sqrt{2} > 2/\pi$.

Using the computer-algebra packages MATHEMATICA and MAPLE, we have verified that the K^2 term in both 2D (15) and 3D (18) *diverges* in the weak coupling limit; curiously, this behavior also occurs for the plasmon, since the quadratic term in (29) can be written as $\hbar K^2/2M_P$ in terms of a "plasmon mass" M_P. This mass

$$M_P \sim \hbar K_{TF}^2/\omega_P \sim \frac{m^{3/2}e}{k_F^{1/2}} \xrightarrow[e\to 0]{} 0, \tag{34}$$

so that it also vanishes in the weak-coupling ($e \to 0$) limit.

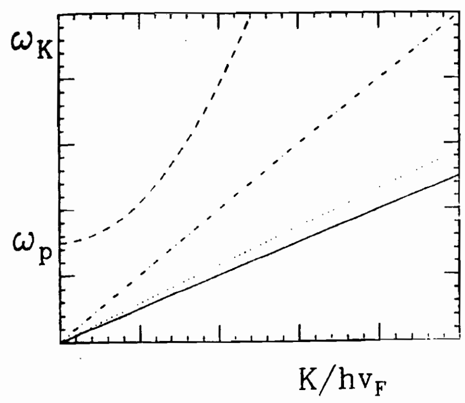

Figure 6: Dispersion curves for: the plasmon (29) (dashed); the weak-coupling (repulsive interaction) zero-sound phonon (30) (dot-dashed); the weak-coupling (attractive interaction) Anderson mode (33) (dotted); and the weak-coupling 3D pairon (19) (full curve).

6. BCS ground state à la Davydov

A surprising property of the BCS theory of the ground ($T = 0$) state of a many-fermion system interacting via the model potential (7) is that the energy shift of the superfluid state E_S with respect to the normal state E_N is, *for any coupling strength*, just the total energy of an *ideal gas* of point bosonic Cooper pairs. This astounding conclusion had not been adequately stressed, to our knowledge, before Davydov [54] (see also Ref. [23] as well as Refs. [30]). It follows directly from the well-known result [55] for the energy shift between the superconducting (S) and normal (N) total many-body energies

$$E_S - E_N = 2g(E_F)\int_0^{\hbar\omega_D} d\epsilon \left[\epsilon - \frac{1}{2}\frac{2\epsilon^2 + \Delta^2}{\sqrt{\epsilon^2 + \Delta^2}}\right], \tag{35}$$

where $g(E_F)$ is defined as in (11). The BCS gap energy is $\Delta = \hbar\omega_D/\sinh[1/g(E_F)V]$, but is *distinct* from the binding energy (11), both being valid for *any* coupling.

As with Eq. (11), in arriving at (35) only $\hbar\omega_D \ll E_F$ has been assumed—*not* weak coupling. Recall, however, that in 2D the assumption $\hbar\omega_D \ll E_F$ is superfluous, since then $g(E_F)$ is a *constant* independent of E_F. The integral (35) can be performed exactly [55], and gives

$$E_S - E_N = g(E_F)(\hbar\omega_D)^2\left[1 - \sqrt{1 + \left(\Delta/\hbar\omega_D\right)^2}\right] \tag{36}$$

$$= -[g(E_F)\hbar\omega_D]\frac{2\hbar\omega_D}{e^{2/g(E_F)V} - 1} \tag{37}$$

$$= -N_{B,0}(0)\Delta_0 \xrightarrow[\lambda\to 0]{} -\tfrac{1}{2}g(E_F)\Delta^2, \tag{38}$$

where Δ was eliminated in the second step, where $g(E_F)\hbar\omega_D$ is precisely the $K = 0$ *pairon* number density $N_{B,0}(0)$ at $T = 0$, provided $\hbar\omega_D \ll E_F$, and use was made of (11) in the last equality. Ironically, (36) and (37) are found in Ref. [55], but *not* the all-important, remarkably simple, and far-reaching equality (38) implying that the BCS ground state corresponds precisely to an ideal gas of composite bosons embedded in an ideal gas of unpaired fermions, for *all* coupling. The final expression in (38) is the well-known and familiar weak-coupling [$\lambda \equiv g(E_F)V \ll 1$] result, where Δ is again the BCS gap energy. Note the crucial difference between Δ and Δ_0 (which in weak coupling reduce to $2\hbar\omega_D e^{-1/\lambda}$ and $2\hbar\omega_D e^{-2/\lambda}$, respectively) that was *required* to arrive at (38), and that is rather frequently neglected in the original and in the textbook literature on the subject. This striking conclusion about the ideal pairon-fermion gas mixture is *not* fully self-consistent since even at $T = 0$ BCS predicts a *smeared* rather than sharp Fermi surface; however, it fullt agrees with a self-consistent result [30] in 2D for *any* interaction describable by an s-wave scattering lenght. This ideal pairon-fermion gas picture supplements the more common interpretation of the BCS *excited* states as an ideal gas of *fermionic* ("bogolon") excitations—a picture, however, valid only in the limit of weak coupling. Note that weak coupling $\lambda \equiv g(E_F)V \ll 1$ is distinct and unrelated to the limit $\hbar\omega_D \ll E_F$.

Besides striking, the conclusion that the BCS ground-state is an ideal (i.e. *interactionless*) gas of *point* pairons embedded in an ideal gas of (unpaired) fermions at *all* couplings, is remarkable because it holds *regardless of how severely the extended pairs actually overlap*. At weak coupling pairs will individually be huge and overlap considerably; for strong coupling pairs are small and well separated. Within the BCS-Bose crossover picture [32] these two extremes are respectively known as the BCS and Bose (or BE) limits.

In essence, therefore, BCS theory is an elegant generalization to *two* particles of the Hartree-Fock (*one*-particle) theory of a many-particle system, both being fundamentally "mean-field" theories.

7. Four-fluid statistical model in 2D

In this section we investigate a statistical model [56] of fermion pairing in two dimensions partly motivated by the Davydov interpretation of the BCS ground state just discussed. This is not a true many-body calculation but nevertheless contains some interesting physics.

The total number of fermions $N = L^2 k_F^2/2\pi$ equals the number of unpaired fermions N_1, plus the number of pairable ones N_2, where $N_2 = 2g(E_F)\hbar\omega_D$. At finite temperature let $N_{20}(T)$ be the number of pairable but (because of thermal and momentum-K pair-breaking) unpaired fermions; this is given by

$$N_{20}(T) = 2\int_{\mu-\hbar\omega_D}^{\mu+\hbar\omega_D} \frac{d\varepsilon\, g(\varepsilon)}{e^{\beta(\varepsilon-\mu_2)}+1} \quad ; \quad \beta \equiv (k_B T)^{-1}, \tag{39}$$

if μ_2 is the fermion chemical potential of the unpaired but pairable fermions. Since $g(\varepsilon)$ is a constant, the integral is exact so that

$$N_{20}(T) = \frac{2g(E_F)}{\beta} \ln\left[\frac{1+e^{-\beta(\mu-\mu_2-\hbar\omega_D)}}{1+e^{-\beta(\mu-\mu_2+\hbar\omega_D}}\right]. \tag{40}$$

The relevant *number equation* is then

$$N_2 = N_{20}(T) + 2[N_{B,0}(T) + N_{B,K>0}(T)] \tag{41}$$

where $N_{B,0}(T)$ is the number of pairons with $K = 0$ at temperature T, while $N_{B,K>0}(T)$ the number with $K > 0$. The latter in turn is just

$$N_{B,K>0}(T) = \sum_{K>0}^{K_0} [e^{\beta\{\bar{\varepsilon}_{B,K}(T)-2\mu_2\}} - 1]^{-1} \tag{42}$$

where

$$\bar{\varepsilon}_{B,K}(T) \equiv 2\mu - \Delta_K \tag{43}$$

is the (thermal) average total energy of a pairon, analogous to the $T = 0$ equation $E_T = 2E_F - \Delta_K$ introduced just below (9). The cutoff K_0 in (42) is defined as

$\Delta_{K_0} \equiv 0$ and is illustrated in Fig. 1. Using (15) and (19), one can define an excitation energy $\varepsilon_K \equiv \Delta_0 - \Delta_K \simeq \frac{2}{\pi}\hbar v_F K$ allowing one to identify $\varepsilon_K - \mu_B$, where μ_B is the bosonic chemical potential, with the factor multiplying β in (42), namely $\varepsilon_{B,K}(T) - 2\mu_2 = \varepsilon_K - \mu_B = \Delta_0 - \Delta_K - \mu_B$ so that (43) leads to

$$\mu_B = 2(\mu_2 - \mu) + \Delta_0. \tag{44}$$

The BEC transition temperature T_c is then given by $\mu_B \simeq 0$ where from (44) $\mu - \mu_2$ in (40) is just $\Delta_0/2$. Hence (41) leads to the equation

$$N_2 = N_{20}(T_c) + 2N_{B,K>0}(T_c). \tag{45}$$

The last quantity in (45) then becomes

$$N_{B,K>0}(T_c) = \frac{L^2}{2\pi}\int_0^{K_{01}} \frac{dK\,K}{e^{\beta_c \frac{2}{\pi}\hbar v_F K} - 1} \tag{46}$$

where $K_{01} = \pi\Delta_0/2\hbar v_F$ in the linear approximation (15) since $\Delta_{K_{01}} \equiv 0$, and $\beta_c \equiv 1/k_B T_c$. Since for weak coupling $\Delta_0 \simeq 2\hbar\omega_D e^{-2/\lambda}$ vanishes, so does K_{01}, allowing the exponential under the integral sign to be expanded to first order, leaving

$$N_{B,K>0}(T_c) \underset{\lambda\to 0}{\simeq} \frac{L^2\pi\Delta_0 m^*}{16\hbar^2}\frac{T_c}{T_F} \tag{47}$$

Note that for quadratic-dispersion-relation bosons one has K^2 instead of K in the exponential, making the integral in (46) diverge in the lower limit so that $T_c \equiv 0$ as expected in 2D. Since $N_2 = 2g(E_F)\hbar\omega_D$, (41) with $\mu - \mu_2 = \Delta_0/2$ in the logarithm in (40) expanded in powers of Δ_0, gives the T_c transcendental equation

$$\tanh(\Theta_D/2T_c) \underset{\Delta_0\to 0}{\simeq} \frac{\pi^2 T_c}{4T_F} \tag{48}$$

or, assuming $\Theta_D/2T_c \ll 1$,

$$T_c \underset{\Delta_0\to 0}{\simeq} \frac{\sqrt{2}}{\pi}\sqrt{\Theta_D T_F} - \ldots \tag{49}$$

the correction term left out tending to *reduce* T_c.

Again using $n_B/n = d\Theta_D/4T_F \equiv d\nu/4$ for $d=2$, (49) becomes

$$\frac{T_c}{T_F} \simeq \frac{2}{\pi}\sqrt{\frac{n_B}{n}} \qquad \text{(boson-fermion gas)} \tag{50}$$

where the bosons are now "breakable" [i.e., the integral (46) is cut off at $K_{01} = \pi\Delta_0/2\hbar v_F$] in contrast to the "unbreakable" bosons in the pure boson gas result (25) following from the integral (4) in (1) which extends over *all* $K > 0$. For this latter model, (25) for $d = 2$, $a(2) = 2/\pi$ and $n_B/n = v/2$ gives

$$\frac{T_c}{T_F} \simeq \left(\frac{2\sqrt{6}}{\pi}\right)\frac{2}{\pi}\sqrt{\frac{n_B}{n}} \qquad \text{(pure unbreakable-boson gas)} \qquad (51)$$

Thus, even though T_c/T_F diverges according to (27) for a pure boson gas of *breakable* pairons, embedding it in a gas of unpaired fermions "tames" this infinite T_c/T_F down to the finite result (50). Figure 7 compares T_c/T_F from (48) (exact) and (49) (approximate) for the boson-fermion four-fluid model, with (25) for the pure unbreakable-boson gas [21], for 2D. The shaded rectangle includes the Uemura *et al.* [15] cuprate data for the empirical range of $v \equiv \Theta_D/T_F$ values given in Ref. [57]. Since in (51) $2\sqrt{6}/\pi \simeq 1.56$, (50) is about 36% lower; this lowering *is due to the presence of unpaired background fermions* plus the fact that pairons break up for $K > K_0$.

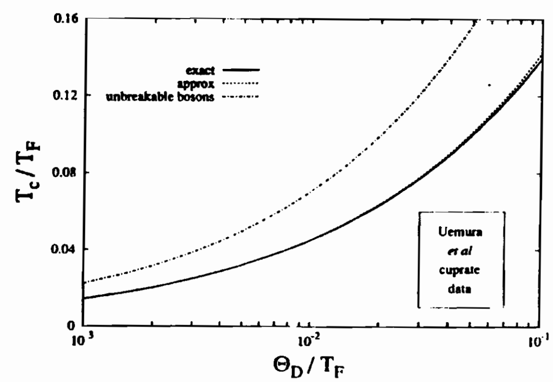

Figure 7: Scaled critical temperatures in 2D BEC for the four-fluid model (48), exact, and (49) approximate, and for the pure unbreakable boson gas model (51), as function of $\nu \equiv \Theta_D/T_F$.

A factor α further reducing T_c can be introduced and called a "pairon-formation suppression factor" (with respect to the ideal spherical, or circular, Fermi surface),

and has three possible origins: i) elemental superconductors have partially hyperboloid Fermi surfaces where electron-electron or hole-hole [22, 23] pairons are formed via acoustic phonons. Since these surfaces are only small parts of the total Fermi surface, α is expected to be quite small;. ii) pairons (either positively- or negatively-charged) are formed in equal numbers from the physical vacuum, meaning that the density of states for the non-predominant [22, 23] fermions, e.g. "holes" in Pb, is the relevant density-of-states entering in (11), thus making α even smaller; iii) "necks" in the Fermi surface are more favorable for pairon formation than "inverted double caps" since the density-of-states is larger around a "neck". This feature appears to explain why face-centered-cubic Pb has a higher T_c than face-centered-cubic Al. For these and possibly other reasons the pairon density n_B in actual superconductors is smaller than that associated with a spherical Fermi surface. On the other hand, if the Fermi surface is spherical or even ellipsoidal but in the first Brillouin zone as in Na, K or other alkali metals, then $\alpha = 0$ exactly; this agrees with the observed fact that alkali metals remain normal down to absolute zero temperature. If a metal Fermi surface is known to contain "necks" and "inverted double caps" (as in Al, Pb, Be, W, etc.) such a metal has a finite, nonzero α and hence $T_c > 0$. An accurate determination of α for a specific substance might conceivably be based on low temperature specific heat comparisons with ideal Fermi gas values corresponding to spherical Fermi surfaces.

8. Conclusions

Bose-Einstein condensation (BEC) T_c-formulae for any positive space dimensionality $d > 0$ are readily derived for a pure gas of bosons with either a quadratic or linear dispersion relation. For the former one recovers the well-known result that $T_c > 0$ only for $d > 2$. On the other hand, for the latter $T_c > 0$ for $d > 1$. This significant difference has a profound impact in the theory of superconductivity. The reason is simply that standard BCS theory can be generalized to include nonzero-center-of-mass-momentum Cooper pairs. Cooper pairs with definite CMM but not definite relative momentum are called *pairons* and move with a *linear* energy-momentum dispersion relation. They undergo BEC not only in 2D but also for any dimension greater than unity. As a result, robustly enhanced T_c values are possible with linear (but not quadratic) dispersion-relation bosons, and the BCS "phonon barrier" of $T_c \leq 46K$ can be "broken" without discarding phonon-mediated interactions and *without assuming strong coupling*.

Not entirely new but scarcely mentioned or understood in the literature is the fact that the BCS ground state describes an *ideal* gas of bosonic Cooper pairs for *all* values of the BCS interaction model coupling. This picture motivates a simple four-fluid mixture statistical model in 2D which continues to give enhanced T_c values even in weak coupling.

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