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THE RESTRICTED DYNAMICS APPROACH FOR $A \leq 60$ NUCLEI AND THE ESTIMATION OF MICROSCOPIC VALUES OF PHENOMENOLOGICAL COLLECTIVE MODEL PARAMETERS

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Abstract

The results of the evaluation of the Generalized Interacting Boson Model (GIBM) Hamiltonian in the framework of Restricted Dynamics Approach have been used in order to estimate the microscopical values for the parameters of two phenomenological collective models - the Bohr-Mottelson Model (BMM) and the standard Interacting Boson Model (IBM). The microscopically derived parameter values are calculated, using the effective NN-interaction potential parameter values obtained from the Strictly Restricted Dynamics Model (SRDM) energy calculations for light α-cluster type nuclei with $8 \leq A \leq 60$, and compared with the phenomenological model parameter values obtained from the direct fit to the experimental level energies of the same nuclei.

1. Introduction

From the point of view of the non-relativistic quantum mechanics one can regard the free nucleus in three-dimensional space as a closed system of well localized A particles - nucleons. The dynamics of this system is defined by its general microscopic Hamiltonian H. At the present stage of nuclear theory development, however, one cannot solve the Schrödinger equation for it. Therefore, H is substituted by the model Hamiltonian H_0, for which it should be possible to solve the corresponding Schrödinger equation.

The traditional way of nuclear model construction is to substitute the original microscopic Hamiltonian H by another - phenomenological Hamiltonian H_{ph}. In most cases H_{ph} is taken from the theory of other physical systems (e.g. as in shell model, or Bohr-Mottelson model), or based on some simplified assumptions about the nuclear structure (IBM, IBFM, etc.). Usually, it is impossible to relate these Hamiltonians with microscopic H. Besides, some of these H_{ph} do not conserve all nuclear integrals of motion.

The Restricted Dynamics Approach (RDA) [1, 2] provides one with the powerful tool for the description of collective phenomena in nucleus via the approximate solution of the microscopic many-body Schrödinger equation. Employing RDA, one can project from the general translationally invariant nuclear Hamiltonian for the system of A nucleons, its collective O_{A-1}-invariant part, depending only on six microscopic collective variables. Although the principles how to obtain microscopic collective models from the general microscopic nuclear Hamiltonian are known, the technique of such evaluations is very complicated. At present the collective part of the microscopic central NN-interaction potential has been evaluated analytically only for the multipole-Gauss type Wigner pontentials. This collective part can be rewritten in the form analogous to the potential energy term of Bohr-Mottelson type models. Again, rewriting it in terms of creation-annihilation operators, one can obtain from it the Generalized Interacting Model Hamiltonian.

The relationships between macroscopic phenomenological collective models and microscopic collective models in RDA for the first time have been studied in Refs.[3, 4] and in more details in Refs.[5, 6].

Let us consider the correspondence chain for nuclear collective Hamiltonians, starting with the General Microscopic Nuclear Hamiltonian (GMNH):

$$\begin{array}{ccccccccc} GMNH & \rightarrow & GRDCM & \rightarrow & GBMM & \rightarrow & GIBM & \rightarrow & IBM \\ & & \downarrow & & \downarrow & & & & \\ & & SRDCM & & BMM & & & & \end{array}$$

where

GRDCM - the general, O_{A-1}-invariant Restricted Dynamics Collective Model;

SRDCM - the Strictly Restricted Dynamics Collective Model - the collective (U_{A-1}-invariant) part of the full Strictly Restricted Dynamics Model (SRDM);

GBMM - the Generalized Bohr-Mottelson Model in 6-dimensional model space;

BMM - the standard phenomenological Bohr-Mottelson Model for nuclear collective motion in 5-dimensional model space (obtained by freezing collective variable ρ^2);

GIBM - the Generalized Interacting Boson Model in 6 dimensions, which conserves U_6 symmetry;

IBM - the standard phenomenological Interacting Boson Model of Arima and Iachello for s, d-bosons in 6 dimensions (denoted in literature as IBM-1).

In the framework of this scheme it is possible to derive the microscopic expressions of the phenomenological collective nuclear model parameters in terms of the nucleon-nucleon interaction potential parameters. The corresponding expressions, accounting for the nuclear kinetic energy as well as the collective potential energy expansion up to the sixth-order terms, have been obtained in Ref.[6]. The aim of present investigation was to estimate numerically the microscopical values of the standard Bohr-Mottelson Model and the standard Interacting Boson Model parameters, using the NN-interaction potential parameters obtained on the basis of SRDM calculations for α-cluster type nuclei with $8 \leq A \leq 60$. The comparison of these estimated parameter values with their values, obtained via the direct level energy fit using phenomenological nuclear models, can give us information about the plausibility of such estimations.

2. General Restricted Dynamics Collective Model (GRDCM) in the case of effective NN-potential of multipole and multipole-Gauss type

Let us consider the general translationally invariant nuclear Hamiltonian for the system of A nucleons, interacting by two-body central forces

$$H^{micr} = T^{micr} + V^{micr} = -\frac{\hbar^2}{2m}\frac{1}{A}\sum_{i<j=2}^{A}\left(\vec{\nabla}_i - \vec{\nabla}_j\right)^2 +$$

$$+ \sum_{i<j=2}^{A}\left(V_W(r_{ij}) + P_{ij}^{(r_{ij})}\cdot V_M(r_{ij}) + P_{ij}^{(\sigma_{ij})}\cdot V_B(r_{ij}) + P_{ij}^{(\tau_{ij})}\cdot V_H(r_{ij})\right), \qquad (1)$$

where for the sake of simplicity we have neglected the difference between protons and neutrons in the operator T^{micr}. The subscripts W, M, B and H in Eq.(1) refer to Wigner, Majorana, Bartlett and Heisenberg interactions, correspondingly. The $P_{ij}^{(\ldots)}$ are the projection operators of spatial, spin and isospin states of nucleon pairs and $r_{ij} = \mid \vec{r}_i - \vec{r}_j \mid$.

Now we shall project the collective part from the Hamiltonian Eq.(1), which, according to RDA, is its O_{A-1}-invariant part. The projection will be carried out in three steps:

1) Instead of the interparticle position vectors $\mid \vec{r}_i - \vec{r}_j \mid$, we shall use the $A-1$ translationally invariant Jacobi vectors [7]

$$\vec{\rho}_i = \frac{1}{\sqrt{i(i+1)}}\left(\sum_{j=1}^{i}\vec{r}_j - i\vec{r}_{i+1}\right); \qquad i = 1, 2, \ldots, A-1. \qquad (2)$$

2) Next, we shall express these vectors in terms of Dzublik-Zickendraht variables [8, 9]

$$\rho_i^{(s)} = \sum_{k=1}^{3}\rho^{(k)}D_{ks}^{(1_3)}(\Theta,\Phi,\Psi)D_{A-4+k,i}^{(1_{A-1})}(\theta_1,\theta_2,\ldots,\theta_{3A-9};\sigma) \qquad (3)$$
$$s = 1,2,3; \qquad i = 1,2,\ldots,A-1.$$

Here $D^{(1_3)}$ and $D^{(1_{A-1})}$ are Wigner matrices of groups O_3^+ and O_{A-1}, correspondingly. These matrices are associated with the defining representations of these groups. $\rho^{(k)}$ are three radial variables; Θ, Φ, Ψ are three Euler angles; $\theta_1, \theta_2, \ldots, \theta_{3A-9}$ is a set of internal angles, and σ denotes two discrete parameters, referring to O_{A-1} group [1, 2]. Let us note, that from all $3(A-1)$ variables, introduced in Eq.(3), only six:

$$\rho^{(1)}, \quad \rho^{(2)}, \quad \rho^{(3)}, \quad \Theta, \quad \Phi, \quad \Psi \qquad (4)$$

(and only these), are scalars with respect to the symmetric group $S_A^{(r)}$, acting on the space variables $\vec{r}_i$ (Ref.[10]). Therefore, they are the collective microscopic variables.

3) Finally, the resulting collective part of the Hamiltonian (Eq.(1)), depending only on six collective variables, one can obtain if one integrates Eq.(1) over $(3A-9)$ internal variables $\theta_1, \theta_2, \ldots, \theta_{3A-9}$, i.e.

$$H_{coll} = H_{coll}\left(\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \Theta, \Phi, \Psi, \sigma_{ij}, \tau_{ij}\right) =$$

$$= \int H^{micr}\left(\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \Theta, \Phi, \Psi; \theta_1, \theta_2, \ldots, \theta_{3A-9}; \sigma_{ij}, \tau_{ij}\right) dV_{\theta_1} dV_{\theta_2}, \ldots, dV_{\theta_{3A-9}} =$$
$$= T_{coll}^{micr} + V_{coll}^{micr}. \qquad (5)$$

The Hamiltonian H_{coll} (Eq.(5)) is the General Restricted Dynamics Collective Model Hamiltonian. The evaluation of the kinetic energy term T_{coll}^{micr} is still an open problem. In order to obtain its expression one should integrate the T^{micr} expression in Dzublik-Zickendraht variables over internal variables, which is a formidable task.

The problem, how to obtain from Eq.(5) explicit expression for the collective part V_{coll}^{micr} of the general microscopic central potential V^{micr}, is still open as well. At present it is solved only for central Wigner potentials presented as

$$V_W^{micr} = V_0 \sum_{i<j=2}^{A}\sum_{t=0,1,2,\ldots} b_t r_{ij}^{2t}. \qquad (6)$$

Expression Eq.(6), despite its simple form, encompasses quite a large class of interactions. Among others the potential Eq.(6) includes also the superposition of multipole-Gauss type interactions:

$$V_{mGW}^{micr} = V_0 \sum_{i<j=2}^{A} (r_{ij})^{2a} e^{-\alpha (r_{ij})^2}, \tag{7}$$

where $a = 0, 1, 2, \ldots$ and α are the potential parameters. In the case of potential Eq.(7) the coefficients b_t of Eq.(6) have the form

$$b_t = \frac{(-\alpha)^{t-a}}{(t-a)!}, \tag{8}$$

where $t = a, a+1, a+2, \ldots$

In Refs.[11, 7] the following result was obtained for the collective potential energy term V_{coll}^{micr} with potential Eq.(6):

$$V_{W\,coll}^{micr}\left(A, \rho^{(1)}, \rho^{(2)}, \rho^{(3)}\right) = \frac{A(A-1)}{2} V_0 \sum_{t=0,1,2,\ldots} b_t \frac{(A-3)!!(2t)!!}{(A+2t-3)!!} \times$$

$$\times \sum_{k_1+k_2+k_3=t} \frac{(2k_1-1)!!(2k_2-1)!!(2k_3-1)!!}{k_1!k_2!k_3!} \left(\rho^{(1)}\right)^{2k_1} \left(\rho^{(2)}\right)^{2k_2} \left(\rho^{(3)}\right)^{2k_3}. \tag{9}$$

3. Derivation of the Microscopic Generalized Bohr-Mottelson Model (GBMM) and the standard Bohr-Mottelson Model (BMM) Hamiltonians from the General Restricted Dynamics Collective Model

Now we shall consider how to obtain the Bohr-Mottelson type collective Hamiltonians (in the generalized 6-dimensional and standard 5-dimensional versions) from the General Restricted Dynamics Collective Model (GRDCM) Hamiltonian described in previous section. In order to associate the GRDCM collective potential energy term Eq.(9) with that of the Bohr-Mottelson model [12], first of all we replace $\left(\rho^{(1)}\right)^2$, $\left(\rho^{(2)}\right)^2$ and $\left(\rho^{(3)}\right)^2$ in Eq.(9) by three new variables (see Refs.[13, 3, 4, 7]):

$$\begin{aligned} \rho^2 &= \left(\rho^{(1)}\right)^2 + \left(\rho^{(2)}\right)^2 + \left(\rho^{(3)}\right)^2, \\ \beta \sin\gamma &= \frac{1}{\sqrt{2}}\left[\left(\rho^{(1)}\right)^2 - \left(\rho^{(2)}\right)^2\right], \\ \beta \cos\gamma &= \frac{1}{\sqrt{6}}\left[\left(\rho^{(1)}\right)^2 + \left(\rho^{(2)}\right)^2 - 2\left(\rho^{(3)}\right)^2\right]. \end{aligned} \tag{10}$$

Then one obtains the final expression (see Ref.[6]) for the collective potential energy term via $\rho^2, \beta^2, \beta^3 \cos 3\gamma$ (which are O_3^+-scalars, as we shall see below):

$$V_{W\,coll}^{micr}\left(A, \rho^2, \beta^2, \beta^3 \cos 3\gamma\right) = \frac{A(A-1)}{2} V_0 \sum_{t=0,1,2,\ldots} b_t \frac{(A-3)!!}{(A+2t-3)!!} \times$$

$$\times \sum_{i+2l+3m=t} C_t(i,l,m) \left(\rho^2\right)^i \left(\beta^2\right)^l \left(\beta^3 \cos 3\gamma\right)^m, \tag{11}$$

where coefficients $C_t(i,l,m)$ one can evaluate as

$$C_t(i,l,m) = \left(-\sqrt{\frac{2}{3}}\right)^m \frac{1}{3^i} \frac{(2t+1)!(2l+2m-1)!!}{i!l!m!(2t-2i+1)!!}. \tag{12}$$

The collective part of the central Wigner interaction (Eq.(6)), written in the form of Eq.(11), is the microscopic generalization of the potential energy term of Bohr-Mottelson type phenomenological models in the sense of RDA, or the potential energy term of the Generalized Bohr-Mottelson Model.

The expression for the kinetic energy term T_{coll}^{micr} of the GBMM, as was told above, is yet to be obtained. For this reason we postulate that the GBMM kinetic energy term is the usual kinetic energy operator in 6-dimensional space.

Let us introduce six new spherical coordinates $q^{[l]m} (l = 0, 2)$, replacing six collective variables $\rho^2, \beta, \gamma, \Theta, \Phi, \Psi$ by O_3^+-irreducible bilinear forms of the relative Jacobi vectors, summed on the particle index, namely:

$$q^{[l]m} = \sum_{j=1}^{A-1} \left(\sum_{m_1 m_2} C^{\,1\;\;1\;\;l}_{\,m_1\;m_2\;m} \rho_j^{m_1} \rho_j^{m_2} \right); \qquad l = 0, 2; \quad -l \le m \le l, \tag{13}$$

which turn out to be the only functions of collective variables one needs. Replacing in Eq.(3) Carthesian index s by the spherical normalized index $m = 0, \pm 1$ and using the explicit expressions for $D^{(1_3)}$ matrices, one obtains from Eq.(13) following results:

$$q^{[0]0} = -\frac{1}{\sqrt{3}} \rho^2, \tag{14}$$

$$q^{[2]m} = -\beta \cos\gamma D_{0m}^{(2)}(\Theta, \Phi, \Psi) + \frac{\beta}{\sqrt{2}} \sin\gamma \left(D_{2m}^{(2)}(\Theta, \Phi, \Psi) + D_{-2m}^{(2)}(\Theta, \Phi, \Psi) \right), \tag{15}$$

where $D_{lm}^{(2)}$ are usual O_3^+ Wigner matrices and ρ, β and γ are given by Eq.(10).

Using Eqs.(14) and (15) one can relate $\rho^2, \beta^2, \beta^3 \cos 3\gamma$ to O_3^+-scalars as follows:

$$\begin{aligned} \rho^2 &= -\sqrt{3} q^{[0]0}, \\ \beta^2 &= \sqrt{5} \left[q^{[2]}, q^{[2]}\right]_0^0, \\ \beta^3 \cos 3\gamma &= \sqrt{\frac{35}{2}} \left[\left[q^{[2]}, q^{[2]}\right]^2, q^{[2]}\right]_0^0, \end{aligned} \tag{16}$$

where

$$\left[q^{[l]}, q^{[l']}\right]_{m''}^{l''} = \sum_{mm'} C^{\,l\;\;l'\;\;l''}_{\,m\;m'\;m''} q^{[l]m} q^{[l']m'}. \tag{17}$$

Therefore we see, that the set of six new collective variables $q^{[l]m}$ is equivalent to the set of previous collective variables $\rho^2, \beta, \gamma, \Theta, \Phi, \Psi$.

Now one can define the kinetic energy operator of the GBMM in the canonical way:

$$T^{micr}_{coll} = -\frac{C_k}{2} \sum_{l=0,2} \sum_{m=-l}^{l} (-1)^{l-m} \frac{\partial}{\partial q_{[l]m}} \frac{\partial}{\partial q^{[l]m}}, \tag{18}$$

where C_k is the kinetic energy constant. Using Eqs.(14) and (15) one can present this operator as

$$T^{GBMM}(\rho^2, \beta, \gamma, \Theta, \Phi, \Psi) =$$

$$= -\frac{3}{2} C_k \frac{\partial^2}{\partial(\rho^2)^2} - 2C_k \left[\sum_{p=1,2,3} \frac{(\hat{L}'_p)^2}{J'_p(\beta,\gamma)} + \frac{1}{\beta^4} \frac{\partial}{\partial\beta} \beta^4 \frac{\partial}{\partial\beta} + \frac{1}{\beta^2} \frac{1}{\sin 3\gamma} \frac{\partial}{\partial\gamma} \sin 3\gamma \frac{\partial}{\partial\gamma} \right] =$$

$$= -\frac{3}{2} C_k \frac{\partial^2}{\partial(\rho^2)^2} + T^{BM}(\beta, \gamma, \Theta, \Phi, \Psi), \tag{19}$$

where $\hat{L}'_p$ are the components of the orbital momentum operator in the internal reference frame of the nucleus. $J'_p(\beta,\gamma)$ is the momentum of inertia

$$J'_p(\beta,\gamma) = 4\beta^2 \sin^2\left(\gamma - p\frac{2}{3}\pi\right); \quad p = 1,2,3, \tag{20}$$

and $T^{BM}(\ldots)$ denotes the usual (5-dimensional) Bohr-Mottelson model kinetic energy operator [13, 3, 12].

Now, the full GBMM Hamiltonian [3] can be presented as

$$H^{GBMM} = T^{GBMM}(\rho^2, \beta, \gamma, \Theta, \Phi, \Psi) + V^{GBMM}(\rho^2, \beta, \gamma), \tag{21}$$

where the potential energy operator is the same one as in the General Restricted Dynamics Model Hamiltonian (see Eq.(11)):

$$V^{GBMM}(\rho^2, \beta, \gamma) \equiv V^{micr}_{W\,coll}\left(A, \rho^2, \beta^2, \beta^3 \cos 3\gamma\right). \tag{22}$$

The standard Bohr-Mottelson Model (BMM) Hamiltonian in 5-dimensional model space one can evaluate from the GBMM Hamiltonian if one "freezes" variable ρ^2. Then one obtains

$$H^{BMM} = T^{BM}(\beta, \gamma, \Theta, \Phi, \Psi) + V^{BM}(\beta, \gamma), \tag{23}$$

where the kinetic energy term $T^{BM}(\ldots)$ is given by the corresponding term in Eq.(19), but $V^{BM}(\ldots)$ has form

$$V^{BM} = \frac{1}{2} C \beta^2, \tag{24}$$

which can be evaluated as a particular case from Eqs.(22),(11). æ

4. Derivation of the Microscopic Generalized Interacting Boson Model (GIBM) Hamiltonian

In order to derive from the GBMM Hamiltonian the microscopic GIBM Hamiltonian let us introduce six creation (η^{lm}) and six annihilation (η^{+lm}) operators as usual:

$$\eta^{lm} = \frac{1}{\sqrt{2}} \left(q^{[l]m} - \frac{\partial}{\partial q_{[l]m}} \right); \quad \eta^{+lm} = \frac{1}{\sqrt{2}} \left(q^{[l]m} + \frac{\partial}{\partial q_{[l]m}} \right). \tag{25}$$

By taking second order expressions in η^{lm} and η^{+lm} one can realize the algebra of the group $Sp(6,R)$, whose compact unitary subgroup U_6 generators are

$$C_{l_1 m_1}{}^{l_2 m_2} = \eta_{l_1 m_1} \eta^{+l_2 m_2}. \tag{26}$$

The contravariant and covariant indices in Eq.(26) are related by

$$\eta_{lm} = (-1)^{l-m} \eta^{l-m}. \tag{27}$$

Taking linear combinations of U_6 generators (Eq.(26)) one constructs all the generators of three U_6 subgroup chains, ending with O_3^+ (Ref.[14]):

$$\begin{array}{lll} U_6 & \supset & U_5 \supset O_5 \supset O_3^+ \\ U_6 & \supset & O_6 \supset O_5 \supset O_3^+ \\ U_6 & \supset & SU_3 \supset O_3^+ \end{array} \tag{28}$$

If one inverts Eq.(25), one gets

$$q^{[l]m} = \frac{1}{\sqrt{2}} \left(\eta^{+lm} + \eta^{lm} \right); \quad \frac{\partial}{\partial q_{[l]m}} = \frac{1}{\sqrt{2}} \left(\eta^{+lm} - \eta^{lm} \right), \tag{29}$$

which allows one to express, using Eq.(16), the invariants $\rho^2, \beta^2, \beta^3 \cos 3\gamma$ in terms of creation and annihilation operators as

$$\begin{aligned} \rho^2 &= -\sqrt{\frac{3}{2}} \left(\eta^{00} + \eta^{+00} \right), \\ \beta^2 &= \frac{1}{2} \sum_m \left(\eta_{2m} + \eta^+_{2m} \right) \left(\eta^{2m} + \eta^{+2m} \right), \\ \beta^3 \cos 3\gamma &= \frac{\sqrt{35}}{4} \left[(\eta_2, \eta_2, \eta_2)^0_0 + \left(\eta^+_2, \eta^+_2, \eta^+_2 \right)^0_0 + 3\left(\eta_2, \eta_2, \eta^+_2 \right)^0_0 + 3\left(\eta_2, \eta^+_2, \eta^+_2 \right)^0_0 \right], \end{aligned} \tag{30}$$

where e.g.

$$(\eta_2, \eta_2, \eta_2)^0_0 = \left[[\eta_2, \eta_2]^2, \eta_2 \right]^0_0 = \frac{1}{\sqrt{5}} \sum_{m_1 m_2 m_3} C^{\,2\ \ 2\ \ 2}_{m_1\ m_2\ m_3} \eta_{2m_1} \eta_{2m_2} \eta^{2m_3}, \tag{31}$$

and similarly for other terms of the $\beta^3 \cos 3\gamma$ expression in Eq.(30).

Now, let us rewrite the GBMM Hamiltonian Eq.(21) in terms of creation-annihilation operators Eq.(25)

$$H^{GBMM}\left(\eta^{lm}, \eta^{+l'm'}\right) = T^{GBMM}\left(\eta^{lm}, \eta^{+l'm'}\right) + V^{GBMM}\left(\eta^{lm}, \eta^{+l'm'}\right), \qquad (32)$$

where

1) the kinetic energy operator is obtained from the canonic expression (Eq.(18)), in which the replacements, given by Eq.(29), are made:

$$T^{GBMM}\left(\eta^{lm}, \eta^{+l'm'}\right) = \frac{C_k}{4}\sum_{lm}\left(1 + 2\eta^{lm}\eta^{+}_{lm}\right) - \frac{C_k}{4}\sum_{lm}\left(\eta^{lm}\eta_{lm} + \eta^{+lm}\eta^{+}_{lm}\right); \qquad (33)$$

2) for the potential energy operator from Eq.(11) one obtains

$$V^{GBMM}\left(\eta^{lm}, \eta^{+l'm'}\right) = \frac{A(A-1)}{2} V_0 \sum_{t=0,1,2,\ldots} b_t R_t \times$$

$$\times \sum_{i+2l+3m=t} C_t(i,l,m)\left[\left(\rho^2(\ldots)\right)^i \left(\beta^2(\ldots)\right)^l \left(\beta^3 \cos 3\gamma(\ldots)\right)^m\right], \qquad (34)$$

where

$$R_t = \frac{(A-3)!!}{(A+2t-3)!!}, \qquad (35)$$

and $\rho^2(\ldots), \beta^2(\ldots), \beta^3 \cos 3\gamma(\ldots)$ denote the corresponding expressions, given by Eq.(30).

Now, we split the microscopic GBMM Hamiltonian (Eq.32)) in two parts, the first of which contains only U_6 generators and constant terms, i.e.,

$$H^{GBMM}\left(\eta^{lm}, \eta^{+l'm'}\right) = H^{GIBM}\left(\eta^{lm}, \eta^{+l'm'}\right) + H'^{GBMM}\left(\eta^{lm}, \eta^{+l'm'}\right). \qquad (36)$$

In Eq.(36) $H^{GIBM}\left(\eta^{lm}, \eta^{+l'm'}\right) = H^{GIBM}$ is the Generalized Interacting Boson Model (GIBM) Hamiltonian - the U_6-restricted part of the GBMM Hamiltonian H^{GBMM}.

Again, one can write H^{GIBM} as

$$H^{GIBM}(\eta^{lm}, \eta^{l'm'}) = T^{GIBM}(\eta^{lm}, \eta^{l'm'}) + V^{GIBM}(\eta^{lm}, \eta^{l'm'}). \qquad (37)$$

The kinetic part easily can be written as U_6-restriction of the kinetic operator (Eq.(33)):

$$T^{GIBM} = \frac{C_k}{4}\sum_{lm}\left(1 + 2\eta^{lm}\eta^{+}_{lm}\right) = \frac{C_k}{2}(\hat{N} + 3). \qquad (38)$$

Here $\hat{N}$ is the operator of the total number of U_6-quanta. Concerning the potential energy operator Eq.(34), when one takes into account Eq.(30), one can easily see, that from the terms $\rho^2(\ldots)$, $\beta^2(\ldots)$, $\beta^3 \cos 3\gamma(\ldots)$ only those with $i + 2l + 3m = t$ =even have U_6-restricted terms.

Finally, from Eq.(37), taking into account Eq.(38) and the $t = 0, 2, 4, 6$ terms of the potential energy expansion Eq.(34), one can write the GIBM Hamiltonian in the following form (for details see Ref.[6]):

$$H^{GIBM} = \frac{1}{2}C_k(\hat{N} + 3) + C_p\Big[b_0 R_0 + b_2 R_2\left(\frac{20}{3}F_2(2,0,0) + 8F_2(0,1,0)\right) +$$

$$+ b_4 R_4\Big(\frac{560}{3}F_4(4,0,0) + 576F_4(0,2,0) + 1344F_4(2,1,0) -$$

$$- 256\sqrt{6}F_4(1,0,1)\Big) + b_6 R_6\Big(\frac{320320}{27}F_6(6,0,0) + \frac{640640}{3}F_6(4,1,0) -$$

$$- \frac{732160\sqrt{6}}{9}F_6(3,0,1) + 549120F_6(2,2,0) - 133120\sqrt{6}F_6(1,1,1) +$$

$$+ 115200F_6(0,3,0) + 20480F_6(0,0,2)\Big)\Big], \qquad (39)$$

where we have used the notation

$$F_t(i,l,m) = \left[\left(\rho^2(\ldots)\right)^i \left(\beta^2(\ldots)\right)^l \left(\beta^3 \cos 3\gamma(\ldots)\right)^m\right]_{|U_6}; \qquad t = i + 2l + 3m, \qquad (40)$$

for the U_6-restriction of the corresponding monomials of O_3^+-scalars. Coefficients R_t in Eq.(39) are given by Eq.(35). Parameters b_t describe the features of the chosen NN-potential, taken in the form of Eq.(6). In the case of multipole-Gauss type potential Eq.(7) b_t are given by Eq.(8). For the superposition of two multipole-Gauss type potentials (Eq.(A.6)) the coefficients $b_t R_t$ are given by Eq.(A.20).

Besides the kinetic energy constant C_k, there is one more constant in Eq.(39) - the potential energy constant C_p. For the Wigner type central NN-interaction it assumes simple form

$$C_p = \frac{A(A-1)}{2}V_0. \qquad (41)$$

One can easily obtain the C_p expression also in more general case, when the central NN-interaction contains Wigner, Majorana, Bartlett and Heisenberg forces of equal potential depth and radial form [4], given by expressions Eq.(6), or Eq.(7):

$$C_p = \left[A_0 c_W + \Lambda(f)c_M + \Lambda(S)c_B + \Lambda(T)c_H\right]V_0, \qquad (42)$$

where $A_0 = A(A-1)/2$ and $\Lambda(f), \Lambda(S), \Lambda(T)$ are the eigenvalues of the symmetric group class operators, consisting of orbital, spin and isospin exchange operators [15, 16]. c_W, c_M, c_B, c_H are the exchange constants of the central interaction.

5. Microscopic derivation of the standard Interacting Boson Model (IBM) Hamiltonian and its microscopic extension

The standard IBM Hamiltonian can be written in several equivalent forms. For our purposes the most convenient form is the one written directly via Casimir invariants (Ref.[4])

$$\begin{aligned} H^{IBM} &= e_0 + e_1\hat{N} + e_2 C_2(U_6) + \varepsilon\hat{n}_d + \alpha C_2(U_5) + \varepsilon_1 \hat{N}\cdot\hat{n}_d + \\ &+ \beta C_2(O_5) + 2\gamma\hat{L}^2 + \delta C_2(SU_3) + \eta C_2(O_6), \end{aligned} \tag{43}$$

where $\epsilon_0, e_1, e_2, \varepsilon, \alpha, \varepsilon_1, \beta, \gamma, \delta, \eta$ are model parameters.

In Eq.(43) all two-boson interaction operators are presented using first and second order Casimir operators (expressed via U_6 generators Eq.(26)) of three U_6 subgroup chains (Eq.(28)). Then,

$$\begin{aligned} C_1(U_6) &= \sum_{lm} C_{lm}{}^{lm} = \sum_{lm} \eta_{lm}\eta^{+lm} = \eta_{00}\eta^{+00} + \sum_m \eta_{2m}\eta^{+2m} = \\ &= \hat{n}_0 + \hat{n}_d = \hat{N}; \end{aligned} \tag{44}$$

$$\begin{aligned} C_2(U_6) &= \sum_{\substack{l_1m_1\\ l_2m_2}} C_{l_1m_1}{}^{l_2m_2} C_{l_2m_2}{}^{l_1m_1} = \sum_{\substack{l_1m_1\\ l_2m_2}} \eta_{l_1m_1}\eta^{+l_2m_2}\eta_{l_2m_2}\eta^{+l_1m_1} = \\ &= \hat{N}(\hat{N}+5); \end{aligned} \tag{45}$$

$$C_1(U_5) = \sum_m C_{2m}{}^{2m} = \sum_m \eta_{2m}\eta^{+2m} = \hat{n}_d; \tag{46}$$

$$\begin{aligned} C_2(U_5) &= \sum_{m_1m_2} C_{2m_1}{}^{2m_2} C_{2m_2}{}^{2m_1} = \sum_{m_1m_2} \eta_{2m_1}\eta^{+2m_2}\eta_{2m_2}\eta^{+2m_1} = \\ &= \hat{n}_d(\hat{n}_d+4); \end{aligned} \tag{47}$$

Four remaining Casimir operators in Eq.(43) one can express via combinations of boson operators in the similar way

$$(C_2(O_6), C_2(O_5), C_2(SU_3), C_2(O_3^+) = \hat{L}^2) = f_{1,2,3,4}(\eta_{lm}, \eta^{+lm}) \tag{48}$$

(see expressions (49)-(54) in Ref.[6]).

The next, most difficult, step in the study of relationships between GIBM and IBM Hamiltonians (Eq.(39) and Eq.(43)) is to obtain, in terms of Casimir invariants of the U_6 subgroup chains Eq.(28) (Eqs.(44)-(48)), the explicit expressions for U_6-restricted monomials $F_t(i,l,m)$ (see Eq.(40)) of O_3^+-scalars, entering in the potential energy terms of Eq.(39):

$$F_t(i,l,m) = f_{t;i;l;m}(\hat{N}, \hat{n}_d, C_2(U_6), C_2(U_5), C_2(O_5), \hat{L}^2, C_2(SU_3), C_2(O_6)). \tag{49}$$

The calculation methods, examples and results of $F_t(i,l,m)$ evaluation for $t = 2,4,6$ values are given in the appendix of our paper [6].

If one inserts the $F_t(i,l,m)$ expressions for $t = 0,2,4,6$ values into the GIBM Hamiltonian (Eq.(39)) and compares the coefficients at corresponding Casimir invariants in Eq.(43) and H^{GIBM} (Eq.(39)) one can derive (see Ref.[6]) following microscopic expressions for the parameters of the standard IBM Hamiltonian (Eq.(43)), via $b_t(t = 0,2,4,6)$ values associated with NN-interaction potential:

$$\begin{aligned} e_0 &= \frac{3}{2}C_k + C_p(b_0R_0 + 30b_2R_2 + 11340b_4R_4 + 15947400b_6R_6); \\ e_1 &= \frac{1}{2}C_k + 20\,C_p(b_2R_2 - 36b_4R_4 + 1797420b_6R_6); \\ e_2 &= 40\,C_p(63b_4R_4 + 38310b_6R_6); \\ \varepsilon &= -12\,C_p(b_2R_2 + 378b_4R_4 + 2566880b_6R_6); \\ \alpha &= -8\,C_p(81b_4R_4 - 240130b_6R_6); \\ \varepsilon_1 &= -16\,C_p(81b_4R_4 - 140690b_6R_6); \\ \beta &= 216\,C_p(b_4R_4 + 1700b_6R_6); \\ \gamma &= -16\,C_p(9b_4R_4 - 11260b_6R_6); \\ \delta &= 48\,C_p(9b_4R_4 - 10180b_6R_6); \\ \eta &= -24\,C_p(9b_4R_4 + 56720b_6R_6). \end{aligned} \tag{50}$$

These results generalize the microscopic IBM Hamiltonian parameter calculations presented in Ref.[4], because we have accounted for the contribution of the kinetic energy term (characterized by the kinetic energy constant C_k), included the $t = 6$ order terms in the potential energy expansion, as well as corrected some errors in $t = 4$ order terms.

6. The Strictly Restricted Dynamics Model (SRDM)

It is a rather difficult task to solve the Schrödinger equation for the full Restricted Dynamics Model Hamiltonian H_{RDM} with realistic NN-interaction potentials. Because of this, in Refs.[1, 17] the simplified version of H_{RDM} – the so called Strictly Restricted Dynamics Model (SRDM) Hamiltonian, was proposed. In this case O_{A-1} group is enlarged to the unitary group U_{A-1} and SO_3 group – to U_3. The detailed description of SRDM and its application for α-cluster nuclei with $A \leq 40$, employing effective NN-interaction potential of multipole-Gauss-exponent type, one can find in our review paper [18].

The general structure of SRDM Hamitonian is

$$H_{SRDM} \equiv H^0 = H^{0coll} + H^{0a}, \tag{51}$$

where H^{0coll} is the U_{A-1}-collective (scalar) part of the general microscopic nuclear Hamiltonian H (Eq.(1)) and H^{0a} – the U_3-scalar part of H, which is responsible for the anticollective effects in the nucleus. The basis function of SRDM can be written as $\Psi\,(N\Lambda_0 \mid \vec{\rho}_1, \vec{\rho}_2, \ldots, \vec{\rho}_{A-1}; Q)$, where:

$\vec{\rho}_i$ are Jacobi vectors (Eq.(2));

Q the set of A-nucleon spin-isospin variables;

Λ_0 - the set of nuclear integrals of motion, which must be conserved according to the symmetry requirements of the many-nucleon system: $\Lambda_0 = (\pi J M_J M_T A a)$;

N - the additional quantum number, denoting both U_3 and U_{A-1} IRs.

In order to separate the orbital and spin-isospin parts in the basis function of SRDM we shall use the Wigner supermultiplet scheme approximation [19, 20]. For the calculations in the framework of SRDM it is more convenient to use the unitary scheme basis functions [21], characterized by the following group reduction chain:

$$\begin{array}{lll} U_{3(A-1)} & \supset & U_3 \times U_{A-1} \\ & & \cup \qquad \cup \\ & & SO_3 \quad O_{A-1} \supset S_A \end{array} \tag{52}$$

In this case quantum numbers of the orbital part of Wigner supermultiplet scheme basis functions have the following meaning:

N_0 - quantum number of the symmetric representation $[N_0, 0, \ldots, 0]$ of group $U_{3(A-1)}$, determining the parity of SRDM state;

$N \equiv (N_1 N_2 N_3)$ - quantum numbers corresponding to IRs of groups U_3 and U_{A-1} in the chain $U_{3(A-1)} \supset U_3 \times U_{A-1}$, where $N_1 + N_2 + N_3 = N_0$;

L - quantum number corresponding to IR of group SO_3, with K denoting it's multiplicity in the chain $U_3 \supset SO_3$;

$\omega = (\omega_1 \omega_2 \omega_3)$ - quantum numbers denoting the IR of O_{A-1} group, and δ - the multiplicity index for ω in the chain $U_{A-1} \supset O_{A-1}$. If $N_0 = N_{min}$, then $\omega_1 = N_1, \omega_2 = N_2, \omega_3 = N_3$, and, therefore, δ is redundant;

f - denotes the IRs of group S_A in the chain $O_{A-1} \supset S_A$ with multiplicity α.

For orbital basis functions of Wigner supermultiplet scheme the reduction rules in the chain $U_{3(A-1)} \supset U_3 \times U_{A-1}$ are well-known [15, 7]. In the $U_3 \supset SO_3$ chain the reduction rules are equivalent to those of $SU_3 \supset SO_3$ chain [7]. If IRs of U_3 group are labelled by $[N_1 N_2 N_3]$, then one can introduce for SU_3 group IRs the Elliott's notation

$$(\lambda\mu) = (N_1 - N_2, N_2 - N_3). \tag{53}$$

In this case, according to Ref.[22], the multiplicity index K assumes following values:

$$K = \min(\lambda, \mu), \min(\lambda, \mu) - 2, \ldots, 0 \quad \text{or} \quad 1, \tag{54}$$

and

$$L = \begin{cases} \max(\lambda, \mu), \max(\lambda, \mu) - 2, \ldots, 0 \text{ or } 1, & \text{if} \quad K = 0, \\ K, K+1, \ldots, K + \max(\lambda, \mu), & \text{if} \quad K \neq 0. \end{cases} \tag{55}$$

Note, that SU_3 basis functions $|(\lambda\mu)KLM>$ in the chain $SU_3 \supset SO_3 \supset SO_2$ are not orthogonal with respect to K, therefore, one must perform the orthogonalization procedure before the diagonalization of SRDM Hamiltonian matrix.

In the chain $U_{A-1} \supset O_{A-1} \supset S_A$ the reduction rules are more complicated [15]. Taking into account that in the unitary scheme (Eq.(52)) groups U_3 and U_{A-1} are complementary [7] (i.e. IRs of U_3 and U_{A-1} are determined by the same set of quantum numbers $[N_1 N_2 N_3]$), one can use for unitary scheme basis the well-known SU_3 classification scheme for nuclei, introduced by Elliott [22].

The problem of the classification of spin-isospin functions of Wigner supermultiplet scheme is reduced to the classification of (S, T) multiplets, entering in IRs $\tilde{f}$ of group $S_A^{(\sigma\tau)}$. Here, (S, T) denotes the pair of IRs λ_σ and λ_τ of groups $S_A^{(\sigma)}$ and $S_A^{(\tau)}$, which for fixed A value are related by expressions $[\lambda_\sigma] = [(A/2) + S, (A/2) - S], [\lambda_\tau] = [(A/2) + T, (A/2) - T]$. Therefore, the classification of (S, T) multiplets is reduced to the reduction

$$[\tilde{f}] = \sum_{\tilde{\alpha}} \tilde{\alpha}(\lambda_\sigma, \lambda_\tau), \tag{56}$$

where $\tilde{\alpha}$ is the multiplicity index of equal pairs $(\lambda_\sigma, \lambda_\tau)$ in $\tilde{f}$, which can be calculated using formula given in Ref.[23].

According to Eq.(51), the SRDM Hamiltonian consists of two parts, accounting for collective and anticollective effects, correspondingly. The general SRDM Hamiltonian matrix structure in the employed Wigner supermultiplet basis functions is:

$$\begin{aligned} & < \begin{matrix} NK(LS)JM_J \\ \delta\omega\alpha(f\tilde{f})\tilde{\alpha}TM_Ta \end{matrix} |H^0| \begin{matrix} NK'(L'S')JM_J \\ \delta'\omega'\alpha'(f'\tilde{f}')\tilde{\alpha}'T'M_Ta \end{matrix} >= \\ & = \left[H^0(N, J, M_T)\right]_{KLS\delta\omega\alpha f\tilde{\alpha}T, K'L'S'\delta'\omega'\alpha' f'\tilde{\alpha}'T'} = \\ & = \left[H^{0coll}(N, J, f, T, M_T)\right]_{KLS\tilde{\alpha}, K'L'S'\tilde{\alpha}'} \oplus \left[H^{0a}(N, S, M_T)\right]_{\delta\omega\alpha f\tilde{\alpha}T, \delta'\omega'\alpha' f'\tilde{\alpha}'T'}. \end{aligned} \tag{57}$$

Indices, placed in brackets (...) at Hamiltonian terms, denote quantum numbers which are conserved. Therefore, the general SRDM Hamiltonian matrix is numerated by nine quantum numbers: $K, L, S, \delta, \omega, \alpha, f, \tilde{\alpha}, T$.

The evaluation of SRDM Hamiltonian matrix Eq.(57) is described in Ref.[18]. The general structure of its matrix elements is

$$H_{SRDM} = H_{kin}^{coll} + H_e^{coll} + H_c^{coll} + H_v^{coll} + H_t^{coll} + H_{kin}^a + H_e^a + H_c^a, \tag{58}$$

where separate terms denote collective (coll) and anticollective (a) kinetic (kin) and Coulomb (e) energies as well as the central (c), vectorial (v) and tensorial (t) NN-interactions.æ

7. The results of α-cluster type nuclei energy calculations in the framework of SRDM

The α-cluster type nuclei with $A = 4, 8, \ldots$ are characterized by the Young pattern $[f] = [4^{A/4}]$ and the spin-isospin pair $(S, T) = (0, 0)$. The SRDM quantum numbers for α-cluster type nuclei with $A = 4, 8, \ldots, 80$ are given in Table 1.

Table 1: The characteristics of α-cluster type nuclei with $4 \leq A \leq 80$

Nucleus	$[f]$	$[N_1N_2N_3]$	$(\lambda\mu)$	K
^4He	$[4]$	[0,0,0]	(0,0)	0
^8Be	$[4^2]$	[4,0,0]	(4,0)	0
^{12}C	$[4^3]$	[4,4,0]	(0,4)	0
^{16}O	$[4^4]$	[4,4,4]	(0,0)	0
^{20}Ne	$[4^5]$	[12,4,4]	(8,0)	0
^{24}Mg	$[4^6]$	[16,8,4]	(8,4)	0,2,4
^{28}Si	$[4^7]$	[16,16,4]	(0,12)	0
^{32}S	$[4^8]$	[20,16,8]	(4,8)	0,2,4
^{36}Ar	$[4^9]$	[20,20,12]	(0,8)	0
^{40}Ca	$[4^{10}]$	[20,20,20]	(0,0)	0
^{44}Ti	$[4^{11}]$	[32,20,20]	(12,0)	0
^{48}Cr	$[4^{12}]$	[40,24,20]	(16,4)	0,2,4
^{52}Fe	$[4^{13}]$	[48,24,24]	(24,0)	0
^{56}Ni	$[4^{14}]$	[52,32,24]	(20,8)	0,2,4,6,8
^{60}Zn	$[4^{15}]$	[56,36,28]	(20,8)	0,2,4,6,8
^{64}Ge	$[4^{16}]$	[56,48,28]	(8,20)	0,2,4,6,8
^{68}Se	$[4^{17}]$	[56,56,32]	(0,24)	0
^{72}Kr	$[4^{18}]$	[60,56,40]	(4,16)	0,2,4
^{76}Sr	$[4^{19}]$	[60,60,48]	(0,12)	0
^{80}Zr	$[4^{20}]$	[60,60,60]	(0,0)	0

The SRDM Hamiltonian for α-cluster type nuclei, following from Eq.(58) if one notices that for the states given in Table 1 the vectorial and tensorial terms of NN-interaction vanish, has the form:

$$H_{SRDM} = H^{coll}_{kin} + H^{coll}_{e} + H^{coll}_{c} + H^{a}_{kin} + H^{a}_{e} + H^{a}_{c}. \tag{59}$$

Joining together all expressions for particular terms of SRDM Hamiltonian Eq.(59) (for details see Ref.[18]), the following final expression for the SRDM matrix elements in the case of α-cluster type nuclei can be obtained:

$$\frac{E_{kin}}{2}\left(B^{(\lambda\mu)L}_{KK'}+1\right) + \frac{e^2}{8}\sqrt{\nu_A}A(A-2)E^{coll}_{eW}(KL,K'L)+$$
$$+C_1\left[\frac{3A(A+4)}{16}E^{coll}_{cW}(KL,K'L)+E^{+a}_{c}\right]+C_2\left[\frac{5A(A-4)}{16}E^{coll}_{cW}(KL,K'L)+E^{-a}_{c}\right]+$$
$$+\frac{e^2}{6}\left[E^{+a}_{e}+\frac{9}{5}E^{-a}_{e}\right]-EB^{(\lambda\mu)L}_{KK'}=0. \tag{60}$$

The quantities entering in Eq.(60) have the following meaning:

The oscillator frequency parameter ν_A (or the wave function scaling factor r_ψ) is adopted according to Ref.[16], and it is equal to

$$\nu_A = \left(\frac{1}{r_\psi}\right)^2 = \frac{5(N_0+3(A-1)/2)}{3r^2_{00}A^{5/3}}, \tag{61}$$

where r_{00}, entering in the nuclear radius expression $R = r_{00}A^{1/3}$, we shall regard as a model parameter.

The model parameters C_1 and C_2 are defined by expressions:

$$\begin{aligned} C_1 &= V_{c0}\,(c_W + c_M)\,, \\ C_2 &= V_{c0}\left[(c_W - c_M) + \frac{4}{5}\,(c_B + c_H)\right]. \end{aligned} \tag{62}$$

These parameters are two independent combinations of central force exchange constants (subjected to the normalization condition [24] $c_W + c_M + c_B - c_H = -1$) together with the common NN-interaction potential depth V_{c0}, assuming that the effective central NN-interaction potential V^c_{NN} has the same shape for Wigner (W), Majorana (M), Bartlett (B) and Heisenberg (H) forces.

E_{kin} denotes the kinetic energy term

$$E_{kin} = 20.7357\nu_A\,(N_0 + (3A-1)/2))\ [MeV]. \tag{63}$$

The collective submatrix elements of orbital operators for Wigner interaction are given by the expression:

$$E^{coll}_{c,eW}(KL,K'L) = \sum_{s=0}^{N_1} I^{r_\psi}_{ss}(V^c_W, V^e_W)Q^{N,coll}_{ss,W}(KL,K'L), \tag{64}$$

where $Q^{N,coll}_{ss,W}(KL,K'L)$ denotes the collective (U_{A-1}-invariant) density matrices in compact form and $I^{r_s}_{ss}(V^c_W, V^e_W)$ are Talmi integrals of central (V^c_W), or Coulomb (V^e_W) interaction. Some details about collective density matrices are given in Ref.[18], and their calculation technique is described in Refs.[25, 26, 27, 28, 29, 30]. The evaluation of Talmi integrals for the effective NN-potential employed in our calculations is described in Ref.[31] and Appendix A.

$E^{\pm a}_{c,e}$ in Eq.(60) denotes the anticollective (SU_3-invariant) orbital matrix elements of central (c) and Coulomb (e) interactions (see Eq.(B.12)). The details of their evaluation one can find in Appendix B.

The SRDM Hamiltonian matrices are given in the Elliott's basis $|(\lambda\mu)KL>$, and the quantities $B^{(\lambda\mu)L}_{KK'}$, entering in Eq.(60), can be expressed [18] via the overlap integrals $<(\lambda\mu)KL|(\lambda\mu)K'L>$ of SU_3 basis functions (see Ref.[32] and Appendix D of Ref.[18]). Applying to Eq.(60) the orthogonalization and diagonalization procedures, we obtain eigenvalues (energies E) and mixing amplitudes of the SRDM Hamiltonian (Eq.(59)).

The choice of NN-potential is limited by the consideration that it's expression should not be too complicated, so that one can evaluate Talmi integrals. But, on the other hand, it must reproduce the experimentally known features of NN-interaction (e.g. it must be able to describe a hard core at small distances, and it's tail must not extend outside the size of the nucleus). One can see, that the single multipole-Gauss potential Eq.(7) does not satisfy these requirements. The next in complexity potential, which allows one to reproduce partly the characteristics of the realistic NN-interaction potential, is the superposition of two multipole-Gauss type potentials [31], which we have employed for the calculations of α-cluster type nuclei with $A \leq 60$. The analytical expressions of this potential, its parametrization, as well as the details of Talmi integral calculations one can find in Appendix A.

The results for α-cluster type nuclei with $8 \leq A \leq 60$, obtained with the NN-interaction potential taken in the form of Eq.(A.6), are given in Table 2. In this table

$$d = \left[\sum_{i=1}^{m} (E_{calc}(i) - E_{exp}(i))^2/m\right]^{1/2}, \qquad (65)$$

with m – the number of experimentally known levels (including binding energy). The values of following parameters have been obtained by the fit to the available experimental binding and excited level energies:

r_{00} - the nuclear radii parameter (see Eq.(61));

r_0, r_1, κ - the parameters defining the shape of the potential formed by the superposition of two multipole-Gauss type potentials (see Appendix A);

C_1, C_2 - parameters combining potential depth V_{c0} and exchange constants (see Eq.(62)).

These results seem to be quite satisfactory with regard to the fact, that the ground state binding energies (whose values vary in the interval $\sim 30 - 350$ MeV) and the

Table 2: The SRDM parameter values for α-cluster type nuclei with $8 \leq A \leq 60$

Nucleus	m	r_{00} [fm]	r_0 [fm]	r_1 [fm]	κ	C_1 [MeV]	C_2 [MeV]	$d[MeV]$
^8Be	3	1.385	0.75	2.433	0.663	-611.202	1101.430	4.82
^{12}C	3	1.064	0.75	1.876	0.717	-686.108	824.354	5.90
^{20}Ne	5	1.236	0.75	1.770	1.151	-68.401	61.637	4.52
^{24}Mg	15	1.109	0.75	1.376	0.881	-79.653	66.917	4.19
^{28}Si	4	1.387	0.25	2.126	0.500	-52.488	42.150	3.27
^{32}S	8	1.175	0.75	1.386	0.868	-66.801	51.535	2.38
^{36}Ar	4	1.166	0.5	1.679	0.654	-54.072	40.616	3.68
^{44}Ti	4	1.033	0.5	1.491	0.507	-27.569	19.891	1.53
^{48}Cr	10	1.284	0.25	1.630	0.082	-22.179	15.750	2.15
^{52}Fe	4	1.221	0.75	1.668	0.501	-26.041	18.239	1.66
^{56}Ni	11	1.081	0.75	1.457	0.521	-34.951	24.203	2.66
^{60}Zn	5	1.291	0.25	1.599	0.419	-22.958	15.752	2.00

low-lying excited level energies (which are in the energy interval $\sim 1 - 15$ MeV) were calculated together at the same time. Contrary to the results obtained for light α-cluster type nuclei with $A \leq 40$ (Ref.[33, 18]), the effective NN-interaction potential parameter values for $40 < A \leq 60$ region α-cluster nuclei reveal significant dependence from the nuclear mass number A. The obtained values of nuclear radii parameters r_{00} are in reasonable agreement with the known calculation results for other neighbouring nuclei.

8. Numerical estimation of microscopic values for the parameters of phenomenological collective models

The relationship chain between nuclear collective Hamiltonians is based on the General Restricted Dynamics Collective Model (GRDCM) Hamiltonian Eq.(5) evaluated from the General Microscopic Nuclear (GMNH) Hamiltonian Eq.(1). Since GRDCM Hamiltonian remains unsolved as yet in the case of many-nucleon system, we shall employ for our numerical estimation of microscopical values for phenomenological collective model parameters the results of α-cluster type nuclei energy calculations presented in Sect.7.

We shall consider the microscopic estimations for parameters entering in both phenomenological collective models – the standard BMM and the standard IBM Hamiltonians. The microscopical values of two standard Bohr-Mottelson Model (BMM) Hamiltonian parameters (the mass parameter - B and the nuclear stiffness parameter - C), obtained using the results of SRDM energy calculations for α-cluster type nuclei, can be compared with their phenomenological values, which follow from the BMM rotational band energy formula. The Standard Interacting Boson Model (IBM) Hamiltonian is characterized by 10 parameters $e_0, e_1, e_2, \ldots, \eta$. The microscopical values of these parameters, estimated using the NN-interaction potential parameter values obtained from

the SRDM energy calculations for α-cluster type nuclei, can be compared with their phenomenological values, obtained by means of standard IBM energy formulas for limiting cases. The preliminary results of such comparison have been reported in Ref.[34].

First of all, we need to derive the microscopic values of the kinetic energy constant C_k (see Eq.(38)) and the potential energy constant C_p (see Eq.(39)). The rough estimation for the kinetic energy constant microscopic value C_k^{micr} one can obtain assuming that the kinetic energy terms of GIBM and SRDCM Hamiltonians are approximately equal. This assumption gives us the following relationship between corresponding matrix elements:

$$E_{kin}^{GIBM} = \frac{1}{2} E_{kin}^{SRDM} \tag{66}$$

Then

$$\frac{C_k}{2}(N+3) = \frac{1}{2}(N_0 + \frac{3}{2}(A-1))20.7357\nu_A \tag{67}$$

and the microscopical value of kinetic energy constant is

$$C_k^{micr} = \frac{2N_0 + 3(A-1)}{N_0+6} 20.7357\nu_A \quad [\text{MeV}], \tag{68}$$

where we have used the microscopic value of the total number of U_6-quanta (bosons)

$$N = N^{micr} = N_0/2. \tag{69}$$

According to the prescription of V.Vanagas [3, 2] such N^{micr} value can be deduced from the following branching rule for the reduction chain $U_6 \supset SU_3 \supset O_3^+$: if N is IR of U_6, then one can enumerate all SU_3 IRs contained in N by counting all the partitions of number $2N$ into no more than 3 parts, i.e. $2N = 2N^{micr} = N_1 + N_2 + N_3 = N_0$, where $N_1 \geq N_2 \geq N_3$ and all N_1, N_2, N_3 are even. The representations $[N_1 N_2 N_3]$ obtained in such a way denote all SU_3 IRs contained in N.

The microscopic value of the potential energy constant C_p^{micr} also is adopted from the collective central NN-interaction energy term of SRDM Hamiltonian (Eq.(59)). Then, for α-cluster type nuclei we have

$$C_p^{micr} = \frac{(3C_1 - 5C_2)A}{4} + \frac{(3C_1 + 5C_2)A^2}{16} \quad [\text{MeV}], \tag{70}$$

where C_1 and C_2 are parameters, whose values are determined from the SRDM energy fit calculations.

For the microscopic estimation of the standard BMM parameters we shall employ now the phenomenological version of the standard BMM Hamiltonian (see also Eq.(23))

$$H_{ph}^{BMM} = -\frac{\hbar^2}{2B} T_{BM}^0(\beta, \gamma, \Theta, \Phi, \Psi) + \frac{1}{2} C\beta^2, \tag{71}$$

which is characterized by the nuclear mass parameter B and the stiffness parameter C. The solution of H_{ph}^{BMM} in diagonal approximation for axially symmetric quadrupole deformation, characterized by deformation parameter β_0, yields the well known rotational band energy formula [35]:

$$E(I, K, n_\gamma, n_\beta) = (2n_\gamma + 1 + \frac{1}{2}K)E_\gamma + (n_\beta + \frac{1}{2})E_\beta + \left[I(I+1) - K^2\right] A_{rot}, \tag{72}$$

from which one can obtain the phenomenological values of parameters B and C:

$$B^{ph} = \frac{\hbar^2}{6A_{rot}\beta_0^2}; \qquad C^{ph} = \frac{B^{ph}E_\beta^2}{\hbar^2} \approx \frac{B^{ph}E_\gamma^2}{\hbar^2}. \tag{73}$$

The microscopic value B^{micr} of the mass parameter B one can obtain from the comparison of microscopically evaluated (see Eqs.(23),(19)) and phenomenological (see Eq.(71)) kinetic energy terms of the standard BMM Hamiltonian, which gives

$$B^{micr} = \frac{\hbar^2}{4C_k^{micr}}, \tag{74}$$

where C_k^{micr} is determined according to the formula Eq.(68). Analogously, the microscopic value of the nuclear stiffness parameter C^{micr} can be obtained by comparing the phenomenological potential energy term of Eq.(71) with corresponding microscopic potential energy term in the expansion of H^{GBMM} (Eqs.(22),(11)):

$$\frac{1}{2}C\beta^2 = 8C_p^{micr}F_2(0,1,0)b_2R_2 = 8C_p^{micr}\beta^2 b_2 R_2, \tag{75}$$

which gives

$$C^{micr} = 16C_p^{micr}b_2R_2, \tag{76}$$

where values of C_p^{micr}, b_2 and R_2 are determined according to formulae Eqs.(70), (8), (35), correspondingly, or Eqs.(70) and (A.20) in the case of superposition of two multipole-Gauss type potentials (Eq.(A.6)).

The comparison of the microscopical values of the standard BMM Hamiltonian parameters B^{micr}, C^{micr} with their phenomenological counterparts for α-cluster type nuclei is presented in Table 3.

The microscopic values of the standard IBM Hamiltonian parameters $e_0, e_1, \ldots, \eta$ can be calculated using formulae Eqs.(50), inserting microscopic values of kinetic and potential energy constants C_k^{micr}, C_p^{micr} (see Eqs.(68),(70)). The corresponding phenomenological values of the standard IBM Hamiltonian parameters one can obtain using the diagonal energy formulae of standard IBM in SU_5, SU_3 and O_6 limiting cases (see Ref.[36]):

1) in SU_5 case

$$E_{SU_5}(N, n_d, v, L) = A(N) + \varepsilon n_d + \alpha n_d(n_d+4) + \varepsilon_1 N n_d + \beta v(v+3) + 2\gamma L(L+1); \tag{77}$$

2) in SU_3 case

$$E_{SU_3}(N,\lambda,\mu,L) = A(N) + \delta\frac{2}{3}(\lambda^2+\mu^2+3\lambda+3\mu+\lambda\mu) + 2\gamma L(L+1); \qquad (78)$$

3) in O_6 case

$$E_{O_6}(N,\sigma,\tau,L) = A(N) + 2\eta\sigma(\sigma+4) + 2\beta\tau(\tau+3) + 2\gamma L(L+1), \qquad (79)$$

where $A(N) = e_0 + e_1 N + e_2 N(N+5)$.

The results of comparison between microscopical and phenomenological values of the standard IBM Hamiltonian parameters for α-cluster type nuclei one can find in Table 4. In both cases the microscopic value (Eq.(69)) of the total boson number $N = N^{micr}$ was used. ae

9. Conclusions

For the first time the numerical estimation and comparison of microscopical and phenomenological values of the standard BMM and the standard IBM parameters have been performed, using the Restricted Dynamics Approach based relationship chain between collective models. The Strictly Restricted Dynamics Model (SRDM) parameter values, used as input data for the evaluation of microscopical values for BMM and IBM parameters, have been obtained in SRDM energy calculations for α-cluster type nuclei with $A \leq 60$.

The comparison of microscopically derived values of phenomenological model parameters with their values obtained from the direct fit to the experimental level energies shows considerable discrepancies in the case of standard BMM (2-3 orders of magnitude). This fact can be explained remembering that standard BMM usually is employed for the description of heavy nuclei ($A > 100$), characterized by the stable deformation of the nuclear core. In the case of standard IBM the microscopically estimated parameter values agree with their phenomenological counterparts much better.

These results indicate that our approach can give only a first very rough estimation for the parameters of phenomenological collective models. One needs more elaborated efforts for the real success in this direction.

Appendix A. The NN-interaction potentials and Talmi integrals

In order to calculate the collective and anticollective matrix elements of NN-interaction one needs to calculate interaction integrals for the chosen NN-interaction potential. In the collective matrix elements Eq.(64) interaction integrals $I_{\varepsilon l,\varepsilon' l'}$ are already reduced to Talmi integrals I_{ss} (since $I_{sl,s'l'} = I_{ss,ss} \equiv I_{ss}$).

For the evaluation of the anticollective matrix elements we shall use the SU_3-averaged interaction integrals $I_{[00]\varepsilon}$ (see Ref.[15]), depending on the interaction integrals $I_{\varepsilon l}$ ($I_{\varepsilon l} \equiv I_{\varepsilon l,\varepsilon l}$):

$$I_{[00]\varepsilon} = \sum_l \frac{2l+1}{d_\varepsilon} I_{\varepsilon l}, \qquad (A.1)$$

where $d_\varepsilon = (\varepsilon+1)(\varepsilon+2)/2$. Interaction integrals $I_{\varepsilon l}$, in turn, for values $\varepsilon = 0,1,2,3,4,5,6$, which we need in our calculations, can be reduced to Talmi integrals $I_{\varepsilon\varepsilon}$, using formulae of Table 15.1 in Ref.[15]. Then we have (see Ref.[24]):

$$\begin{aligned}
I_{[00]0} &= I_{00};\\
I_{[00]1} &= I_{11};\\
I_{[00]2} &= (I_{00} - 2I_{11} + 5I_{22})/4;\\
I_{[00]3} &= (3I_{11} - 6I_{22} + 7I_{33})/4;\\
I_{[00]4} &= (I_{00} - 4I_{11} + 18I_{22} - 28I_{33} + 21I_{44})/8;\\
I_{[00]5} &= (105I_{11} - 420I_{22} + 1050I_{33} - 1260I_{44} + 693I_{55})/168;\\
I_{[00]6} &= (105I_{00} - 630I_{11} + 4095I_{22} - 12180I_{33} + 21735I_{44} - 20790I_{55} +\\
&+ 9009I_{66})/1344.
\end{aligned} \qquad (A.2)$$

For the convenience of practical calculations, we introduce into chosen NN-interaction potential expression the wave function scaling factor r_ψ (see Eq.(61)), which leads to the replacement

$$V_{W,M,B,H}(\vec{\rho}_a) \longrightarrow V_{W,M,B,H}\left(\frac{r_\psi}{r_p}\sqrt{2}\vec{\rho}_a\right) \qquad (A.3)$$

for the particular exchange components of the central NN-interaction. Talmi integrals of SRDM, taking into account Eq.(A.3), have the form

$$\begin{aligned}
I^{r_\psi}_{\varepsilon\varepsilon}(V_{W,M,B,H}) &= I^{r_\psi}_{\varepsilon\varepsilon,\varepsilon\varepsilon}\left(V_{W,M,B,H}\left(\frac{r_\psi}{r_p}\sqrt{2}\vec{\rho}_a\right)\right) =\\
&= \int_0^\infty \rho_a^2 d\rho_a R^{r_\psi}_{\varepsilon\varepsilon}(\rho_a) V_{W,M,B,H}\left(\frac{r_\psi}{r_p}\sqrt{2}\rho_a\right) R^{r_\psi}_{\varepsilon'\varepsilon'}(\rho_a),
\end{aligned} \qquad (A.4)$$

where $\rho_a = |\vec{\rho}_a|$. Integrals Eq.(A.4) are defined with respect to the unknotted three-dimensional harmonic oscillator radial wave functions

$$R^{r_\psi}_{\varepsilon\varepsilon}(\rho_a) = \sqrt{\frac{2}{(r_\psi)^3\Gamma(\varepsilon+3/2)}}\left(\frac{\rho_a}{r_\psi}\right)^\varepsilon \exp\left[-\frac{1}{2}\left(\frac{\rho_a}{r_\psi}\right)^2\right]. \qquad (A.5)$$

The NN-interaction potential, written as the superposition of two multipole-Gauss type potentials [31], which we have employed for the calculations of α-cluster type nuclei with $A \leq 60$, can be presented as follows:

$$V(\rho_a) = V_{01} r_{1p}^{q_1}\left(\frac{\rho_a}{r_{1p}}\right)^{q_1} \exp\left[-Z_{01}\left(\frac{\rho_a}{r_{1p}}\right)^{\beta_1}\right] + V_{02} r_{2p}^{q_2}\left(\frac{\rho_a}{r_{2p}}\right)^{q_2} \exp\left[-Z_{02}\left(\frac{\rho_a}{r_{2p}}\right)^{\beta_2}\right], \qquad (A.6)$$

where

$$Z_{01} = \mathrm{sign}(\alpha_1); \qquad Z_{02} = \mathrm{sign}(\alpha_2),$$

and

$$r_{1p} = \mid \alpha_1 \mid^{-1/\beta_1}; \qquad r_{2p} = \mid \alpha_2 \mid^{-1/\beta_2}.$$

In Gauss potential case (when $q_1 = q_2 = 0$ and $\beta_1 = \beta_2 = 2$) the potential Eq.(A.6) is characterized by 4 parameters: $r_{1p}, r_{2p}, V_{01}, V_{02}$. Taking into account the replacement Eq.(A.3), the NN-interaction potential expression Eq.(A.6) becomes

$$V\left(\sqrt{2}r_\psi\rho_a\right) = V_{01}(r_{1p})^{q_1}V'(q_1, r_{21}, Z_{01}, \beta_1) + V_{02}(r_{2p})^{q_2}V''(q_2, r_{22}, Z_{02}, \beta_2) =$$

$$= V_{01}(r_{1p})^{q_1}\left(\frac{\rho_a}{r_{21}}\right)^{q_1}\exp\left[-Z_{01}\left(\frac{\rho_a}{r_{21}}\right)^{\beta_1}\right] + V_{02}(r_{2p})^{q_2}\left(\frac{\rho_a}{r_{22}}\right)^{q_2}\exp\left[-Z_{02}\left(\frac{\rho_a}{r_{22}}\right)^{\beta_2}\right], \quad \text{(A.7)}$$

where

$$r_{21} = r_{1p}\sqrt{\frac{\nu_A}{2}}; \qquad r_{22} = r_{2p}\sqrt{\frac{\nu_A}{2}}. \quad \text{(A.8)}$$

Such superposition of two multipole-Gauss type potentials allows to obtain wide variety of potential shapes, which are more convenient to investigate if one introduces new parameters:

$$(r_{1p}, r_{2p}, V_{01}, V_{02}) \longrightarrow (r_0, r_1, V_1, \kappa), \quad \text{(A.9)}$$

where

$$V(\rho_a = r_0) = 0; \qquad V_1 = V(r_1);$$

$$\frac{\partial}{\partial\rho_a}V(\rho_a)\mid_{\rho_a=r_1} = 0; \quad \kappa = \frac{q_2 - \alpha_2\beta_2 r_1^{\beta_2}}{q_1 - \alpha_1\beta_1 r_1^{\beta_1}}.$$

New parameters (r_0, r_1, V_1, κ) have the following meaning:

r_0 - the distance , where $V(\rho_a) = 0$. (In our calculations this parameter was fixed, since the potential shape is defined by the r_0 and r_1 ratio.)

r_1 - the distance, where $\frac{\partial}{\partial\rho_a}V(\rho_a) = 0$.

V_1 - the total potential depth defined as $V_1 = V(r_1)$.

κ - the ratio of the first and second Gauss potential scales $\kappa \sim \frac{r_{1p}^2}{r_{2p}^2}$.

When one introduces parameters r_0, r_1, V_1 and κ, then the old parameters, entering in the potential expression Eq.(A.6), can be evaluated as follows:

$$\alpha_1 = \frac{(\kappa q_1 - q_2)\left(1 - \left(\frac{r_0}{r_1}\right)^{\beta_2}\right) + \beta_2 \ln\left[\kappa\left(\frac{r_0}{r_1}\right)^{q_1-q_2}\right]}{(r_1)^{\beta_1}\left(\kappa\beta_1\left(1 - \left(\frac{r_0}{r_1}\right)^{\beta_2}\right) - \beta_2\left(1 - \left(\frac{r_0}{r_1}\right)^{\beta_1}\right)\right)}; \quad \text{(A.10)}$$

$$\alpha_2 = \frac{(\kappa q_1 - q_2)\left(1 - \left(\frac{r_0}{r_1}\right)^{\beta_1}\right) + \kappa\beta_1 \ln\left[\kappa\left(\frac{r_0}{r_1}\right)^{q_1-q_2}\right]}{(r_1)^{\beta_2}\left(\kappa\beta_1\left(1 - \left(\frac{r_0}{r_1}\right)^{\beta_2}\right) - \beta_2\left(1 - \left(\frac{r_0}{r_1}\right)^{\beta_1}\right)\right)}, \quad \text{(A.11)}$$

and

$$V_{01} = -V_1\frac{(r_0)^{q_2}\exp\left[-\alpha_2(r_0)^{\beta_2}\right]}{U(r_1)}; \quad \text{(A.12)}$$

$$V_{02} = V_1\frac{(r_0)^{q_1}\exp\left[-\alpha_1(r_0)^{\beta_1}\right]}{U(r_1)}, \quad \text{(A.13)}$$

where

$$U(r_1) = (r_0)^{q_1}(r_1)^{q_2}\exp\left[-\left(\alpha_1(r_0)^{\beta_1} + \alpha_2(r_1)^{\beta_2}\right)\right] - (r_0)^{q_2}(r_1)^{q_1}\exp\left[-\left(\alpha_1(r_1)^{\beta_1} + \alpha_2(r_0)^{\beta_2}\right)\right]. \quad \text{(A.14)}$$

For NN-potentials, written as a superposition of two multipole-Gauss type potentials (Eq.(A.6)), corresponding Talmi integrals are obtained as the weighted sum of two Talmi integrals for each multipole-Gauss potential V' and V'' (for details see Ref.[31]):

$$I_{\varepsilon\varepsilon}\left(q, r_p\sqrt{\frac{\nu_A}{2}}, z_1, \beta\right) = \frac{\varepsilon!\,2^{2\varepsilon+1}\left(r_p\sqrt{\nu_A/2}\right)^{2\varepsilon+3}\Gamma\left((2\varepsilon+q+3)/2\right)}{\sqrt{\pi}\,(2\varepsilon+1)!\left(r_p^2\nu_A/2 + z_1\right)^{\varepsilon+(q+3)/2}}, \quad \text{(A.15)}$$

where q, r_p, z_1, β for the first part of the potential assume values $q_1, r_{1p}, Z_{01}, \beta_1$, and for the second part - $q_2, r_{2p}, Z_{02}, \beta_2$, correspondingly. Then, the summed Talmi integrals can be presented as follows:

$$I_{\varepsilon\varepsilon}\left(V(\sqrt{2}r_\psi\rho_a)\right) = V'_{01}I'_{\varepsilon\varepsilon}\left(q_1, r_{1p}\sqrt{\frac{\nu_A}{2}}, Z_{01}, \beta_1\right) + V'_{02}I''_{\varepsilon\varepsilon}\left(q_2, r_{2p}\sqrt{\frac{\nu_A}{2}}, Z_{02}, \beta_2\right), \quad \text{(A.16)}$$

where the weight factors:

$$V'_{01} = V_{01}(r_{1p})^{q_1}; \qquad V'_{02} = V_{02}(r_{2p})^{q_2} \quad \text{(A.17)}$$

one can express using Eqs.(A.12) and (A.13). Since the common part of the potential depth $V_1 \equiv V_{c0}$ we include in parameters C_1 and C_2 (see Eq.(62)), then from Eqs.(A.17), (A.12) and (A.13) we finally obtain following expressions for the weight factors between two parts of the Talmi integral Eq.(A.16):

$$V'_{01} = -(r_{1p})^{q_1}\frac{(r_0)^{q_2}\exp\left[-\alpha_2(r_0)^{\beta_2}\right]}{U(r_1)}; \quad \text{(A.18)}$$

$$V'_{02} = (r_{2p})^{q_2}\frac{(r_0)^{q_1}\exp\left[-\alpha_1(r_0)^{\beta_1}\right]}{U(r_1)}, \quad \text{(A.19)}$$

which should be used also when evaluating the potential part of H^{GIBM} (Eq.(39)). Then, in the case of potential Eq.(A.6), we have

$$b_t R_t = (b'_t + b''_t)R_t = \left(V'_{01}\frac{(-\alpha_1)^{t-a}}{(t-a)!} + V'_{02}\frac{(-\alpha_2)^{t-a}}{(t-a)!}\right)\frac{(A-3)!!}{(A+2t-3)!!}, \quad \text{(A.20)}$$

where α_1 and α_2 are defined by Eqs.(A.10) and (A.11).

Appendix B. The expressions of SRDM anticollective density matrices

The resulting formulae for the anticollective orbital matrix elements of Wigner and Majorana interactions are following [24]:

$$E^a_W(V_W) = \sum_{\varepsilon=0}^{2\varepsilon_0+i_{ex}} I_{[00]\varepsilon}(V_W) Q_{[00]\varepsilon}(\delta\omega\alpha f, \delta'\omega'\alpha' f'), \tag{B.1}$$

$$E^a_M(V_M) = \sum_{\varepsilon=0}^{2\varepsilon_0+i_{ex}} (-1)^\varepsilon I_{[00]\varepsilon}(V_M) Q_{[00]\varepsilon}(\delta\omega\alpha f, \delta'\omega'\alpha' f'), \tag{B.2}$$

where $\varepsilon_0 = 0, 1, 2, \ldots$ denotes SU_3-shells $(s, p, sd, \ldots)$ and i_{ex} – ground ($i_{ex} = 0$) and excited ($i_{ex} = 1, 2, \ldots$) SU_3-configurations. In Eqs.(B.1) and (B.2) $Q_{[00]\varepsilon}(\ldots)$ denotes the anticollective (U_3-scalar) two-particle density matrix. Its general expressions are known only in the case of p-shell ($\varepsilon_0 = 1$) and sd-shell ($\varepsilon_0 = 2$) nuclei. One can find them in Refs.[15, 1, 24].

For other shells ($\varepsilon_0 \geq 3$), which we need in the case of α-cluster nuclei with $A > 40$, we must use the approximate expressions for the two-particle anticollective density matrix components, which have been evaluated employing the approach described in Refs.[15, 37]. According to this method, the two-particle anticollective density matrix $Q_{[00]\varepsilon} \equiv Q_\varepsilon$ can be approximately expressed using "four universal relationships" (see Eqs.(25.14), (25.15), (25.17) and (25.32) in Ref.[15]):

$$\sum_\varepsilon Q_\varepsilon = \frac{A(A-1)}{2} = f_1; \tag{B.3}$$

$$\sum_\varepsilon (-1)^\varepsilon Q_\varepsilon = \Lambda(f) = f_2; \tag{B.4}$$

$$\sum_\varepsilon \varepsilon Q_\varepsilon = \frac{A}{2} N_0 = f_3; \tag{B.5}$$

$$\sum_\varepsilon \varepsilon^2 Q_\varepsilon = \frac{1}{3} N_0^2 + \frac{A-4}{4} N_0 + \frac{3}{2} G + \frac{A}{4} X(\varepsilon_0) =$$
$$= \left[\tilde{k}_3 + 4 S_1^{(Z,N)} (S_1^{(Z,N)} - 3)/3 + A(S_1^{(Z,N)} + S_2(Z,N))\right]/4 = f_4. \tag{B.6}$$

In Eq.(B.4) $\Lambda(f)$ is the eigenvalue of the orbital coordinate exchange operator, its expression one can find, e.g. in Ref.[18]. Quantity G in Eq.(B.6) is the eigenvalue of the second (quadratic) Casimir operator of the SU_3 group:

$$G = [(\lambda+\mu)(\lambda+\mu+3) - (\lambda\mu)]/9. \tag{B.7}$$

Expressions for functions $S_1^{(Z,N)}, S_2^{(Z,N)}, \tilde{k}_3$ one can find in Refs.[15, 37].

From the sums of both sides of these four relationships with coefficients b_1^W, b_2^W, b_3^W, b_4^W and b_1^M, b_2^M, b_3^M, b_4^M, correspondingly, one obtains:

1) the approximate equation in the case of Wigner forces:

$$\sum_\varepsilon Q_\varepsilon^W \left(b_1^W + (-1)^\varepsilon b_2^W + \varepsilon b_3^W + \varepsilon^2 b_4^W\right) \approx \sum_\varepsilon Q_\varepsilon^W I_{[00]\varepsilon}; \tag{B.8}$$

2) and, analogously, in the case of Majorana forces:

$$\sum_\varepsilon Q_\varepsilon^M \left(b_1^M + (-1)^\varepsilon b_2^M + \varepsilon b_3^M + \varepsilon^2 b_4^M\right) \approx \sum_\varepsilon Q_\varepsilon^M (-1)^\varepsilon I_{[00]\varepsilon}. \tag{B.9}$$

Coefficients b_i^W, b_i^M ($i = 1-4$) one can find solving the overdetermined linear equation systems [37]:

$$b_1^W + (-1)^\varepsilon b_2^W + \varepsilon b_3^W + \varepsilon^2 b_4^W = I_{[00]\varepsilon} \tag{B.10}$$

and

$$b_1^M + (-1)^\varepsilon b_2^M + \varepsilon b_3^M + \varepsilon^2 b_4^M = (-1)^\varepsilon I_{[00]\varepsilon}, \tag{B.11}$$

where for $A \leq 80$ $\varepsilon = 0, 1, \ldots, 6$. The expressions for SU_3-averaged integrals $I_{[00]\varepsilon}$ one can find in Appendix A. Then, taking into account relationships Eqs.(B.3)-(B.7) and using evaluated coefficients b_i^W, b_i^M, we can express the total anticollective energies entering in Eq.(60) as follows:

$$E^{\pm a}_{e,c} = \frac{E^W_a \pm E^M_a}{2} = \frac{1}{2}\left[(b_1^W \pm b_1^M) f_1 + (b_2^W \pm b_2^M) f_2 + (b_3^W \pm b_3^M) f_3 + (b_4^W \pm b_4^M) f_4\right], \tag{B.12}$$

using Coulomb and central NN-interaction potentials, correspondingly.

For the first time the approximate expressions for the SU_3-invariant two-particle density matrix have been obtained in Ref.[37]. Though, empolying these expressions in our SRDM calculations of $44 \leq A \leq 80$ nuclei, we have found that these expressions contain errors. Therefore, we have solved systems Eqs.(B.10) and (B.11) again and obtained the corrected formulae for the coefficients b_i^W, b_i^M, which we have used in our calculations (in the following expressions $\varepsilon = \varepsilon_{max} = 2\varepsilon_0 + i_{ex}$ for $40 < A \leq 80$, since $\varepsilon_0 = 3$ and $i_{ex} = 0$, therefore, $\varepsilon_{max} = 6$):

$$\begin{aligned} b_1^W &= \frac{3}{(\varepsilon-2)\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+4)} \sum_{i=0}^{\varepsilon} [3\varepsilon^4 - 4\varepsilon^2 - 8\varepsilon - 1 - \\ &- (\varepsilon-1)(\varepsilon+1)(2\varepsilon+1)(-1)^i + 10i^2(\varepsilon^2 - \varepsilon + 1) - \\ &- 6\varepsilon(2\varepsilon+1)(\varepsilon-1)i] I_i \qquad (\text{for } \varepsilon - \text{even}); \end{aligned} \tag{B.13}$$

$$\begin{aligned} b_1^W &= \frac{3}{(\varepsilon-1)(\varepsilon+2)(\varepsilon+1)(\varepsilon+3)} \sum_{i=0}^{\varepsilon} [3\varepsilon^3 + 2\varepsilon - 2 - \varepsilon(\varepsilon+2)(-1)^i + \\ &+ 2i^2(-5+5\varepsilon) + 6i\varepsilon(1-2\varepsilon)] I_i \qquad (\text{for } \varepsilon - \text{odd}); \end{aligned} \tag{B.14}$$

$$\begin{aligned} b_2^W &= \frac{1}{(\varepsilon-2)\varepsilon(\varepsilon+2)(\varepsilon+4)} \sum_{i=0}^{\varepsilon} [-3(\varepsilon-1)(2\varepsilon+1) + \\ &+ (\varepsilon-1)(\varepsilon+1)(\varepsilon+3)(-1)^i + 30i(\varepsilon-i)] I_i \qquad (\text{for } \varepsilon - \text{even}); \end{aligned} \tag{B.15}$$

$$b_2^W = \frac{1}{(\varepsilon-1)(\varepsilon+1)(\varepsilon+3)}\sum_{i=0}^{\varepsilon}[-3\varepsilon+6i+\varepsilon(\varepsilon+2)(-1)^i]I_i \quad (\text{for } \varepsilon - \text{odd}); \tag{B.16}$$

$$b_3^W = \frac{6}{(\varepsilon-2)\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+4)}\sum_{i=0}^{\varepsilon}[\varepsilon(-6\varepsilon^2+(\varepsilon+1)(3+5(-1)^i))+ + 2i(16\varepsilon^2+2\varepsilon-8-15\varepsilon i)]I_i \quad (\text{for } \varepsilon - \text{even}); \tag{B.17}$$

$$b_3^W = \frac{6}{(\varepsilon-1)(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}\sum_{i=0}^{\varepsilon}[\varepsilon(-6\varepsilon+3+32i+(-1)^i)+ + 2i(2-15i)+2(-1)^i]I_i \quad (\text{for } \varepsilon - \text{odd}); \tag{B.18}$$

$$b_4^W = \frac{30}{(\varepsilon-2)\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+4)}\sum_{i=0}^{\varepsilon}[\varepsilon^2-\varepsilon(6i+1+(-1)^i)+ + 6i^2+1-(-1)^i]I_i \quad (\text{for } \varepsilon - \text{even}); \tag{B.19}$$

$$b_4^W = \frac{30}{(\varepsilon-1)\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}\sum_{i=0}^{\varepsilon}[\varepsilon^2-\varepsilon(6i+1)+6i^2]I_i \quad (\text{for } \varepsilon - \text{odd}). \tag{B.20}$$

Coefficients b_i^M one can obtain replacing in all expressions I_i with $(-1)^i I_i$. æ

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Table 4: The microscopical and phenomenological values of the standard IBM Hamiltonian parameters for α-cluster type nuclei

Nucl.	N	$A(N)$		$\varepsilon + N\varepsilon_1$		α		β		δ		η		γ	
	micr	micr	ph	micr	ph	micr	ph	micr	ph	micr	ph	micr	ph	micr	ph
^{8}Be	2	-29	-27	15.0	-	0.17	-	-0.186	-	-0.217	-0.100	0.313	-	0.070	0.299
^{12}C	4	-134	-82	27.0	-2.6	0.29	1.07	-0.286	0.046	-0.342	-0.258	0.472	-0.260	0.111	0.272
^{20}Ne	10	-150	-151	12.8	2.6	0.03	-0.10	-0.143	0.028	-0.126	-0.037	0.274	-0.039	0.040	0.039
^{24}Mg	14	-203	-184	9.5	1.7	-0.18	0.01	-0.153	-0.010	-0.049	-0.029	0.363	-0.044	0.013	0.057
^{28}Si	18	-235	-222	10.7	3.4	0.08	-0.19	-0.063	0.022	-0.082	-0.019	0.099	-0.026	0.027	0.051
^{32}S	22	-252	-259	5.8	1.8	-0.10	-0.05	-0.090	0.005	-0.031	-0.013	0.212	-0.017	0.008	0.050
^{36}Ar	26	-300	-287	10.5	3.5	0.07	-0.26	-0.055	0.027	-0.070	-0.017	0.087	-0.017	0.023	0.083
^{44}Ti	36	-362	-361	10.3	1.9	0.05	-0.10	-0.045	0.004	-0.056	-0.006	0.071	-0.007	0.018	0.020
^{48}Cr	42	-390	-390	7.5	2.4	0.04	-0.13	-0.030	0.010	-0.039	-0.006	0.048	-0.009	0.013	0.017
^{52}Fe	48	-425	-428	6.7	2.3	0.02	-0.10	-0.029	-0.005	-0.033	-0.004	0.050	-0.006	0.011	0.015
^{56}Ni	54	-465	-462	6.9	1.9	0.02	-0.06	-0.030	0.008	-0.032	-0.004	0.054	-0.005	0.010	0.024
^{60}Zn	60	-480	-480	5.7	4.5	0.02	-0.34	-0.021	0.025	-0.024	-0.005	0.036	-0.007	0.008	0.007

Table 3: The microscopical and phenomenological values of the standard BMM Hamiltonian parameters for α-cluster type nuclei

Nucleus	$[N_1N_2N_3]$	N	C_k^{micr}	C_p^{micr}	β_0	B		C	
		micr				B^{micr}	B^{ph}	C^{micr}	C^{ph}
^{8}Be	[4,0,0]	2	23.679	12.661	-	0.011	-	9.423	-
^{12}C	[4,4,0]	4	41.582	30.713	0.574	0.006	0.760	17.571	11.825
^{20}Ne	[12,4,4]	10	27.758	7.583	0.722	0.009	3.162	5.381	20.942
^{24}Mg	[16,8,4]	14	32.327	1.299	0.607	0.008	3.445	2.285	22.687
^{28}Si	[20,8,8]	18	19.384	33.450	0.411	0.013	8.101	6.164	40.503
^{32}S	[20,16,8]	22	25.427	1.017	0.315	0.010	11.640	1.500	36.347
^{36}Ar	[20,20,12]	26	24.378	22.358	0.256	0.011	14.624	5.072	33.040
^{44}Ti	[32,20,20]	36	28.230	22.711	0.260	0.009	53.639	4.493	80.162
^{48}Cr	[40,24,20]	42	17.527	15.006	0.368	0.014	27.024	2.892	50.782
^{52}Fe	[48,24,24]	48	18.661	7.722	0.266	0.013	54.400	2.263	107.191
^{56}Ni	[52,32,24]	54	22.965	5.407	0.174	0.011	80.574	2.087	267.581
^{60}Zn	[56,36,28]	60	15.562	10.090	0.223	0.016	80.759	1.742	142.365