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GAUGE INVARIANT TREATMENT OF CONSTRAINED DYNAMICS AND FIELD QUANTIZATION ON THE LIGHT FRONT

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The method of removing constraints by integrating explicitly over constraint variables (i.e. lacking time derivative in the Lagrangian) in the Feynman path integral is presented. With its help scalar, Dirac and vector fields are quantized canonically on the light front. In the latter case one recovers the results of Coulomb (t=const case) and A =0 (on the light front) gauges. For non abelian fields a new, explicitly gauge invariant Hamiltonian is obtained.

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INTRODUCTION

The use of light front coordinates has now a long history. Suggested first by Susskind [1], Bardakci and Halpern [2] and Cheng and Ma [3], they led to important simplifications in such areas as current algebra [4] and deep inelastic processes [5]. Also Kogut and Soper [6], Bjorken, Kogut and Soper [7], Neville and Rohrlich [8] and Chang, Rost and Yan [9] showed that one can formulate consistently quantum electrodynamics and other theories on the surface $x^+ = z + t = const$, called the light front. Dynamics is assumed to develop in this variable, rather than in t. Those authors showed also that the first few terms of the x^+ ordered perturbation theory reproduce automatically Weinberg rules [10], i.e. important simplifications of the "old fashioned" time ordered perturbation theory in the infinite momentum frame.

This is the illustration of an early and prophetic work by Dirac [11], who showed that constant time surface is not the only possibility for formulating Poincare invariant theories and attracted attention to $x^+ = const$ dynamics. He emphasized the fact that in this form dynamics has only 3 interaction dependent Poincare generators as opposed to 4 in the case of $t = 0$ dynamics.

The usual approach to the quantization of fields was to make use of the Schwinger quantum action principle [12], which permits one to guess the commutation relations of the fields and the construction of Poincare generators. Those problems are nontrivial on the light front: in general the dynamics has constraints; the fields have half as many degrees of freedom compared with $t = 0$ dynamics and often the commutation relations are non local.

In the case of gauge theories additional difficulties appear: in the natural axial gauge the Feynman propagator for the photon (gluon) has additional singularities and one has to face the problem of proper regularization. While several prescriptions exist in the literature [13,14,15,16], for the time being there is no consensus. In this work I decided to investigate quantum field dynamics in a systematic and the most conservative way possible: looking for classical canonically conjugated variables and quantizing by passing from Poisson brackets to (anti)commutators. For all cases investigated (scalar, Dirac, photon) except the non-abelian gauge field the technique used was to exploit the translational invariance to decompose fields into independent Fourier components. As each component presented a finite dimensional problem, the analysis was much simplified. The problem of constraints was solved in a new and simple way, based on the observation that constraint variable

lacks time derivative in the Lagrangian. The usual treatment is to use the complicated formalism of Dirac brackets [17,18]. In my treatment one writes the transition amplitudes in the form of Feynman path integral and one integrates over constraints: this is always possible to do explicitly since integrations at different times are independent. For scalar and Dirac fields one recovers known results. For the electromagnetic field one finds the results of the Coulomb gauge in the case of $t = 0$ dynamics and of $A^+ = 0$ gauge in the case of light front dynamics. In both cases this way of treating gauge degrees of freedom produced gauge invariant Hamiltonians. The spurious singularity was to be understood in the principal value sense.

In the important case of non-abelian gauge fields, studied for simplicity for the SU(2) group, new and unexpected results emerged: the final hamiltonian is explicitly gauge invariant and there are indications that the Faddeev-Popov determinant is free of Gribov ambiguities. In the first chapters I present the notation, kinematics and the analysis of a classical particle and free scalar, classical field. Next scalar and Dirac fields are quantized. To elucidate the complications of gauge degrees of freedom photon field is studied by my method both in $t = const$ and $x^+ = const$ cases. Least not last SU(2) Yang-Mills field is studied.

1 Notation

We shall define our null-plane coordinates as follows

$$\begin{aligned} x^p &= \tau := \frac{1}{\sqrt{2}}(t+z) \\ x^m &= \xi := \frac{1}{\sqrt{2}}(t-z) \\ x^i &= x^i \qquad i = 1,2 \end{aligned} \tag{1.1}$$

In those coordinates the metric tensor is

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.2}$$

with Greek indices $\mu,\ \nu = (p, m, 1, 2)$. Latin indices of vector quantities will take only the values (1,2). We use indices p (plus) and m (minus) instead of the more customary $+$, $-$ for improved legibility of the formulae. With our metric tensor (1.2) we have the scalar product of two four-vectors

$$a^\mu b_\mu = a^p b^m + a^m b^p - a^j b^j \tag{1.3}$$

since the covariant components are

$$\begin{aligned} x_p &= x^m \\ x_m &= x^p \\ x_i &= -x^i \end{aligned} \tag{1.4}$$

Notice that we also have, due to non-diagonal nature of (1.2)

$$\partial^p \Psi = \frac{\partial \Psi}{\partial x_p} = \frac{\partial \Psi}{\partial x^m} \tag{1.5}$$

$$\partial^m \Psi = \frac{\partial \Psi}{\partial x_m} = \frac{\partial \Psi}{\partial x^p} \tag{1.6}$$

In what follows we shall use the usual methods of mechanics (Lagrange and Hamiltonian functions etc.) to analyze various systems. The coordinate τ eq. 1.1 will be the new time. We must of course have the crucial property of time: each world line of a particle should cross once and only once the plane $\tau = const.$ Our variable τ fulfills this requirement with a single exception: a massless particle travelling in the z direction. Since it is but one direction from a 2-dim continuum one does not expect difficulties associated with this point. We first check the feasibility of such an approach by looking at a classical free particle.

2 Classical particle

Let us write the world line element in our variables

$$ds^2 = dx_\mu dx^\mu = 2d\tau d\xi - (dx^i)^2 = d\tau^2(2\dot{\xi} - \dot{x}_\perp^2) \tag{2.1}$$

where the dot over a variable means differentiation with respect to τ. The usual action now is

$$S = -m \int_a^b ds = -m \int_{\tau_a}^{\tau_b} d\tau (2\dot{\xi} - \dot{x}_\perp^2)^{\frac{1}{2}} \tag{2.2}$$

so the Lagrangian is

$$\tilde{L} = -m(2\dot{\xi} - \dot{x}_\perp^2)^{\frac{1}{2}} \tag{2.3}$$

and we have the canonical momenta

$$p^p = \frac{\partial \tilde{L}}{\partial \dot{x}^m} = \frac{\partial \tilde{L}}{\partial \dot{\xi}} = \frac{-m}{(2\dot{\xi} - \dot{x}_\perp^2)^{1/2}} < 0 \tag{2.4}$$

$$p^i = \frac{\partial \tilde{L}}{\partial \dot{x}_i} = \frac{m\dot{x}^i}{(2\dot{\xi} - \dot{x}_\perp^2)}^{1/2} \tag{2.5}$$

Notice that the momentum p^p is always negative. The Hamiltonian takes the form

$$\tilde{H} = p^p \dot{\xi} + p_\perp \dot{x}_\perp - \tilde{L} = \frac{-(m^2 + p_\perp^2)}{2p^p} \tag{2.6}$$

This gives us the correct equations of motion, the simplest way to see this is to consider the Hamilton-Jacobi equation which follows from (2.6)

$$\frac{\left(m^2 + \left(\frac{\partial S}{\partial x_\perp}\right)^2\right)}{2\frac{\partial S}{\partial \xi}} + \frac{\partial S}{\partial \tau} = 0 \tag{2.7}$$

which can be rewritten in the form

$$-m^2 - \left(\frac{\partial S}{\partial x_\perp}\right)^2 + 2\frac{\partial S}{\partial \xi}\frac{\partial S}{\partial \tau} = -m^2 + S_{,\mu} S^{,\mu} = 0 \tag{2.8}$$

We see that in this case, the unfamiliar form of $\tilde{L}$, $\tilde{H}$ notwithstanding, we reconstruct the dynamics of a free relativistic particle. We can now proceed to the analysis of the massive scalar field.

3 Free real scalar field

It is well known [9] that a field, when analyzed on the null plane exhibits only half of the degrees of freedom as compared with the usual formalism i.e., at $t = 0$. We demonstrate this first for a classical, scalar field with the Lagrangian density

$$\tilde{L} = \frac{1}{2}(\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2) \tag{3.1}$$

The Euler-Lagrange equations

$$\partial^\mu \frac{\partial \tilde{L}}{\partial(\partial^\mu \Phi)} = \frac{\partial \tilde{L}}{\partial \Phi} \tag{3.2}$$

lead to the Klein-Gordon equation

$$\partial^\mu \partial_\mu \Phi + m^2 \Phi = 0 \tag{3.3}$$

Using time as evolution parameter we can write this

$$\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi + m^2 \Phi = 0 \tag{3.4}$$

If we now seek the solutions in form of plane waves

$$\Phi(\boldsymbol{x}, t) = exp(-ik^\mu x_\mu) = exp(-iEt + i\boldsymbol{k}\boldsymbol{x}) \tag{3.5}$$

the Klein-Gordon eq. (3.3) leads to the constraint

$$k^\mu k_\mu - m^2 = E^2 - \boldsymbol{k} \cdot \boldsymbol{k} - m^2 = 0 \tag{3.6}$$

which, when solved for E gives an ambiguity of sign

$$E = \pm\sqrt{m^2 + \boldsymbol{k}^2} \tag{3.7}$$

As a consequence we have a Fourier analysis of the field $\Phi(\boldsymbol{x}, t)$ which has two terms

$$\begin{aligned} &\Phi(\boldsymbol{x}, t) = \\ &= \frac{1}{(2\pi)^{2/3}} \int d_3k (c_1(\boldsymbol{k}) e^{-Iet + i\boldsymbol{k}\boldsymbol{x}} + c_2(\boldsymbol{k}) e^{iEt + i\boldsymbol{k}\boldsymbol{x}}) \end{aligned} \tag{3.8}$$

and, in order to determine $c_1(\boldsymbol{k})$ and $c_2(\boldsymbol{k})$, we also need the time derivative of $\Phi(\boldsymbol{x},t)$

$$\frac{\partial\Phi(\boldsymbol{x},t)}{\partial t} = \\ = \frac{1}{(2\pi)^{2/3}}\int d_3k(-iEc_1(\boldsymbol{k})e^{-iEt+i\boldsymbol{kx}} + iEc_2(\boldsymbol{k})e^{iEt+i\boldsymbol{kx}} \tag{3.9}$$

In effect let us introduce the Fourier transforms of the initial conditions which we take at $t=0$

$$\begin{aligned} \tilde{\Phi}_0(\boldsymbol{k}) &= \frac{1}{(2\pi)^{2/3}}\int d_3xe^{-i\boldsymbol{kx}}\Phi(\boldsymbol{x};t)\mid_{t=0} \\ \dot{\tilde{\Phi}}_0(\boldsymbol{k}) &= \frac{1}{(2\pi)^{2/3}}\int d_3xe^{-i\boldsymbol{kx}}\frac{\partial\Phi}{\partial t}(\boldsymbol{x});t)\mid_{t=0} \end{aligned} \tag{3.10}$$

Comparing eq's (3.8) and (3.9) we obtain

$$\begin{aligned} \tilde{\Phi}_0(\boldsymbol{k}) &= c_1(\boldsymbol{k}) + c_2(\boldsymbol{k}) \\ \dot{\tilde{\Phi}}_0(\boldsymbol{k}) &= iEc_1(\boldsymbol{k}) - iEc_2(\boldsymbol{k}) \end{aligned} \tag{3.11}$$

which determines $c_1(\boldsymbol{k})$ and $c_1(\boldsymbol{k})$.

The same procedure carried on a null-plane is rather different. Firstly we note that the Klein-Gordon eq. (3.3) in the null-plane variables is linear in our new "time" τ

$$2\frac{\partial^2}{\partial\tau\partial\xi}\Phi - \nabla_\perp^2\Phi + m^2\Phi = 0 \tag{3.12}$$

Second, looking again for plane wave solutions which we write in our new variables

$$\Phi(\xi, x_\perp;\tau) = e^{ik_\mu x^\mu} = exp(ik^m\tau + ik^p\xi - ik_\perp x_\perp) \tag{3.13}$$

we obtain a linear constraint for k^m, the variable associated with τ

$$-2k^mk^p + k_\perp^2 + m^2 = 0 \tag{3.14}$$

or, explicitly

$$k^m(k^p, k_\perp) = \frac{k_\perp^2 + m^2}{2k^p} \tag{3.15}$$

which uniquely determines k^m. In fact the negative energy solutions obtained at $\tau = 0$ map into our new solutions with $k^p < 0$ while the positive energy

solutions correspond to solutions with $k^p > 0$. The field at arbitrary "time" τ can now be expressed in terms of Fourier components $c(k^p, k_\perp)$

$$\begin{aligned}\Phi(\xi, x_\perp; \tau) &= \\ &= \frac{1}{(2\pi)^{2/3}} \int_{-\infty}^{\infty} dk^p d_2 k_\perp c(k^p, k_\perp) \\ &exp \left(-i \frac{(m^2 + k_\perp^2)}{2k^p} \tau - ik^p \xi + ik_\perp x_\perp \right) \end{aligned} \tag{3.16}$$

The field at $\tau = 0$

$$\begin{aligned}&\Phi(\xi, x_\perp; \tau = 0) = \\ &= \frac{1}{(2\pi)^{2/3}} \int_{-\infty}^{\infty} dk^p \int d_2 k_\perp c(k^p, k_\perp) exp(-ik^p \xi + ik_\perp x_\perp) \end{aligned} \tag{3.17}$$

determines the function $c(k^p, k_\perp)$

$$\begin{aligned}&c(k^p, k_\perp) = \\ &= \frac{1}{(2\pi)^{2/3}} \int d\xi d_2 x_\perp \Phi(\xi, x_\perp; \tau = 0) exp(ik^p \xi - ik_\perp x_\perp) \end{aligned} \tag{3.18}$$

We see that in this case we need the field only at $\tau = 0$, without any reference to its time derivative; as we shall subsequently find on the quantum level this implies halving of the degrees of freedom.

4 Equations of motion for Fourier components

Before attempting the quantization of various fields to be discussed later we will simplify the problem as for as possible, by using the translational invariance. This will reduce the problem of the field (infinitely dimensional) to the far simpler problem of a few degrees of freedom. The interaction will couple the various modes, but will not have any influence on the quantization process itself. Let us start by decomposing a classical scalar field into its Fourier space components, with the "time" τ playing the role of evolution

parameter,

$$\begin{aligned}
\varphi(x;\tau) &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty} dk^p d_2k_\perp e^{i(k^p x^m - k^j x^j)} \\
&\times \varphi_k(\tau) \\
&= \frac{1}{2\pi)^{3/2}}\int_0^{\infty} dk^p d_2k_\perp (e^{ikx}\varphi_k(\tau) + e^{-ikx}\varphi_{-k}(\tau)) \\
&= \frac{1}{(2\pi)^{3/2}}\int [dk](e^{ikx}\varphi_k + e^{-ikx}\varphi_{-k}) \qquad (4.1)
\end{aligned}$$

in obvious notation.

The free field Lagrangian (3.1) can now be written as

$$L = \frac{1}{2}\int [dk]\left\{ik^p(\varphi_k\dot{\varphi}_{-k} - \dot{\varphi}_k\varphi_{-k}) - \mu^2(k^jk^j)\varphi_k\varphi_{-k}\right\} \qquad (4.2)$$

As expected Fourier transform decoupled modes with different $k's$, leaving coupled modes k and $-k$ only. Introduce the customary notations

$$\begin{aligned}
k^p &\equiv x \qquad (4.3) \\
m_\perp^2 &\equiv \mu^2 + k^ik^i \qquad (4.4)
\end{aligned}$$

The Lagrangian for the mode k is now

$$\mathcal{L}_k = i\frac{x}{2}(\varphi_k\dot{\varphi}_{-k} - \dot{\varphi}_k\varphi_{-k}) \qquad (4.5)$$

where it is to be noticed that the dynamical degrees of freedom φ_k, φ_{-k} are complex. Since the field $\varphi(x,\tau)$ is real we have

$$\varphi_k = \varphi_{-k}^* \qquad (4.6)$$

We can therefore restrict ourselves to the momenta with positive $k^p > 0$. It is convenient to treat real, non complex degrees of freedom; let us use therefore the real and imaginary parts of φ explicitly

$$\begin{aligned}
\varphi_k &= a_k + ib_k \quad , \\
\varphi_{-k} &= a_k - ib_k \quad . \qquad (4.7)
\end{aligned}$$

Dropping the index k from now can we obtain finally the Lagrangian to be analyzed

$$\mathcal{L} = x(a\dot{b} - b\dot{a}) - \frac{m_\perp^2}{2}(a^2 + b^2) \tag{4.8}$$

This does not look familiar at all! Had we performed the same procedure at $t = 0$ we would have obtained for a and b decoupled harmonic oscillator Lagrangian; notice that a and b are symmetric and antisymmetric parts of the field respectively. In our case a and b are coupled and exhibit a kind of gauge symmetry

$$a \longrightarrow a\cos\alpha - b\sin\alpha \tag{4.9}$$

$$b \longrightarrow a\sin\alpha + b\cos\alpha \tag{4.10}$$

which mixes symmetric and antisymmetric parts of the field.

The canonical momenta are

$$\begin{aligned} \pi_a &= \frac{\partial\mathcal{L}}{\partial\dot{a}} = -xb \quad , \\ \pi_b &= \frac{\partial\mathcal{L}}{\partial\dot{b}} = xa \quad . \end{aligned} \tag{4.11}$$

and we obtain the equations of motion

$$\begin{aligned} \dot{b} &= \frac{m_\perp^2}{2x}a \quad , \\ \dot{a} &= \frac{-m_\perp^2}{2x}b \quad . \end{aligned} \tag{4.12}$$

The equations (4.11) represent <u>one</u> oscillator with frequency

$$w = \frac{m_\perp^2}{2x} \tag{4.13}$$

We have equations representing a harmonic oscillator but the canonical Hamiltonian is not available since eq's (4.10) would be incompatible with canonical Poisson brackets. It is possible to analyze this situation by using the Dirac brackets [17 ,18] but we will follow a simpler and better route. The constrained system à la Dirac will be illustrated in chapter 9. The problem with the Lagrangian (4.8) is that b seems to be proportional to π_a and vice versa.

In order to obtain a simple, conventional hamiltonian formalism I choose to modify the Lagrangian (4.8) without modifying the equations of motion by adding to (4.8) a total derivative

$$\tilde{\mathcal{L}} = \mathcal{L} + \frac{d}{d\tau}(xab) \quad , \tag{4.14}$$

i.e.

$$\tilde{\mathcal{L}} = 2xa\dot{b} - \frac{m_\perp^2}{2}(a^2 + b^2) \tag{4.15}$$

We now have only one canonical momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{b}} = 2xa \tag{4.16}$$

and the canonical Hamiltonian is simply

$$\begin{aligned} H &= \pi\dot{b} - \tilde{\mathcal{L}} = m_\perp^2(a^2 + b^2) \quad , \\ H &= \frac{m_\perp^2}{2}\left(\frac{\pi^2}{4x^2} + b^2\right) \quad . \end{aligned} \tag{4.17}$$

Thus we have obtained <u>one</u> harmonic oscillator with correct frequency (4.13) and it will be our starting point for quantizing the scalar field. It is to be stressed that the Hamiltonian (4.17), although obtained by a route not followed in usual treatment of the light front quantization [9], is fully in accord with other treatments. The approach pursued here is the most conservative one, from Lagrangian to Hamiltonian, identifying canonical variables etc.

In order to recover the full field Hamiltonian let us reintroduce momentum indices and use eq. (4.16)

$$H_k = m_\perp^2(a_k^2 + b_k^2) = \frac{m_\perp^2}{2} \mid \varphi_k \mid^2 \tag{4.18}$$

and

$$H = \int [dk] H_k = \int [dk] \frac{1}{2}(\mu^2 + k^i k^i) \mid \varphi_k \mid^2 \tag{4.19}$$

and returning to configuration space via eq. (4.1)

$$H = \int d_3x \frac{1}{2}(\varphi_{,i}\varphi_{,i} + \mu^2\varphi^2) \tag{4.20}$$

This is the well known form of the τ-translation generator [9]

5 Canonical Quantization

In order to quantize our system we will use the Hamiltonian (4.17),

$$H = \frac{m_\perp^2}{2}\left(\frac{\pi^2}{4x^2} + b^2\right) \tag{5.1}$$

with π and b being the canonically conjugated variables. I therefore impose the natural commutation relations

$$[b, \pi] = i \tag{5.2}$$

from which it is easy to find the commutation relations of the field itself; we have, by using eq. (4.16)

$$[b, a] = \frac{i}{2x} \tag{5.3}$$

and by using eq. (4.7) we obtain

$$[\varphi_k, \varphi_{-q}] = -\frac{\delta_3(q-k)}{2q^p} \tag{5.4}$$

In configuration space is convenient to compute the equal τ commutator

$$\begin{aligned} &\left[\frac{\partial\varphi(x,\tau)}{\partial x^m}, \varphi(y,\tau)\right] \\ = &\frac{1}{2\pi)^3}\int [dk][dq](ik^p)\left[e^{ikx}\varphi_k + e^{-ikx}\varphi_{-k}\ ,\ e^{iqx}\varphi_q + e^{-iqx}q_{-q}\right] \\ = &-\frac{i}{2}\delta_3(x-y) \end{aligned} \tag{5.5}$$

Integrating this over x^m we have the well known [7,8,9] non local commutation relation

$$[\varphi(x,\tau), \varphi(y,\tau)] = -\frac{i}{2}\theta(x-y) \tag{5.6}$$

which is usually postulated heuristically by the use of the Schwinger variational principle [12]

Notice that field components φ_k have commutation relations, eq. (5.4) without i, characteristic of creation and annihilation operators. It follows that the field components φ_k with momenta having their "plus" component

positive $k^p > 0$ are proportional to creation operators, while negative "plus" component field $k^p < 0$ is proportional to annihilation operator.

Let us introduce new operators

$$A_q = \sqrt{2q^p}\ \varphi_{-q} \quad , \tag{5.7}$$

$$A_q^\dagger = \sqrt{2q^p}\ \varphi_q \quad . \tag{5.8}$$

We have now

$$\left[A_q, A_k^\dagger\right] = \delta_3(q-k) \tag{5.9}$$

The field itself has a simple decomposition

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty \frac{dq^p}{\sqrt{2q^p}} \int d_2q_\perp (A_q e^{-iqx} + A_q^\dagger e^{iqx}) \tag{5.10}$$

Compared to equal time formalism we notice halving of degrees of freedom in our system; in eq. (5.10) we integrate only over momenta having positive "plus" component. Field hamiltonian eq. (4.19) is now simply

$$H = \int [dq] \left(\frac{\mu^2 + q_\perp^2}{2q^p}\right) (A_q^\dagger A_q + \frac{1}{2}) \tag{5.11}$$

Another important difference between equal time and light front is that the relations (5.7,5.8) between the field $\varphi(x)$ and the creation and annihilation operators does not depend on the mass μ of the field. On one hand this makes the relations (5.7,5.8) mass renormalization independent, on the other hand, however, it makes impossible to use Ritz variational principle estimates. In effect in this approach [19,20] one uses the renormalized mass as a variational parameter to construct a vacuum wave functional for the interacting field and to obtain the non-perturbative mass-gap estimate. Fortunately methods have been developed [21] to perform correctly variational calculation on the light-front.

6 A novel way to eliminate constraints

In this chapter I will present a new alternative to Dirac treatment of constraints. This mathematical trick will allow me, as I will show later, to quantize gauge fields without gauge fixing. This procedure is in a sense the

reversal of the one used to study field condensates where one introduces a fictitious field, whose value is equal at all times to the condensate itself. I will illustrate this approach on the example with one degree of freedom x with the Lagrangian

$$L = \frac{\dot{x}^2}{2} - \frac{g^2}{4}x^4 \quad , \tag{6.1}$$

this is the well known quartic oscillator and we have the classical equation of motion

$$\ddot{x} = -g^2 x^3 \tag{6.2}$$

Let us introduce a fictitious variable σ with the Lagrangian

$$L_c = \frac{\dot{x}^2}{2} + \frac{g}{\sqrt{2}}\sigma x^2 + \frac{\sigma^2}{2} \tag{6.3}$$

The equations of motion are now

$$0 = \frac{g}{\sqrt{2}}x^2 + \sigma \tag{6.4}$$

$$\ddot{x} = \sqrt{2}g\sigma x = -g^2 x^3 \tag{6.5}$$

i.e. the Lagrangians (6.1) and (6.3) give the same classical equations of motion. Moreover the variable σ is now proportional to x^2 at all times. Notice that the Lagrangian (6.3) has constraints since the time derivative of σ does not appear, so σ lacks its conjugate momentum. We have to use Dirac's procedure [17,18] or, alternatively, proceed as follows. I will use the Feynman path integral to define the transition amplitude [22]

$$\begin{aligned} U &= \int \mathcal{D}[x(t)]\mathcal{D}[\sigma(t)] exp\left\{ i \int_0^t \mathcal{L}(x, \dot{x}, \sigma) dt \right\} \\ &= \int \mathcal{D}[x(t)]\mathcal{D}[\sigma(t)] exp\left\{ i \int_0^t dt \left[\frac{\dot{x}^2}{2} + \frac{g}{\sqrt{2}}\sigma x^2 + \frac{\sigma^2}{2} \right] \right\} \end{aligned} \tag{6.6}$$

As usual I will skip the details of the normalization. In the expression (6.6) it is possible to integrate out independently the variable $\sigma(t)$ at each point of the time because the time derivative of σ is lacking. The time derivative couples σ's at different times. Therefore the source of difficulty for canonical treatment in Lagrangian (6.3) simplifies the Feynman path integral approach!

In effect after integrating at each t the variable $\sigma(t)$ (a simple task : the integral is Gaussian) we obtain

$$U = \int \mathcal{D}[x(t)] exp\left\{ i \int_0^t dt \left[\frac{\dot{x}^2}{2} - \frac{g^2 x^2}{4} \right] \right\} \tag{6.7}$$

i.e. an amplitude corresponding to the starting Lagrangian (6.1). This procedure of integrating out the constraint has permitted us to recover the original unconstrained Lagrangian (6.1). Applying the same treatment to the scalar field Lagrangian (4.15) we have

$$\begin{aligned} U &= \int \mathcal{D}[a(\tau)]\mathcal{D}[b(\tau)] \\ &\times \; exp\left\{ i \int_{\tau_1}^{\tau_2} d\tau \left[xa\dot{b} - \frac{m_\perp^2}{2}(a^2 + b^2) \right] \right\} \end{aligned} \tag{6.8}$$

We can now integrate at each τ (the light front "time") the variable $a(\tau)$

$$U = \int \mathcal{D}[b(\tau)] exp\left\{ i \int_{\tau_1}^{\tau_2} d\tau \left[\frac{2x^2}{m_\perp^2}\dot{b}^2 - \frac{m_\perp^2}{2} b^2 \right] \right\} \tag{6.9}$$

Therefore the Lagrangian (4.15) is equivalent to the harmonic oscillator Lagrangian

$$L = \frac{2x^2}{m_\perp^2}\dot{b}^2 - \frac{m_\perp^2}{2} b^2 \tag{6.10}$$

with the Hamiltonian

$$H = \frac{m_\perp^2}{2}\left(\frac{\pi^2}{4x} + b^2 \right) \tag{6.11}$$

We can now quantize canonically the Hamiltonian (6.11); one recovers all the results of sec 5.

7 The Dirac Field

Let us write the Dirac field Lagrangian density

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - \mathcal{M})\psi(x) \tag{7.1}$$

in the light front variables

$$\mathcal{L} = \bar{\psi}\left(i\gamma^p\frac{\partial}{\partial\tau} + i\gamma^m\frac{\partial}{\partial\xi} - i\gamma^j\nabla^j - \mathcal{M}\right)\psi(x) \tag{7.2}$$

where I have introduced γ matrices with "plus" and "minus" indices

$$\gamma^p := \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^3) \tag{7.3}$$

$$\gamma^m := \frac{1}{\sqrt{2}}(\gamma^0 - \gamma^3) \tag{7.4}$$

Before proceeding in our now usual way a few remarks are in order. The halving of degrees of freedom we observed earlier on the light front as compared with $t = 0$ dynamics is the consequence of the fact that the Einstein energy-momentum relation

$$p_\mu p^\mu = m^2 \tag{7.5}$$

has <u>two</u> solutions for the normal energy $E = p_0$

$$E = \pm\sqrt{\vec{p}^2 + m^2} \tag{7.6}$$

while on the light front the "energy" $\mathcal{E} = p^p$ is unambiguous

$$\mathcal{E} = \frac{m^2 + \vec{p}_\perp^2}{2p^m} \tag{7.7}$$

For the scalar field this had the consequence that the equations of motion (3.12) were of the first order in the "time" τ. In case of the Dirac field the equations of motion are linear in normal time; this was, incidentally, the very reason for their introduction. Nevertheless in view of the universality of the relation (7.5) in this case too we expect halving of degrees of freedom.

In order to simplify our analysis let us introduce the Fourier transform of the field $\psi(x,\tau)$ defined by

$$\begin{aligned}\psi(x,\tau) &:= \frac{1}{(2\pi)^{3/2}}\int dp^p d_2k_\perp \\ &\times \quad exp(ik^p\xi - ik_\perp x_\perp)\psi_k(\tau)\end{aligned} \tag{7.8}$$

The Lagrangian for one mode k is now

$$L_k = \psi_k^\dagger\left(i\gamma^0\gamma^p\frac{d}{d\tau} - k^p\gamma^0\gamma^m - k^j\gamma^0\gamma^j - \mathcal{M}\gamma^0\right)\psi_k \tag{7.9}$$

Introducing

$$\Lambda^c = \sqrt{2}\gamma^0\gamma^c \qquad , \quad c = p, m \tag{7.10}$$

and

$$\mathcal{M}_\perp = \gamma^0(\mathcal{M} + k^j\gamma^j) \tag{7.11}$$

it is easily checked that with the definitions (7.3,7.4) the Dirac matrices satisfy anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad , \ \mu, \nu = p, m, i \tag{7.12}$$

with the metric tensor given by (1.2). Matrices Λ^c are in fact projection operators

$$\Lambda^c\Lambda^d = \Lambda^c\delta^{dc} \quad , \ c, d = p, m \tag{7.13}$$

and they have simple commutation relations with $\mathcal{M}_\perp$

$$\Lambda^c\mathcal{M}_\perp = \mathcal{M}_\perp\Lambda^{\bar{c}} \ ; \tag{7.14}$$

where bar on the index means the replacement $p \to m$, $m \to p$. The Lagrangian (7.9) is now

$$L_k = \psi_k^\dagger \left(i\sqrt{2}\Lambda^p \frac{d}{d\tau} - \sqrt{2}k^p\Lambda^m - M_\perp \right) \psi_k \tag{7.15}$$

which prompts us to define 2 component quantities

$$\psi_k^c = \Lambda^c\psi_k \quad , \quad c = p, m \tag{7.16}$$

so that eq. (7.15) is

$$\begin{aligned} L_k &= i\sqrt{2}(\psi_k^p)^\dagger \frac{d\psi_k^p}{d\tau} - \sqrt{2}k^p(\psi_k^m)^\dagger\psi_k^m \\ &- (\psi_k^p)^\dagger\mathcal{M}_\perp\psi_k^m - (\psi_k^m)^\dagger\mathcal{M}_\perp\psi_k^p \end{aligned} \tag{7.17}$$

where we have made use of eq. (7.14). We recognize that ψ_k^m does not have a time derivative in eq. (7.17); it is therefore a constraint variable, not a dynamical one. The equations of motion are

$$i\sqrt{2}(\psi_k^p)^\dagger = -(\psi_k^m)^\dagger\mathcal{M}_\perp \tag{7.18}$$

$$0 = -\sqrt{2}k^p(\psi_k^m)^\dagger - (\psi_k^p)^\dagger\mathcal{M}_\perp \tag{7.19}$$

$$0 = i\sqrt{2}\dot{\psi}_k^p - \mathcal{M}_\perp\psi_k^m \tag{7.20}$$

$$0 = -\sqrt{2}k^p\psi_k^m - \mathcal{M}_\perp\psi_k^p \tag{7.21}$$

Dynamics is contained equations (7.18) and (7.20), equations (7.19) and (7.21) can be used to eliminate ψ_k^m and $(\psi_k^m)^\dagger$:

$$\psi_k^m = -\frac{\mathcal{M}_\perp}{\sqrt{2}k^p}\psi_k^p \tag{7.22}$$

$$(\psi_k^m)^\dagger = -(\psi_m^p)^\dagger\frac{\mathcal{M}_\perp}{\sqrt{2}k^p} \tag{7.23}$$

We have the equations of motion for dynamical quantities ψ_k^p , $\psi_k^{m\dagger}$

$$i\dot{\psi}_k^p = -\frac{\mathcal{M}_\perp^2}{2k^p}\psi_k^p \tag{7.24}$$

$$-i(\dot{\psi}_k^p)^\dagger = (\psi_k^p)^\dagger\frac{\mathcal{M}_\perp^2}{2k^p} \tag{7.25}$$

These equations are indeed correct since they predict oscillation with frequency $\pm(\mathcal{M}+k_\perp^2)/2k^p$, as seen by looking for the eigenvalues of the matrix $\mathcal{M}_\perp$. Let us also notice that in the case of a gauge coupling to a vector field A^μ obtained by the replacement (minimal coupling)

$$\partial_\mu \to \partial_\mu - ieA_\mu \tag{7.26}$$

the term k^p would be modified by A^p, the "plus" component of A_μ, which would complicate the inversion (7.24,7.25). For this reason gauge fields are quantized almost uniquely in the "axial" gauge

$$A^p(x) = 0 \tag{7.27}$$

The passage to Hamiltonian form is straightforward. The only non zero canonical momentum is

$$\pi_k^p = \frac{\partial L_k}{\partial\dot{\psi}_k^p} = i\sqrt{2}(\psi_k^p)^\dagger \tag{7.28}$$

We obtain

$$H_k = \pi_k^p\dot{\psi}_k^p - L_k = i\pi_k^p\frac{\mathcal{M}_\perp^2}{3k^p}\psi_k^p \tag{7.29}$$

where I used eq.'s (7.24,7.25) to eliminate ψ_k^m , $(\psi_k^m)^\dagger$. It is now natural to impose the canonical anticommutation relations

$$\{\pi_k^p\ ,\ \psi_{k'}^p\} = -i\Lambda^p\delta_3(k-k') \tag{7.30}$$

$$\{\pi_k^p\ ,\ \pi_{k'}^p\} = 0 \tag{7.31}$$

$$\{\psi_k^p\ ,\ \psi_{k'}^p\} = 0 \tag{7.32}$$

Returning to configuration space we have

$$\{(\psi^p)^\dagger(x,\tau)\ ,\ \psi^p(y,\tau)\} = \frac{1}{\sqrt{2}}\Lambda^p\delta_3(x-y) \tag{7.33}$$

with other anticommutators vanishing

Total Hamiltonian is obtained by integrating eq. (7.29) over all k's. One can eliminate $\mathcal{M}_\perp^2/2k^p$ by the use of eq.'s (7.19,7.21)

We finally obtain

$$A = \int d_3x i\sqrt{2}\left\{\psi^\dagger(x)\frac{\partial\psi(x)}{\partial\tau} - \frac{\partial\psi^\dagger(x)}{\partial\tau}\psi(x)\right\} \tag{7.34}$$

This is precisely the Hamiltonian (P^-generator) and the anticommutation relation postulated in [9].

8 Electromagnetic field in the temporal gauge

I will present the quantization of the gauge fields starting with electromagnetic field quantized at the $t = 0$ surface. In general gauge fields have a large group of symmetries: gauge transformations. In the canonical approach gauge symmetry can be used to obtain a simple canonical scheme without constraints; on the level of the Feynman path integral approach this symmetry was a source of difficulty solved only by Faddeev-Popov theory [23]. The whole story can be summarized by the "equation"

$$2 = 4 - 1 - 1 \tag{8.1}$$

In effect vector field A^μ has 4 degrees of freedom at each point in space; on the other hand 2 degrees only are physical. We will see different ways of eliminating those 2 spurious components. Consider the electromagnetic field $A^\mu(x)$ coupled to a conserved current j^μ, whose exact nature need not be specified at this point. The Lagrangian is

$$L = \int\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A^\mu j_\mu\right)d_3x \tag{8.2}$$

where

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{8.3}$$

is the usual electromagnetic tensor. Lagrangian density (8.2) has a gauge symmetry, it is namely invariant under the replacement

$$A^\mu \to A^\mu + \partial^\mu \chi \tag{8.4}$$

where χ is an arbitrary function. The term coupling A^μ to j^μ is not invariant under (8.4); it picks up a piece

$$\chi_{,\mu} j^\mu \quad ; \tag{8.5}$$

however, if a current is conserved

$$j^\mu{}_{,\mu} = 0 \tag{8.6}$$

we can add eq. (8.6) multiplied by χ to eq. (8.5)

$$\chi_{,\mu} j^\mu + \chi j^\mu{}_{,\mu} = \partial_\mu(\chi j^\mu) \tag{8.7}$$

So the transformed Lagrangian differs from the original one by a total divergence which does not affect the equations of motion. It is convenient to introduce non-covariant notation since we will use $t = 0$ formalism anyway. Let us denote

$$\varphi := A^0 \tag{8.8}$$

and

$$\rho := j^0 \tag{8.9}$$

We have the Lagrange density (8.2) in new variables

$$\begin{aligned} \mathcal{L} &= \frac{1}{4}\varphi_{,j}\,\varphi_{,j} + \varphi_{,j}\,\dot{A}^j \\ &+ \frac{1}{2}\dot{A}^i\dot{A}^i - \frac{1}{2}A^i{}_{,j}\,(A^i{}_{,j} - A^j{}_{,i}) \\ &- \rho\varphi + j^i A^i \end{aligned} \tag{8.10}$$

This can be partially diagonalized by the use of modes (Fourier components)

$$\varphi(\boldsymbol{k}, t) = \frac{1}{(2\pi)^{3/2}} \int d_3x e^{-i\boldsymbol{k}\boldsymbol{x}} \varphi(\boldsymbol{x}; t) \tag{8.11}$$

and similarly for other quantities.
Our Lagrangian (8.2) is now a sum over modes

$$L = \int d_3\boldsymbol{k} L_k \tag{8.12}$$

with

$$\begin{aligned}
\mathrm{L}_k = & \left(\frac{\boldsymbol{k}^2}{2}\right)\varphi(\boldsymbol{k})\varphi(-\boldsymbol{k}) + i\varphi(\boldsymbol{k})(k^j\dot{A}^j(-\boldsymbol{k})) \\
& + \frac{1}{2}\dot{A}^i(\boldsymbol{k})\dot{A}^i(-\boldsymbol{k}) \\
& - \left(\frac{\boldsymbol{k}^2}{2}\right)(A^i(\boldsymbol{k})A^i(-\boldsymbol{k})) + \frac{1}{2}(k^iA^i(\boldsymbol{k}))(k^jA^j(-\boldsymbol{k})) \\
& + \rho(\boldsymbol{k})\varphi(-\boldsymbol{k}) + j^i(\boldsymbol{k})A^i(-\boldsymbol{k})
\end{aligned} \tag{8.13}$$

We can decompose various vector quantities into components parallel to the vector $\boldsymbol{k}$

$$k^iA^i = kA_{\|} \quad , \quad k^ik^i = k^2 \tag{8.14}$$

and components perpendicular to it

$$\boldsymbol{k}\boldsymbol{A}_{\perp} = 0 \tag{8.15}$$

The Lagrangian (8.13) splits into 2 independent parts

$$L_k = L_k^{\|} + L_k^{\perp} \tag{8.16}$$

with

$$\begin{aligned}
L_k^{\|} = & \frac{k^2}{2}\varphi(k)\varphi(-k) + ik\varphi(k)\dot{A}_{\|}(-k) + \rho(k)\varphi(-k) \\
& + \frac{1}{2}\dot{A}_{\|}(k)\dot{A}_{\|}(-k) + j_{\|}(k)A_{\|}(-k)
\end{aligned} \tag{8.17}$$

and

$$L_k^{\perp} = \frac{1}{2}\dot{\boldsymbol{A}}_{\perp}(k)\dot{\boldsymbol{A}}_{\perp}(-k) - \frac{k^2}{2}\boldsymbol{A}_{\perp}(k)\boldsymbol{A}_{\perp}(-k) + \boldsymbol{j}_{\perp}(k)\boldsymbol{A}_{\perp}(-k) \tag{8.18}$$

Remembering that our field A^{μ} is real i.e.

$$A^{\mu}(-k) = A^{\mu}(k)^{\star} \tag{8.19}$$

we recognize immediately in (8.18) 2 harmonic oscillators with frequency

$$w = k \tag{8.20}$$

coupled linearly to the current $\boldsymbol{j}_\perp$.
This is exactly the physical content of (8.2): for each momentum $\boldsymbol{k}$ we have 2 degrees of freedom $\boldsymbol{A}_\perp(k)$ (representing,say, two helicities).

The formalism has somehow to get rid of the spurious components φ and $A_\parallel$. In order to analyze the Lagrangian (8.17) let us write explicite the real and imaginary parts of the fields, exactly as we did with the scalar field (cf. eq. 4.7); we will also drop the momentum index

$$\begin{aligned} \varphi(k) &= a + ib \\ \varphi(-k) &= a - ib \quad , \end{aligned} \tag{8.21}$$

$$\begin{aligned} A_\parallel(k) &= A + iB \\ A_\parallel(-k) &= A - iB \end{aligned} \tag{8.22}$$

In the same manner the current is now

$$\begin{aligned} \rho(k) &= f_0 + ig_0 \\ \rho(-k) &= f_0 - ig_0 \end{aligned} \tag{8.23}$$

and

$$\begin{aligned} j_\parallel(k) &= f_\parallel + ig_\parallel \\ j_\parallel(-k) &= f_\parallel - ig_\parallel \end{aligned} \tag{8.24}$$

Current conservation, eq. (8.6) links longitudinal and temporal components only

$$\dot{\rho} - ik^j j^j = 0 \tag{8.25}$$

or

$$\dot{\rho} - ikj_\parallel = 0 \tag{8.26}$$

With the help of our definitions eq. (8.24) it can be written

$$\begin{aligned} \dot{f}_0 &= kg_\parallel \ , \\ \dot{g}_0 &= -kf_\parallel \ . \end{aligned} \tag{8.27}$$

The Lagrangian to be analyzed, eq. (8.17) is now

$$\begin{aligned} L^{\|} &= \frac{k^2}{2}(a^2+b^2) + k(a\dot{B} - b\dot{A}) - f_0 a - g_0 b \\ &+ \frac{\dot{A}^2}{2} + \frac{\dot{B}^2}{2} + f_{\|}A + g_{\|}B \end{aligned} \tag{8.28}$$

We notice immediately that the temporal variables a, b lack time derivative: they are constraint variables.
One obtains usually the simplest dynamics by choosing a gauge which makes the passage to Hamiltonian dynamics simplest possible, i.e. without constraints. Here it is obviously

$$a = b = 0 \tag{8.29}$$

which can be written

$$A^0(x) = 0 \tag{8.30}$$

By this gauge fixing we have then eliminated one degree of freedom (A^0) The residual Lagrangian is now

$$L = \frac{\dot{A}^2}{2} + \frac{\dot{B}^2}{2} + f_{\|}A + g_{\|}B \tag{8.31}$$

The gauge fixing eq. (8.29) is not complete, we have freedom to perform a <u>time independent gauge transformation.</u> It follows that Lagrangian eq. (8.31) has an additional symmetry, whose generator is the <u>Gauss law</u>

$$\nabla \boldsymbol{E} - \rho = 0 \tag{8.32}$$

where $\boldsymbol{E}$ is the electric field

$$\boldsymbol{E} = -\nabla\varphi + \dot{\boldsymbol{A}} \tag{8.33}$$

In our gauge $\varphi = 0$, so we can write eq. (8.32) in Fourier components

$$\begin{aligned} k\dot{A} + g_0 &= 0 \\ k\dot{B} - f_0 &= 0 \end{aligned} \tag{8.34}$$

In order to exhibit this symmetry let us replace in eq.(8.31) $f_{\|}$ and $g_{\|}$ by time derivatives given by eq's (8.27)

$$L = \left(\frac{\dot{A}^2}{2} - \frac{\dot{g}_0 A}{k}\right) + \left(\frac{\dot{B}^2}{2} + \frac{\dot{f}_0 B}{k}\right) \tag{8.35}$$

We modify now Lagrangian (8.35) without changing the equations of motion by adding a total derivative

$$L_1 = L + \frac{d}{dt}(Ag_0) - \frac{d}{dt}(Bf_0) \quad , \tag{8.36}$$

we have

$$L_1 = \left(\frac{\dot{A}^2}{2} + \frac{g_0\dot{A}}{k}\right) + \left(\frac{\dot{B}^2}{2} - \frac{f_0\dot{B}}{k}\right) \tag{8.37}$$

In this form we see that L_1 is translation invariant. Time independent gauge transformation is

$$A^i \to A^i + \nabla^i\chi(x) \tag{8.38}$$

or, in momentum representation

$$A_{||}(k) \to A_{||}(k) + ik\chi(k) \tag{8.39}$$

and in terms of A and B

$$\begin{aligned} A \to A &- kIm\chi(k) \\ B \to B &+ kRe\chi(k) \end{aligned} \tag{8.40}$$

so it is indeed an arbitrary translation of variables A, B. Passing to the Hamiltonian we have

$$\pi_A = \frac{\partial L_1}{\partial \dot{A}} = \dot{A} + \frac{g_0}{k} \tag{8.41}$$

and

$$\pi_B = \frac{\partial L_1}{\partial \dot{B}} = \dot{B} - \frac{f_0}{k} \tag{8.42}$$

and the Hamiltonian is

$$H = \frac{1}{2}\left(\pi_A - \frac{g_0}{k}\right)^2 + \frac{1}{2}\left(\pi_B + \frac{f_0}{k}\right)^2 \tag{8.43}$$

It is clear that this is translation invariant. The appropriate generator of this symmetry is of course the momentum. Notice that the Gauss law, eq. (8.32) states that

$$\pi_A = \pi_B = 0 \tag{8.44}$$

This cannot, of course, be understood as an operator equation, it would conflict with commutation relations. We must therefore impose the Gauss law as a condition on the wave function.

$$\pi_A \mid \psi >= \pi_B \mid \psi >= 0 \tag{8.45}$$

We therefore see that the Hamiltonian (8.43) has many eigenstates, but the only physically relevant are those fulfilling eq. (8.45). A different way of stating this fact is to say that the wave function of the system has to be invariant under time-independent gauge transformations. Those remarks are valid for every momentum k, so we have in reality an infinity of conditions for the whole wave functional.

With condition (8.42) the Hamiltonian is simply the Coulomb energy of charge distribution ρ,

$$H_{Coul} = \frac{(g_0^2 + f_0^2)}{2k^2} = \frac{\mid \rho(\boldsymbol{k}) \mid^2}{2k^2} \tag{8.46}$$

and any reference to longitudinal degrees of freedom $A_{\|}$ has disappeared.

9 Dirac treatment of constraints

Dirac's approach to the canonical Hamiltonian function of singular Lagrangians [17,18] is presented here not only for the sake of completeness but above all to underscore that this sophisticated approach is a fancy way to fix the gauge. Additionally, while excellent treatments of the subject exist [18], I believe that treating simple system described by Lagrangian (8.28) is a transparent and amusing application of this machinery. Examining eq. (8.28) we notice that it splits into two independent parts, one coupling a (real part of $\varphi(k)$) to B (imaginary part of $A_{\|}(k)$) and a second part coupling b (imaginary part of $\varphi(k)$) and A (real part of $A_{\|}(k)$). Treatment of both is identic, we choose for definiteness

$$L = \frac{k^2}{2}b^2 - kb\dot{A} - g_0 b + \frac{\dot{A}^2}{2} + f_{\|}A \tag{9.1}$$

In order not to be confused by the fancy footwork to follow, let as examine the physical content of eq. (9.1). The Lagrange equations are

$$(kb - \dot{A}) \;=\; \frac{g_0}{k}$$

$$\frac{d}{dt}(\dot{A} - kb) = f_{\|} \tag{9.2}$$

Firstly, the only physically observable quantity is the longitudinal part of the electrical field

$$E(k) = -ik\varphi(k) - \dot{A}(k) \tag{9.3}$$

whose imaginary longitudinal part is exactly $kb - \dot{A}$. Equation (9.2) states then

$$ImE_{\|}(k) = \frac{g_0}{k} \tag{9.4}$$

i.e. this is the electric field generated by change distribution g_0. Differentiating eq. (9.2) over time and inserting in second eq. (9.2) we obtain

$$-\frac{\dot{g}_0}{k} = f_{\|} \tag{9.5}$$

So this is simply the current conservation (cf. eq. 8.27). It is also follows that the system of equations (9.2) is in reality only one equation for quantities b and A. We can for instance choose to take $b(t)$ to be completely arbitrary, we have then

$$A(t) = \int_{t_0}^{t} dt' \left[kb(t') - \frac{g_0(t')}{k} \right] \tag{9.6}$$

Returning to the canonical analysis of eq. (9.1) we compute the canonical momenta

$$\begin{aligned} \pi_A &= \frac{\partial L}{\partial \dot{A}} = \dot{A} - kb \\ &\text{and} \\ \pi_b &= \frac{\partial L}{\partial \dot{b}} \approx 0 \end{aligned} \tag{9.7}$$

The π_b momentum vanishes weakly, giving the first primary constraints, obviously incompatible with canonical Poisson brackets

$$\begin{aligned} \{A, \pi_A\} &= 1 \\ \{b, \pi_b\} &= 1 \end{aligned} \tag{9.8}$$

Using eq. (9.7) we see that the equation (9.2) is also a weak equation

$$\pi_A + g_0/k \approx 0 \tag{9.9}$$

The canonical Hamiltonian is

$$\begin{aligned} H_c &= \pi_A \dot{A} + \pi_b \dot{b} - L \\ &= \frac{\pi_A^2}{2} + kb\pi_A + g_0 b - f_{\|} A \end{aligned} \tag{9.10}$$

We have also two first class constraints (their Poisson bracket vanishes)

$$\begin{aligned} \psi_1 &= \pi_b \approx 0 \\ \psi_2 &= k\pi_A + g_0 \approx 0 \end{aligned} \tag{9.11}$$

The most general Hamiltonian, according to Dirac, is the canonical one with constraints added with arbitrary coefficients

$$\begin{aligned} H &= H_c + v_1\psi_1 + v_2\psi_2 \\ &= \frac{\pi_A^2}{2} kb\pi_A + g_0 b - f_{\|} A \\ &+ v_1\pi_b + v_2(k\pi_A + g_0) \end{aligned} \tag{9.12}$$

The equations of motion are

$$\begin{aligned} \dot{A} &= \{A, H\} = \pi_A + kb \\ \dot{b} &= \{b, H\} = v_1 \\ \dot{\pi}_A &= \{\pi_A, H\} = f_{\|} + kv_2 \\ \dot{\pi}_b &= \{\pi_b, H\} = -k\pi_A - g_0 - v_1 \end{aligned} \tag{9.13}$$

One can set therefore $v_1 = \dot{b}$ and $v_2 = 0$ and the final Hamiltonian is

$$H = \frac{\pi_A^2}{2} + kb\pi_A + g_0 b - f_{\|} A + \dot{b}\pi_b \tag{9.14}$$

Here b is essentially an arbitrary function, to be eliminated only after <u>a choice of gauge has been made.</u>

Now I will show that the radiation gauge recovers our previous results. I will use the freedom of gauge to make constraints (9.11) second class (i.e. non commuting among themselves). Since $\pi_b \approx 0$ it is logical to choose

$$\psi_3 = b \approx 0 \tag{9.15}$$

similarly eq. (9.11) suggests

$$\psi_4 = kA \approx 0 \tag{9.16}$$

The matrix of Poisson brackets of constraints $\psi_1,\ \psi_2,\ \psi_3,\ \psi_4$ is

$$C_{ij} = \{\psi_i\ \psi_j\} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k^2 \\ -1 & 0 & 0 & 0 \\ 0 & -k^2 & 0 & 0 \end{bmatrix} \tag{9.17}$$

The inverse is

$$C^{-1}_{ij} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1/k^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1/-k^2 & 0 & 0 \end{bmatrix} \tag{9.18}$$

and the Dirac bracket between two any dynamical quantities r, s is

$$\{r,\ s\}^\star = \{r,\ s\} - \sum_{n,m=1}^{4} \{r,\ \psi_n\} C^{-1}_{nm} \{\psi_m,\ s\} \tag{9.19}$$

At this stage we can set our constraints ψ_i strongly to zero (i.e. identically). The Hamiltonian (9.14) reduces to

$$H = \frac{\pi_A^2}{2} \tag{9.20}$$

which in view of the eq. (9.14) is just the Coulomb energy of the charge ρ

$$H = \frac{g_0^2}{2k^2} \tag{9.21}$$

Finally let us remark that the Dirac brackets (9.19) imply

$$\{b,\ \pi_b\}^\star = \{A,\ \pi_A\}^\star = 0 \tag{9.22}$$

so these variables should not be quantized (we have eliminated them in our final Hamiltonian (9.20)).

10 Quantization of the electromagnetic field without gauge fixing

The material presented here, due to the remarkable simplicity of the method will be short but, in view of the importance of the result presented, it is felt that it deserves a separate chapter. Recall that the electromagnetic Lagrangian (8.10) after decomposition into modes split into a transverse part (eq. 8.18) wholly conventional and a longitudinal part (eq. 8.17) where A^0 component played the rôle of constraint. Instead of fixing the gauge, we will, as advertised in chapter 6, integrate over all possible A^0 (a and b variables in eq. (8.17)). The transition amplitude for the longitudinal part is defined by the Feynman path integral

$$\begin{aligned} U &= \int D[a(t)]D[b(t)]D[A(t)]D[B(t)] \\ &\times \; exp\left\{ i \int dt \left[\frac{k^2}{2}(a^2+b^2) + k(a\dot{B} - b\dot{A}) - f_0 a - g_0 b \right.\right. \\ &+ \left.\left. \frac{\dot{A}^2}{2} + \frac{\dot{B}^2}{2} + f_{\|}A + g_{\|}B \right] \right\} \end{aligned} \tag{10.1}$$

Again we can easily integrate over $a(t)$ and $b(t)$ at each time independently. The result is an effective Lagrangian.

$$\tilde{L} = \frac{-f_0}{2k^2} - \frac{g_0^2}{2k^2} + \frac{1}{k}\frac{d}{dt}(Bf_0 - Ag_0) \tag{10.2}$$

It contains simply the Coulomb energy (eq. 8.46) and a total derivative not changing the equations of motion. The longitudinal variables A, B have been eliminated. In this simple manner we have decomposed the electromagnetic field into dynamical transverse components and an instantaneous Coulomb interaction. No gauge was singled out: we have integrated over all gauges. The form of the final dynamics (i.e transverse degrees + Coulomb energy) was the consequence of choosing $t = 0$ hyperplane for quantization, or, in other words, singling out the t integration in the action appearing in the amplitude eq. (10.1). The total amplitude for a field is the path integral of a manifestly covariant quantity

$$exp\{i \int d_4x \mathcal{L}(\varphi, A^\mu, \text{etc})\} \tag{10.3}$$

It is the splitting of the integration in (10.3) into time (the evolution parameter) and position variables (enumerating the degrees of freedom) which breaks the manifest Lorenz invariance of (10.3). On the level of canonical quantization it corresponds to imposing the commutation relations on hyperplane $t = 0$. What has been achieved here is the demonstration of the fact that for electromagnetic field this suffices for quantization without any gauge fixing. In this particular case we recover the results of quantization in $A^0 = 0$ gauge. We will show later that for non-abelian fields we will discover new, nontrivial way of quantizing gauge fields.

11 Electromagnetic field on the light front

We will quantize the electromagnetic field with the Lagrangian (8.2) which in the light front coordinates takes the form

$$\begin{aligned} L &= \int dx^- d_2x^i \left[\frac{1}{2}(A^m{}_{,m})^2 - A^m{}_{,m} A^p{}_{,p}\right. \\ &+ A^m{}_{,i}(A^p{}_{,i} + A^i{}_{,p}) + \frac{1}{2}(A^p{}_{,p})^2 + A^p{}_{,i} A^i{}_{,p} \\ &+ A^i{}_{,m} A^i{}_{,p} - \frac{1}{2}A^{i,j}(A_{i,j} - A_{j,i}) \\ &+ \left. A^m j^p + A^p j_m - A^i j^i\right\} \end{aligned} \tag{11.1}$$

when we have suppressed the arguments of the field $A^i \equiv A^i(x^+, x^-, x^i)$ and $i = 1, 2$, we have also included the coupling to a conserved current j^μ. In order to simplify the formulae to follow we shall write the Fourier transform of the field A^μ with indices below and with the momentum dependence suppressed

$$A^\mu(\boldsymbol{x}, \tau) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\tau} dk^p d_2k_\perp A^\mu(\boldsymbol{k}) e^{i(k^p x^m - k^j x^j)} \tag{11.2}$$

with

$$\begin{aligned} A^\mu(k) &= a_\mu + ib_\mu \qquad , \ k^p > 0 \\ &= a_\mu - ib_\mu \qquad , \ k^p < 0 \end{aligned} \tag{11.3}$$

With the introduction of the notation

$$\begin{aligned} x &\equiv k^p \\ k &= \left[(k^1)^2 + (k^2)^2\right]^{1/2} \end{aligned} \tag{11.4}$$

eq. (11.1) gives us the mode Lagrangian

$$
\begin{aligned}
L_q &= \frac{x^2}{2}(a_m^2 + b_m^2) + x(b_m \dot{a}_p - a_m \dot{b}_p) \\
&+ a_m(k^2 a_p - xka_{\|}) + b_m(k^2 b_p - xkb_{\|}) \\
&+ \frac{1}{2}(\dot{a}_p^2 + \dot{b}_p^2) - ka_p \dot{b}_{\|} + kb_p \dot{a}_{\|} \\
&+ x(a_{\|}\dot{b} - b_{\|}\dot{a}_{\|}) \\
&- x(a_\perp \dot{b}_\perp - b_\perp \dot{a}_\perp) - \frac{k^2}{2}(a_\perp^2 + b_\perp^2) \\
&+ a_p f_m + b_p g_m + a_m f_p + b_m g_p \\
&- a_{\|} f_{\|} - b_{\|} g_{\|} - a_\perp f_\perp - b_\perp g_\perp
\end{aligned} \tag{11.5}
$$

where we have also used the Fourier components of the current

$$
j^\mu(k) = f_\mu + ig_\mu \tag{11.6}
$$

and introduced components parallel ($a_{\|}$) and perpendicular ($a_\perp$) to the 2 dimensional vector (k^1, k^2). The dot means the x^p derivative ("time" derivative). The continuity equation for the current takes now form

$$
\begin{aligned}
\dot{f}_p &= xg_m - kg_{\|} \quad , \\
\dot{g}_p &= -xf_m + kf_{\|} \quad .
\end{aligned} \tag{11.7}
$$

The Lagrangian (11.5) looks and is considerably more complicated than its equal time counterpart (eq. 8.17,8.18). One component only is uncoupled from the others, the Lagrangian for this perpendicular component is

$$
\begin{aligned}
L_\perp &= x(a_\perp \dot{b}_\perp - b_\perp \dot{a}_\perp) - \frac{k^2}{2}(a_\perp^2 + b_\perp^2) \\
&- a_\perp f_\perp - b_\perp g_\perp
\end{aligned} \tag{11.8}
$$

This is simply the Lagrangian for the scalar field analyzed in chapter 4 and 5. It displays the halving of degrees of freedom ($a_\perp$ is conjugated to $b_\perp$, cf. eq. 5.3). Note also that the continuity equations (11.7) do not restrict $f_\perp$ and $g_\perp$.

The remaining variables are coupled. The variables a_m and b_m lack the time derivative: those are constraint variables and the analogue of the Coulomb gauge at the light front is the gauge

$$
A^m = 0 \tag{11.9}
$$

a fact known [18] but seldom used. The variables $a_{\|}$ and $b_{\|}$ have their time derivatives entering linearly to the Lagrangian (11.5) as expected from light-front formalism since this leads to the halving of degrees of freedom. The variables a_p and b_p are the most surprising, since their time derivatives enter quadratically so, a priori, no halving of degrees of freedom is apparent. The canonical analysis would base on the gauge $A^m = 0$, which eliminates the constraint variables. However the resulting Lagrangian has still a large group of symmetries corresponding to time independent gauge transformations; the variables a_p, b_p and $a_{\|}$ and $b_{\|}$ enter asymmetrically and, last not least, if the vector field is coupled to the Dirac field, important simplifications occur when $A^p = 0$, as noted earlier. Again integrating out the constraint variables clarifies the situation completely. After integrating out variables a_m and b_m we are simply left with

$$\begin{aligned} \tilde{L} &= x(a_n\dot{b}_n - b_n\dot{a}_n) - \frac{k^2}{2}(a_n^2 + b_n^2) \\ &+ a_n(f_{\|} - \frac{k}{x}f_p) + b_n(g_{\|} - \frac{k}{x}g_p) \\ &- \frac{f_p^2 + g_p^2}{2x^2} \quad , \end{aligned} \tag{11.10}$$

where we have introduced new variables

$$\begin{aligned} a_n &= a_{\|} + \frac{k}{x}a_p \\ b_n &= b_{\|} + \frac{k}{x}b_p \end{aligned} \tag{11.11}$$

It is seen that a_n and b_n are the natural counterparts of $a_\perp$, $b_\perp$: they represent the other photon polarization. We have also obtained the light front Coulomb term: the last line of (eq. 10.10). The most remarkable is the coupling of a_n, b_n to the current: the coupling is only to the combination

$$j_{\|} - \frac{k}{x}j^p \tag{11.12}$$

There is no coupling to j^m components, and in the case of Dirac field all the simplifications of $A^p = 0$ gauge occur. Our procedure has therefore permitted us to identify the correct dynamical variables for quantization. We can

project out these variables from the full field $A^\mu(k)$ by using 2 polarization vectors $e_\mu(\lambda)$, $\lambda = 1, 2$

$$\begin{aligned} A_\perp(k) &= e_\mu(1) A^\mu(k) \\ A_n(k) &= e_\mu(2) A^\mu(k) \end{aligned} \tag{11.13}$$

where

$$\begin{aligned} e_\mu(1) &= \left(0, 0, -\frac{k^1}{k}, \frac{k^2}{k}\right) \\ e_\mu(2) &= \left(\frac{k}{x}, 0, \frac{k^1}{k}, \frac{k^2}{k}\right) \end{aligned} \tag{11.14}$$

If we introduce the vector

$$\eta^\mu = (1, 0, 0, 0) \tag{11.15}$$

and the on shell photon four momentum

$$k^\mu = \left(\frac{k^2}{2x}, x, k^1, k^2\right) \tag{11.16}$$

we see that $e_\mu(\lambda)$ are purely transverse

$$\eta^\mu e_\mu(\lambda) = k^\mu e_\mu(\lambda) = 0 \tag{11.17}$$

We can express the projection operator on the physical degrees of freedom (the polarization sum)

$$\begin{aligned} d_{\mu\nu}(k) &= \sum_{\lambda=1,2} e_\mu(\lambda) e_\nu(\lambda) \\ &= -g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{\eta_\alpha k^\alpha} \end{aligned} \tag{11.18}$$

This is precisely what one uses to define Feynman rules [6]. Equation (11.18) has a spurious singularity at $\eta_\alpha k^\alpha = 0$, i.e. when $k^p = 0$. At this point relations (11.11) become also singular and the light front energy eq. 3.15 becomes infinite. Therefore it is an infinite energy endpoint problem. In the variational approach to field theory on the light front [21] this point is excluded. From the point of view presented here, since all quantities are taken at positive k^p the singularity is of end point type. At the level of Feynman graphs it means that this singularity is to be regulated by principal value prescription.

12 Yang-Mills gauge field, canonical treatment

We will consider here, for simplicity the $SU(n)$ Yang-Mills fields without quarks; when needed the generalization is immediate. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a \quad , \tag{12.1}$$

where, in addition to summation over Lorentz indices μ, ν additional summation over group index a is implied. We have by definition

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g f_a^{bc} A_b^\mu A_c^\nu \tag{12.2}$$

The group index takes values $a = 1, 2 \cdots \quad (n^2-1)$ for $SU(n)$, g is the coupling constant and f_a^{bc} are structure constants of the $SU(n)$ group defined in terms of (n^2-1) generators T_a

$$i[T_a,\ T_b] = -f_{ab}^c\ T_c \tag{12.3}$$

We will use mostly SU(2) as illustration, we have then

$$f^{abc} = \mathcal{E}^{abc} \tag{12.4}$$

where $\mathcal{E}^{abs}$ is the totally antisymmetric symbol in 3 dimensions (for $n = 2$ $n^2 - 1 = 3$). The Lagrangian density (12.1) is invariant under infinitesimal gauge transformations

$$A_a^\mu \to A_a^\mu + g f_a^{bc} \Lambda_b(x) A_c^\mu(x) + \frac{\partial \Lambda_a}{\partial x_\mu} \tag{12.5}$$

where $\Lambda_a(x)$ are (n^2-1) arbitrary functions. Since $F_a^{\mu\nu}$ (eq. 12.2) is antisymmetric in μ, ν the time derivatives of $A_a^{\mu=0}$ do not enter the Lagrangian: we have again constraints. The dynamical content is best discussed in the temporal gauge

$$A_a^{\mu=0}(x) = 0 \tag{12.6}$$

This is a partial gauge fixing since (12.5) allows still <u>time-independent</u> gauge transformations. It is now easy to get the canonical Hamiltonian. The canonical momenta

$$\pi_a^i = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial A_i}{\partial t}\right)} \tag{12.7}$$

are clearly given in terms of time derivatives of the potentials. The canonical Hamiltonian density is

$$\begin{aligned} \mathcal{H} &= \pi_a^i \left(\frac{\partial A_i^a}{\partial t} \right) - \mathcal{L} \\ \mathcal{H} &= \frac{1}{2}(\pi_a^i \pi_i^a + B_a^i B_i^a) \end{aligned} \tag{12.8}$$

where we have introduced

$$B_a^i = \frac{1}{2}\mathcal{E}^{ijk} F_{a,jk} \tag{12.9}$$

The theory can be simply quantized, on the formal level, by imposing the commutation relations

$$[A_i^a \ , \ \pi_b^j] = i\delta_b^a \delta_j^i \tag{12.10}$$

The system has still an infinity of symmetries corresponding to time independent gauge transformations whose generators are

$$G^a(\boldsymbol{x}) = \frac{1}{g}\partial_i \pi_i^a + f^{abc} A_i^b \pi_i^c \tag{12.11}$$

i.e. we have a conservation law in every point of space

$$[H, G^a(\boldsymbol{x})] = 0 \tag{12.12}$$

The generators themselves satisfy

$$[G^a(\boldsymbol{x}), G^b(\boldsymbol{y})] = i\delta_3(\boldsymbol{x} - \boldsymbol{y}) f^{abc} G^c(x) \tag{12.13}$$

Setting all $G^a(\boldsymbol{x})$ to zero is obviously incompatible with the commutation relations (12.10). We have then <u>to impose on the wave functional</u> $\mid \psi >$ that for every x we have

$$G^a(x) \mid \psi >= 0 \tag{12.14}$$

This condition will be preserved in time and is compatible with relations eq. (12.13). It is often called "the Gauss law" and serves to reduce the number of degrees of freedom at each point $\boldsymbol{x}$ from $3(m^2 - 1)$ ($A_i^a(x)$ are the dynamical variables) to $2(n^2 - 1)$. In the abelian case (electromagnetism) this eliminates the longitudinal degrees of freedom. Thus presented the theory is very simple but one has to regularize it without destroying gauge and Lorentz invariance.

Changing to other gauges is best accomplished by using the Feynman path integral. This, as well known [23,24] introduces ghosts in some gauges. We present here the approach of Feynman in order to underscore the fact that standard quantization in other gauges is simply the change of variables in the path integral written in $A_a^0 = 0$ gauge. The Feynman path integral in this gauge is

$$\int exp\left\{-\frac{i}{4}\int d_4x F_a^{\mu\nu} F_{\mu\nu}^a\right\} \delta(A_0^a(\boldsymbol{x},t))\mathcal{D}[A_\mu^a(x,t)] \tag{12.15}$$

where $\delta(A_0^a(\boldsymbol{x},t))$ is a shorthand notation for

$$\prod_{a,\boldsymbol{x},t} \delta(A_0^a(\boldsymbol{x},t)) \tag{12.16}$$

In the expression (12.15) the action is manifestly gauge invariant. Integration measure

$$\mathcal{D}[A_\mu^a] = \Pi_{\mu\boldsymbol{x},t} dA_\mu^{a=1}(\boldsymbol{x},t) dA_\mu^{a=2} dA_\mu^{a=3} \tag{12.17}$$

is also gauge invariant, since in the transformation (12.5) we have a translation and a rotation in the group space and both transformations preserve the volume element (12.17). It appears then that the only object in (12.15) which changes with gauge is the δ functions (12.16).

For illustration purposes let us transform eq. (12.15) to the Landau gauge

$$\partial_\mu A_a^\mu = 0 \tag{12.18}$$

The δ eq. (12.16) get now an infinitely dimensional Jacobian

$$\delta(\partial_\mu A_a^\mu) = \delta(A_a^0)\frac{\Delta(\partial_\mu A_a^\mu)}{\Delta(A_a^0)} \tag{12.19}$$

The determinant in eq. (12.19) is difficult to compute directly since it involves a finite gauge transformation. The trick [23] is to calculate it using infinitesimal gauge transformations only. The idea is to use an indirect technique: computing the change of both $\partial_\mu A_a^\mu$ and A_a^0 under the same infinitesimal gauge transformation

$$\begin{aligned} \tilde{A}_0^a &= A_0^a + \partial_0\Lambda^a + \mathcal{E}^{abc}\Lambda_b A_0^c \\ \partial_\mu \tilde{A}_\mu^a &= \partial^\mu A_\mu^a + \partial^\mu\partial_\mu\tilde{\Lambda}^a \\ &+ \mathcal{E}^{abc}\tilde{\Lambda}^b(\partial^\mu A_\mu^c) \end{aligned} \tag{12.20}$$

where $\tilde{\Lambda}^a$ is the transform of Λ^a under the finite gauge transformation which transforms A_0^a gauge to $\partial_\mu A_a^\mu$ gauge. By the rule of chain differentiation we have then

$$\frac{\Delta(\partial_\mu A_a^\mu)}{\Delta(A_a^0)} = \frac{\Delta(\partial_\mu A_a^\mu)}{\Delta(\tilde{\Lambda}_a)} \cdot \frac{\Delta(\Lambda_a)}{\partial(A_a^0)} \tag{12.21}$$

The second factor is a constant (i.e. field independent) since it is the Jacobian of a translation of A_a^0 by $\partial_0 \Lambda^a$; the second term in (12.20, first line) disappears because $A_0^a = 0$. The first term is genuinely field dependent. It can be shown that it is exactly the result of an integration over grassman (anticommuting) complex variables $P_a(\boldsymbol{x}, t)$ of a gaussian

$$\begin{aligned} \frac{\delta(\partial_\mu A_a^\mu)}{\partial(\tilde{\Lambda}_a)} &= \int \mathcal{D}(P_a(\boldsymbol{x}, t)) \\ &\times \; exp\left\{\frac{i}{2}\int d_4 x P_a^+ (\partial_\mu \partial^\mu \delta_{ac} - \mathcal{E}_{abc} A_b^\mu \partial_\mu) P_c\right\} \end{aligned} \tag{12.22}$$

Thus we have introduced new scalar Dirac fields to the path integral (12.15). Those are, of course, the famous Faddeev-Popov ghosts. In general if we want to transform the gauge $A_o^a = 0$ to a gauge

$$f(A_\mu^a) = 0 \tag{12.23}$$

we first expand to first order

$$f(A_\mu^a + D_\mu \Lambda^a) = f(A_\mu^a) + \frac{\partial f}{\partial(A_\mu^a)} D_{ab}^\mu \Lambda^{de} \tag{12.24}$$

where we used the gauge derivative

$$D_{ac}^\mu = [\delta_{ac}\partial^\mu + g\mathcal{E}_{abc} A_b^\mu] \tag{12.25}$$

and this leads to the replacement

$$\begin{aligned} \delta(A_a^0) \;&\rightarrow\; \delta(f(A_a^\mu)) \\ &\times \; \mathcal{D}[P^a(\boldsymbol{x}, t] exp\left\{\frac{i}{2}\int d_4 x P_a^+ \left[\frac{\partial f}{\partial(A_\mu^0)} D_\mu^b\right] P_c\right\} \end{aligned} \tag{12.26}$$

In cases when $\partial f / \partial(A_\mu^a)$ is field independent (e.g. $A_a^3 = 0$ gauge) the integration over $P_a(\boldsymbol{x}, t)$ can be made independently of A_μ^a: we have a ghostless gauge.

This treatment is perfectly correct for perturbative expansions. In effect the path integral (12.15) must operate on states which fulfill the Gauss law (cf. eq. 12.14). In the perturbative approach the state is usually the perturbative vacuum which fulfills eq. (12.14). Stated differently, even after gauge fixing we have $3(n^2-1)$ variables in each point in space instead of $2(n^2-1)$ we have to choose appropriate "transverse" variables. In $A_a^0 = 0$ gauge a popular choice is the Coulomb fixing

$$\partial^j A_j^a = 0 \tag{12.27}$$

This choice is obviously correct for small A_a^μ: this is the usual transversality condition for abelian field. For nonperturbative approach eq. (12.27) presents a difficulty. As noted by Gribov [25] there are fields satisfying eq. (12.27) and eq. (12.6) which are gauge transform one of another, the so called Gribov copies. This problem appears whenever the operator

$$\boldsymbol{\Delta}_{ab} = \partial_i D_{ab}^i(A_a^\mu) \tag{12.28}$$

has zero eigenvalues

$$\boldsymbol{\Delta}\chi_b = 0 \tag{12.29}$$

where χ_b has to have correct boundary conditions. The set of all A_a^μ's which allow zero eigenvalue (eq. 12.29) is the Gribov horizon. Gribov has shown that there exist fields A_a^μ being gauge transforms one of another which lie on both sides of the horizon. Worse still, Singer [26] has shown that this holds true for any reasonable generalization, hence recent attempts to modify the path integral (12.15) to include only these A_a^μ's which lie inside the Gribov horizon [27,28].

13 Yang-Mills gauge field without gauge fixing

I will present here new results which follow from the method of chapter 9. In this case the treatment seems to be genuinely new and leads to expression which, albeit tantalizingly close to these obtained by Zwanziger [27], are nevertheless different. As noted by everyone in the field the obvious simple path integral

$$U = \int \mathcal{D}(A_\mu^a) exp\left\{-\frac{i}{4}\int d_4x F_{\mu\nu}^a F_a^{\mu\nu}\right\} \tag{13.1}$$

in meaningless since there are directions in $A^a_\mu(\boldsymbol{x},t)$ space along which the exponent of (13.1) is constant giving as a result infinity. In my approach, recall, one integrates first over constraints, i.e. the components which lack time derivative in the Lagrangian. Then one notices that some variables disappear from the Lagrangian and, of course, one does not integrate over them. The effective Lagrangian was therefore expressed solely in terms of physical variables (e.g. in constant time electrodynamics the transverse variables). In the present case we shall follow exactly the same route The crucial observation is that the Yang-Mills Lagrangian eq. (12.1), although in general it contains the fields A^μ_a up to quartic terms, the constraint fields (i.e. without time derivative) appear at most quadratically. Therefore our idea of integrating over constraints is perfectly doable: the integral is a Gaussian. In the case of Yang-Mills fields the use of Fourier components does not diagonalize the Lagrangian eq. (12.1). So we have to use matrix notation. Recall that for a general quadratic form with coefficients $\mathcal{M}_{ij}$, $i,j = 1\ldots\mathcal{N}$ and a constant vector b_i we have

$$\begin{aligned}\int d_{\mathcal{N}}y \quad exp \quad & \left\{-\frac{1}{2}y_i\mathcal{M}_{ij}y_j - b_iy_i\right\} \\ = \quad & \mathcal{D}et(\mathcal{M}_{ij})^{1/2}\ exp\ \left\{+\frac{1}{2}b_i\mathcal{M}^{-1}_{ij}b_j\right\}\end{aligned} \tag{13.2}$$

Let us discuss first the $t = 0$ case. For clarity of notation introduce

$$\varphi^a(\boldsymbol{x},t) \equiv A^a_0(\boldsymbol{x},t) \quad , \tag{13.3}$$

and the gauge gradient

$$D^{ac}_i = \partial_i\delta^{ac} + g\mathcal{E}^{abc}A^b_i \tag{13.4}$$

The Lagrangian eq. (12.1) is then given in terms of electric fields

$$E^a_i = F^a_{0i} = \dot{A}^a_i - D^{ab}_i\varphi^b \tag{13.5}$$

and magnetic fields

$$B^a_i = \mathcal{E}_{ijk}F^a_{jk} \tag{13.6}$$

where in our notation

$$F^a_{ij} = A^a_{j,i} - A^a_{i,j} - g\mathcal{E}^{abc}A^b_iA^c_j \tag{13.7}$$

so eq. (12.1) is simply

$$\mathcal{L} = \frac{1}{2}(E_i^a E_i^a - B_i^a B_i^a) \tag{13.8}$$

We notice immediately that time derivatives and the fields φ^a occur in the electric field (eq. 13.5) only. Furthermore the electric field depends on φ^a only linearly hence L is quadratic in φ^a. In order to integrate, at each point in time the amplitude (13.1) it is therefore necessary to take into account electric field only

$$I = \int \mathcal{D}[\varphi^a(\boldsymbol{x},t)]\ exp\ \left\{-\frac{i}{2}\int d_4x(E_i^a E_i^a)\right\} \tag{13.9}$$

The exponent is now, at each t

$$f(\varphi_0^a) = -\frac{i}{2}(\dot{A}_i^a - \varphi^c D_i^{\dagger ca})(\dot{A}_i^a - D_i^{ad}\varphi^d) \tag{13.10}$$

By using eq. (13.2) we obtain

$$\begin{aligned} I &= \mathcal{D}et[D_i^{\dagger ca} D_i^{ad}]^{-1/2} \\ &\times\ exp\left\{-\frac{i}{2}\int d_4x E_{tr}^{ai}(\boldsymbol{x},t)E_{tr}^{ai}(x,t)\right\} \end{aligned} \tag{13.11}$$

where the "transverse" electric field is defined

$$E_{tr}^{ai} = \dot{A}^{ia} - D_{ab}^i[D_k^\dagger D_k]_{bc}^{-1}(D_j^{\dagger cd}\dot{A}_d^j) \tag{13.12}$$

The field E_{tr}^{ac} is transverse in the sense that it fulfills

$$D_{ab}^{\dagger i} E_{tr}^{bi} = 0 \tag{13.13}$$

as easily seen from (13.12).

We see that exactly as in abelian case the integration over constraints (φ^a fields) eliminated (n^2-1) variables at each point. The condition (13.13) is the natural generalization of the Coulomb condition

$$\partial_i E_i = 0 \tag{13.14}$$

of the electromagnetic field. In contrast to usual treatment [23,28,27,29] where one imposes (13.14) on the fields in order to eliminate the unphysical

variables, the formalism presented here dictates the condition (13.13) which explicitly depends on A_a^i though the gauge derivative D_{ab}^i. Similarly the usual Coulomb gauge has the unsymmetrical Faddeev-Popov determinant

$$\mathcal{D}et[\partial_i D_i^{ab}] \tag{13.15}$$

whereas in our treatment the relevant determinant is

$$\mathcal{D}et[D_i^\dagger D_i] \tag{13.16}$$

The passage to Hamilton function is facilitated by the fact that the operator appearing in eq. (13.12) is a projection operator. To see this notice first that the gauge gradient eq. (13.4) is purely antihermitian and can therefore be diagonalized (its eigenvalues are purely imaginary). The eigenvalue equation for our operator is (v^i is the eigenfunction)

$$D_{ab}^i [D_k^\dagger D_k]_{bc}^{-1} D_{cd}^{\dagger j}\ v_d^j = \lambda v_a^i \tag{13.17}$$

and by multiplying by $D_{ab}^{\dagger i}$

$$D_{ab}^{\dagger i}\ v_b^i = \lambda D_{ab}^{\dagger i}\ v_b^i \tag{13.18}$$

So either $\lambda = 1$ or $\lambda = 0$, in the latter case v^i fulfills a simpler equation

$$D_{ab}^{\dagger i}\ v_b^i = 0 \tag{13.19}$$

The kinetic part of our final Lagrange density is (cf. eq. 13.11)

$$L_{kin} = \frac{1}{2} E_{tr}^{ai} E_{tr}^{ai} \tag{13.20}$$

By going to a basis which diagonalizes the operator (13.17) we derive the canonical Hamiltonian density

$$H_0 = \frac{1}{2} \pi_{tr}^{ai} \pi_{tr}^{ai} \tag{13.21}$$

where

$$\pi_{tr}^{ai} = \pi^{ia} - D_{ab}^i [D_k^\dagger D_k]_{bc}^{-1} (D_{cd}^{\dagger j} \pi_d^j) \tag{13.22}$$

and π^{ia} are variables canonically conjugated to A^{ia} (cf. eq. 12.10). The final Hamiltonian density which, let us stress it once again, depends on transverse, i.e. physical variables, can be written in following form

$$\begin{aligned} H &= \frac{1}{2}\left[\pi^i_a \pi^i_a + B^i_a B^i_a\right] \\ &- \frac{1}{2}(\pi^i_a D^i_{ab})[D^\dagger_k D_k]^{-1}_{bc}(D^{\dagger j}_{cd}\pi^j_d) \end{aligned} \tag{13.23}$$

The last term eliminates "longitudinal" variables and it can be written in terms of gauge transform generators G^a (12.11)

$$\frac{1}{2}G^a[D^\dagger_k D_k]^{-1}_{ab}G^b \tag{13.24}$$

The effect of this term is zero on gauge invariant wave functionals, it is also gauge invariant.

In the case of non-abelian fields we observe that our formalism forced us to use <u>curvilinear</u> coordinates E_{tr} and B_{tr} similar to those used in rigid body dynamics. The other difference with the Coulomb gauge is that, although the non-abelian field carries charge, no explicit Coulomb interaction appears due to this charge.

In the light front formalism the situation is somewhat more complicated. There are 3 types of variables

$$\varphi^a = A^m_a \tag{13.25}$$

which is a constraint, A^p_a whose time derivatives appear quadratically and A^i_a, $i = 1,2$ whose time derivatives appear linearly leading to halving of degrees of freedom. In the abelian case the integration over A^m restored the symmetry between A^p and A^i. In the present case we have explicitly

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= 2F^{pm}F_{pm} + 2F^{pi}F_{pi} \\ &+ 2F^{mi}F_{mi} + F^{ij}F_{ij} \end{aligned} \tag{13.26}$$

Introducing gauge derivatives

$$D^{ac}_m = [\delta_{ac}\partial_m + g\mathcal{E}^{abc}A^p_b] \tag{13.27}$$

and

$$D^{ac}_i = [\delta_{ac}\partial_i + g\mathcal{E}^{abc}A^b_i] \tag{13.28}$$

we can write

$$
\begin{aligned}
F^a_{pm} &= -F^{pm}_a = \dot{A}^p_a - D^{ab}_m \varphi^b \\
F^a_{pi} &= -F^{mi}_a = \dot{A}^i_a - D^{ab}_i \varphi^b \\
F^a_{mi} &= -F^{pi}_a = -\partial_i A^p_a - D^{ab}_m A^b_i
\end{aligned} \tag{13.29}
$$

and

$$
F^a_{ij} = F^{ij}_a = D^{ab}_i A^b_j - D^{ab}_j A^b_i \tag{13.30}
$$

Notice that in eq's. (13.29 last line) and (13.30) neither time derivatives nor φ's appear. The Lagrange density (12.1) is now

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left(\dot{A}^p_a - \varphi^b (D^\dagger_m)^{ba}\right)\left(\dot{A}^p_a - D^{ac}_m \varphi^c\right) \\
&- F^a_{mi}\left(\dot{A}_i - D^{ab}_i \varphi^b\right) - \frac{1}{4} F^a_{ij} F^{ij}_a
\end{aligned} \tag{13.31}
$$

In order the integrate the above expression over φ^b it is judicious to make first a shift

$$
\varphi^b = \chi^b + (D^{-1}_m)^{ab} \dot{A}^p_b \tag{13.32}
$$

We have now

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\chi^a [D^\dagger_m D_m]^{ab} \chi^b - (F^a_{mi} D^{ab}_i)\chi^b \\
&- F^a_{mi}[\dot{A}_i - (D_i D^{-1}_m)^{ab} \dot{A}^p_b] - \frac{1}{4} F^a_{ij} F^{ij}_a
\end{aligned} \tag{13.33}
$$

The integration over χ^a produces a Faddeev-Popov determinant

$$
\mathcal{D}et[D^\dagger_m D_m]^{1/2} \tag{13.34}
$$

and a Coulomb energy term

$$
E_{Coul} = \frac{1}{2} F^a_{mi} [D^\dagger_i (D^\dagger_m D_m)^{-1} D_j]^{ab} F^b_{mj} \tag{13.35}
$$

The second line of eq. (13.31) remains unchanged and is the new Lagrange density Notice that now all time derivatives $\dot{A}^a_i, \dot{A}^p$ appear linearly and that they are projected on a 2 dimensional subspace by contraction with 2-dimensional vector F^a_{mi}. These results are therefore the generalization of

the abelian case studied in chapter 10. Compared with $t = 0$ approach no decisive simplifications occur. The Faddeev-Popov determinant (13.25) is somewhat easier to study since it contains derivatives over x^m only. Whereas in $t = 0$ case the Faddeev-Popov determinant can be studied only perturbatively in the present case because eq. (13.25) depends only on one variable a WKB (semiclassical) analysis seems possible.

14 Infrared region of non-abelian fields

Our previous results can be analyzed on two extreme cases according to whether in the gauge transform eq. (11.5) dominate

a) the gradient term $\frac{\partial \Lambda}{\partial x_\mu}$

b) the non-abelian term $g f^{abc} \Lambda^b A_c^\mu$

The first case is evidently important when the gradients are important: this the high momentum region of the theory. One can treat the non-abelian additions as perturbation; the transversality condition eq. (12.13) becomes the usual Coulomb gauge, a situation often discussed in the literature [29].

The second and very interesting case is b): the fields A_i^μ are large and slowly varying: this corresponds to the infrared region of the theory which we will discuss in the extreme case of constant fields. In this case we have important simplifications. The gauge "derivative" now becomes

$$D_i^{ac} = g\mathcal{E}^{abc} A_i^b \tag{14.1}$$

The Lagrangian is, as before

$$L = \frac{1}{2}((\boldsymbol{E}_i)^2 + (\boldsymbol{B}_i)^2) \tag{14.2}$$

where we use the vector notation in the group space. We have now

$$\begin{aligned} \boldsymbol{E}_i &= \dot{\boldsymbol{A}}_i - D_i \varphi & (14.3) \\ &= \dot{\boldsymbol{A}}_i - g\boldsymbol{A}_i \times \varphi & (14.4) \end{aligned}$$

and

$$\boldsymbol{B}_i = \mathcal{E}_{ijk} \quad \boldsymbol{A}_j \times \boldsymbol{A}_k \tag{14.5}$$

The potential (i.e. the square of magnetic field) is now

$$\sum_i (B_i^2) = \frac{g^2}{2}\left\{\left(\sum_i A_i^2\right)^2 - \sum_{ij=1}^{3}(A_i A_j)^2\right\} \tag{14.6}$$

This potential is quartic in fields and it depends on relative positions of vectors $\boldsymbol{A}_i$ only: it is invariant against global notations in group space. It is also highly anisotropic it attains its minimum (0) for all fields parallel to each other,

$$\boldsymbol{A}_x \parallel \boldsymbol{A}_y \parallel \boldsymbol{A}_z \tag{14.7}$$

The integration over φ^a projects out the "longitudinal" part of the velocities $\dot{\boldsymbol{A}}_i$. We have

$$D_i \dot{\boldsymbol{A}}_i = g\boldsymbol{A} \times \dot{\boldsymbol{A}}_i \tag{14.8}$$

(i.e.) this is proportional to the rotation as a whole of the 3 vectors $\boldsymbol{A}_i$. Therefore, in this case also, the integration over constraint eliminated the unphysical variables: the Euler angles defining the notation of Lab frame to the body frame. The whole Lagrangian depends only on absolute magnitude of the vectors $\boldsymbol{A}_i$ and their relative angles. This is also the case for the determinant eq. (12.16), which is now, explicitly

$$det[D_i D_i]^{ab} = \begin{bmatrix} A_y^2 + A_z^2 & , & -A_x Ay & , & -A_x Az \\ -A_1 A_2 & , & A_x^2 + A_z^2 & , & -A_2 A3 \\ -A_1 A_3 & , & -A_2 A_3 & , & A_x^2 + A_y^2 \end{bmatrix} \tag{14.9}$$

It is a rather complicated function of the fields. However it has two important properties:

a) it is zero for parallel fields, i.e. fields fulfilling eq. (14.7)

b) for non parallel fields it behaves as r^6 where r is a order of magnitude of the fields.

To discuss the physical implications it is convenient to write the transition amplitude after Wick rotation, i.e. $t \to it$. We have then

$$U = \int D[\boldsymbol{A}_i(t)] Det[D_i D_i]^{1/2} exp\left\{-\frac{1}{2}\int_0^T dt(\boldsymbol{E}_i^2 + \boldsymbol{B}_i^2)\right\} \tag{14.10}$$

The determinant is local so we can add its logarithm to the exponent of (13.9). We obtain in this way an effective potential

$$V_{eff} = \frac{1}{2} B_i^a B_i^a - \frac{\Lambda^4}{2} ln\ Det[D_i D_i] \tag{14.11}$$

The arbitrary constant Λ had to be introduced for dimensional reasons: it depends explicitly on the way we normalize the integration in (14.10). The appearance of a scale Λ is the well known dimensional transmutation [30]. The most important fact about the effective potential (14.11) is that it has no more a minimum at $A_i^a = 0$! since the determinant then is zero. Our effective theory has not only correct variables but predicts also that

$$< B_i^a B_i^a > \neq 0 \tag{14.12}$$

Without making any detailed calculation we see immediately that it implies that

$$< F_{\mu\nu}^a F_a^{\mu\nu} > \neq 0 \tag{14.13}$$

and is of the order of Λ^4.

15 CONCLUSIONS

The method of integrating out the constraint variables in the Feynman path integral is a powerful one. It has permitted us not only to find all previously known results in a simple way, but, in the case of non abelian gauge theory, to write down explicitly the gauge invariant hamiltonian. This can be of use, e.g., to analyze the theory in strong coupling regime by the use of Ritz variational methods. Preliminary estimates indicate that in case of non-abelian gauge theories the vacuum is spontaneously broken by infrared modes.

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