



# Feigenbaum networks

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## Abstract

We study dynamical systems composed of a set of linearly coupled quadratic maps which, if uncoupled, would be on the Feigenbaum accumulation point.

For two units we prove the existence of an infinite number of sinks for an open set of coupling parameters. In the limit of many units a mean field analysis also implies the stabilization in periodic orbits of, at least, a subset of the coupled units.

Possible applications in the fields of control of chaos, signal processing through complex dynamics and as models of self-organization, are discussed.

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# 1 Introduction

In this paper we study dynamical systems composed of a set of linearly coupled quadratic maps

$$x_i(t+1) = 1 - \mu_* \left\{ \sum_j W_{ij} x_j(t) \right\}^2 \quad (1)$$

with  $x \in [-1, 1]$ ,  $\sum_j W_{ij} = 1 \forall i$  and  $W_{ij} > 0 \forall i, j$  and  $\mu_* = 1.401155\dots$ . The value chosen for  $\mu_*$  implies that, in the uncoupled limit ( $W_{ii} = 1, W_{ij} = 0$   $i \neq j$ ), each unit transforms as a one-dimensional quadratic map in the accumulation point of the Feigenbaum period-doubling bifurcation cascade. The coupling constants  $W_{ij}$  resembling the synaptic connections of a neural network, we call this system a Feigenbaum network. The system (1) has a rich dynamical behavior and linearly coupled systems of this type have been used in the past to model the dynamics of interacting populations[1] [2].

The quadratic map at the Feigenbaum accumulation point is not in the class of chaotic systems (in the sense of having positive Lyapunov exponents) however, it shares with them the property of having an infinite number of unstable periodic orbits. Therefore, before the interaction sets in, each elementary map possesses an infinite diversity of potential dynamical behaviors. As we will show later, the interaction between the individual units is able to selectively stabilize some of the previously unstable periodic orbits. The selection of the periodic orbits that are stabilized depends both on the initial conditions and on the intensity of the interaction coefficients  $W_{ij}$ . As a result Feigenbaum networks appear as systems with potential applications in the fields of control of chaos, information processing and as models of self-organization.

Control of chaos or of the transition to chaos has been, in recent years, a very active field (see for example Ref.[3] and references therein). Several methods were developed to control the unstable periodic orbits that are embedded within a chaotic attractor. Having a way to select and stabilize at will these orbits we would have a device with infinite storage capacity (or infinite pattern discrimination capacity). However, an even better control might be achieved if, instead of an infinite number of unstable periodic orbits, the system possesses an infinite number of periodic attractors. The basins of attraction would evidently be small but the situation is in principle more favorable because the control need not be as sharp as before. As long as the

system is kept in a neighborhood of an attractor the uncontrolled dynamics itself stabilizes the orbit.

The creation of systems with infinitely many sinks near an homoclinic tangency was discovered by Newhouse[4] and later studied by several other authors[5] [6] [7] [8] [9]. In the Newhouse phenomenon infinitely many attractors may coexist but only for special parameter values, namely for a residual subset of an interval. Another system, different from the Newhouse phenomena, which also displays many coexisting periodic attractors is a rotor map with a small amount of dissipation[10].

In Sect. 2 we prove that with only two units and symmetrical couplings one obtains a system which has an infinite number of sinks for an open set of coupling parameters. Then, in Sect.3, we analyze the behavior of a Feigenbaum network in the limit of a very large number of units. Using a mean field analysis we show how the interaction between the units generates distinct periodic orbit patterns throughout the network. The framework and motivation for studying chaotic (or chaotic-like) networks for signal processing and pattern recognition is then discussed, a simple example being presented in the Feigenbaum network setting. Finally, in Sect.4, we add a few conclusions and discuss some other systems, with similar behavior, which might be useful for practical designing purposes.

## 2 A simple system with an infinite number of sinks

Here we consider two units with symmetrical positive couplings ( $W_{12} = W_{21} = c > 0$ )

$$\begin{aligned} x_1(t+1) &= 1 - \mu_* ((1-c)x_1(t) + cx_2(t))^2 \\ x_2(t+1) &= 1 - \mu_* (cx_1(t) + (1-c)x_2(t))^2 \end{aligned} \quad (2)$$

The mechanism leading to the emergence of periodic attractors from a system that, without coupling, has no stable finite-period orbits is the permanence of the unstabilized orbits in a flip bifurcation and the contraction effect introduced by the linear coupling. The structure of the basins of attraction is also understood from the same mechanism. The result is:

**Theorem:** For sufficiently small  $c$  there is an  $N$  such that the system (2) has stable periodic orbits of all periods  $2^n$  for  $n > N$ .

**Proof:**

The bifurcations leading to the Feigenbaum accumulation point at  $\mu_*$  are flip bifurcations. This means that, after each bifurcation, the orbit that loses stability remains as an unstable periodic orbit. Therefore, (for  $c = 0$ ) at  $\mu = \mu_*$  the system (2) has an infinite number of unstable periodic orbits of all periods  $p = 2^n$ .

The proof has two basic steps. First we need to prove that, for sufficiently small  $c \neq 0$ , these periodic orbits still exist in the system (2). Second, that for any such  $c$ , no matter how small, there is an  $N$  such that there is at least one stable orbit for all periods  $p = 2^n$  with  $n > N$ . For both steps an important role is played by the instability factor, given by  $f^{(p)'}(x_p)$  at the fixed points  $x_p$  of the  $p$ -iterated map. Therefore we will first find the behavior of this derivative for the periodic unstable orbits at  $\mu = \mu_*$  and  $c = 0$ .

Iterating once the Feigenbaum-Cvitanovic functional equation

$$-\frac{1}{\lambda} f \circ f(-\lambda x) = f(x) \quad (3)$$

and taking derivatives one concludes

$$f^{(4)'}(x) = f' \left( \frac{x}{\lambda^2} \right) \quad (4)$$

Because  $\lambda$  is the scaling factor that controls distances, if a fixed point of  $f^{(p)}$  is located at  $x^*$  the corresponding fixed point for  $f^{(4p)}$  will be at  $x^* \lambda^2$ . Eq.(4) means that for the scaling function  $f$ , solution of the functional equation, the instability factor for all unstable periodic orbits is the same at  $\mu = \mu_*$ . The logistic function is not a solution of the functional equation, however it shares the scaling properties of this solution at  $\mu = \mu_*$ . Therefore the instability factor of the orbits at  $\mu_*$  will also converge to a fixed value. Numerically one finds this value to be around  $-1.6$ . The exact value is not important however, because what matters for the proof of our result is that it is a non-zero uniformly bounded value for all orbits. We may now proceed to the proof of the theorem.

*First step:* Permanence of the periodic orbits for small  $c$

From the fixed point equation

$$x^* = f_{\mu_*, c}^{(p)}(x^*)$$

for a  $p$ -periodic orbit one obtains

$$\frac{\partial x_i^*}{\partial c} = - \left( \frac{Df_{\mu_*, c}^{(p)}}{Dx} - 1 \right)_{ik}^{-1} \frac{\partial f_{\mu_*, c}^{(p)k}}{\partial c} \quad (5)$$

From the result above one knows that, at  $c = 0$ , the Jacobian  $\frac{Df_{\mu_*, c}^{(p)}}{Dx}$  is negative definite and bounded. Therefore, by the inverse function theorem, there is a  $c^* > 0$  such that for  $c < c^*$  there are  $p$ -periodic orbits for all periods  $p = 2^n$ .

*Second step:* Stabilization of at least one orbit for all periods  $p = 2^n$  with  $n > N(c)$

The stability of the periodic orbits is controlled by the eigenvalues of the Jacobian  $J_p = \frac{Df_{\mu_*, c}^{(p)}}{Dx}$  in the fixed point of  $f_{\mu_*, c}^{(p)}$ . The map (2) is a composition of two maps  $f_1 \circ f_2$

$$f_1 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 - \mu_* x_1^2 \\ 1 - \mu_* x_2^2 \end{pmatrix}$$

$$f_2 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} (1-c)x_1 + cx_2 \\ cx_1 + (1-c)x_2 \end{pmatrix}$$

By the chain rule the Jacobian is

$$J_p = \prod_{k=1}^p \left\{ \begin{pmatrix} -2\mu_* x_1(k) & 0 \\ 0 & -2\mu_* x_2(k) \end{pmatrix} \begin{pmatrix} 1-c & c \\ c & 1-c \end{pmatrix} \right\} \quad (6)$$

with determinant

$$\det J_p = (1-2c)^p (-2\mu_*)^{2p} \prod_{k=1}^p x_1(k)x_2(k) \quad (7)$$

Because of the permanence of the periodic orbits, for small  $c$ , in the neighborhood of the original coordinates (those for  $c = 0$ ) and the boundedness of the instability factor, the product of the last two factors in (7) is uniformly bounded for all  $p$ . Then for all sufficiently large  $p$ ,  $|\det J_p| < 1$ . The question

is how this overall contraction is distributed among the two eigenvalues of  $J_p$ .

For a periodic orbit of period  $p = 2^n$  we define

$$X_{l,q}^{(i)} = \begin{cases} (-2\mu_*)^{q-l+1} \prod_{k=l}^q x_i(k) & \text{if } q \geq l \\ 1 & \text{if } l > q \end{cases} \quad (8)$$

For small  $c$  consider the linear approximation to the Jacobian  $J_p$

$$\begin{pmatrix} (1-pc)X_{1,p}^{(1)} & c \sum_{k=1}^p X_{1,k}^{(1)} X_{k+1,p}^{(2)} \\ c \sum_{k=1}^p X_{1,k}^{(2)} X_{k+1,p}^{(1)} & (1-pc)X_{1,p}^{(2)} \end{pmatrix}$$

The eigenvalues are

$$\lambda_{\pm} = \frac{1}{2}(1-pc) (X_{1,p}^{(1)} + X_{1,p}^{(2)}) \pm \frac{1}{2} \sqrt{(1-pc)^2 (X_{1,p}^{(1)} - X_{1,p}^{(2)})^2 + 4c^2 \sum_{k=1}^p X_{1,k}^{(1)} X_{k+1,p}^{(2)} \sum_{k'=1}^p X_{1,k'}^{(2)} X_{k'+1,p}^{(1)}} \quad (9)$$

If the periodic orbit runs with the two coordinates  $x_1$  and  $x_2$  synchronized then

$$\begin{aligned} \lambda_+ &= X_{1,p} \\ \lambda_- &= (1-2pc) X_{1,p} \end{aligned}$$

and the orbit being unstable for  $c = 0$  it remains unstable for  $c \neq 0$ . However if the two coordinates are out of phase by  $\frac{p}{2}$  steps the radical in Eq.(9) is

$$\sqrt{\prod_{k=1}^p x(k)} \left| \sum_{i=1}^p x(i)x(i+1) \cdots x(i + \frac{p}{2} - 1) \right|$$

The existence of a superstable orbit for all periods  $p = 2^n$  implies that at  $\mu = \mu_*$  the product  $\prod_{k=1}^p x(k)$  has an odd number of negative-valued coordinates. Therefore the two eigenvalues are complex conjugate and, for small  $c$ , the contraction implicit in (7) is equally distributed by the two eigenvalues. Therefore for sufficiently large  $N$  all orbits of this type with  $p > 2^N$  become stable periodic orbits.  $\square$

The attracting periodic orbits of the coupled system being associated to the unstable periodic orbits of the Feigenbaum cascade, it is clear that the basins of attraction will be controlled by neighborhoods of these orbits in

each coordinate. Therefore one expects a checkerboard-type structure for the basins of attraction.

Fig.1 shows the structure of the basins of attraction of the orbits for  $c = 4 \times 10^{-4}$ . We have used a mesh of  $4000 \times 4000$ , each point being an initial condition for the two units. The coupled system is allowed to evolve for  $4 \times 10^4$  time steps after which the sequence is analyzed and its period determined. Periods are detected up to  $p_{\max} = 2^{12}$ . The basins, labelled by the period of the orbits, are displayed using the color map in Table I. The detected orbits have all periods from  $2^5$  to  $p_{\max}$  but some of the basins are so narrow that they cannot be seen in the figure.

Period	Colour	Period	Colour
$2^4$	lightred	$2^9$	cyan
$2^5$	red	$2^{10}$	lightblue
$2^6$	yellow	$2^{11}$	blue
$2^7$	lightgreen	$2^{12}$	brown
$2^8$	green	<i>others</i>	darkgray

Table I

### 3 Feigenbaum networks with many units

#### 3.1 Mean-field analysis

For the calculations below it is convenient to use as variable the net input to the units,  $y_i = \sum_j W_{ij}x_j$ . Then Eq.(1) becomes,

$$y_i(t+1) = 1 - \mu_* \sum_{j=1}^N W_{ij}y_j^2(t) \quad (10)$$

For practical purposes some restrictions have to be put on the range of values that the connection strengths may take. First, it is clear that for information processing (pattern storage and pattern recognition) it is important to preserve, as much as possible, the dynamical diversity of the system. That means, for example, that a state with all the units synchronized is undesirable insofar as the effective number of degrees of freedom is drastically

reduced. From

$$\delta y_i(t+1) = -2\mu_* y(t)(W_{ii}\delta y_i(t) + \sum_{j \neq i} W_{ij}\delta y_j(t)) \quad (11)$$

one sees that instability of the fully synchronized state implies  $|2\mu_* y(t)W_{ii}| > 1$ . Therefore, the interesting case is when the off-diagonal connections are sufficiently small to insure that

$$W_{ii} > \frac{1}{\mu_*} \quad (12)$$

For large  $N$ , provided there is no large scale synchronization effect, a mean-field analysis is appropriate, at least to obtain qualitative estimates on the behavior of the network. For the unit  $i$  the average value  $\langle 1 - \mu_* \sum_{j \neq i} W_{ij} y_j^2(t) \rangle$  acts like a constant and the mean-field dynamics is

$$z_i(t+1) = 1 - \mu_{eff} z_i^2(t) \quad (13)$$

where

$$z_i = \frac{y_i}{\langle 1 - \mu_* \sum_{j \neq i} W_{ij} y_j^2 \rangle} \quad (14)$$

and

$$\mu_{i,eff} = \mu_* W_{ii} \langle 1 - \mu_* \sum_{j \neq i} W_{ij} y_j^2 \rangle \quad (15)$$

$\mu_{i,eff}$  is the effective parameter for the mean-field dynamics of unit  $i$ . From (12) and (15) it follows  $\mu_* W_{ii} > \mu_{i,eff} > \mu_* W_{ii}(2 - \mu_*)$ . The conclusion is that the effective mean-field dynamics always corresponds to a parameter value below the Feigenbaum accumulation point, therefore, one expects the interaction to stabilize the dynamics of each unit in one of the  $2^n$ - periodic orbits. On the other hand to keep the dynamics inside an interesting region we require  $\mu_{i,eff} > \mu_2 = 1.3681$ , the period-2 bifurcation point. With the estimate  $\langle y^2 \rangle = \frac{1}{3}$  one obtains

$$\mu_* W_{ii} (1 - \frac{\mu_*}{3} (1 - W_{ii})) > \mu_2 \quad (16)$$

which, together with (12), defines the interesting range of parameters for  $W_{ii} = 1 - \sum_{j \neq i} W_{ij}$ .

### 3.2 Feigenbaum networks as signal processors

Recent studies of biological systems suggest that complex and chaotic dynamics plays an important role in biological information processing[11]. In the biological models the chaotic units are large interacting aggregates of neurons not the individual neurons themselves. It is therefore these aggregates that should be identified with the chaotic units in the mathematical models.

In particular the evidence from the study of the mammalian olfactory system[12] [13] [14] [15] [16] is very interesting. The researchers that studied this system propose a strange attractor with multiple wings, each one being accessed depending on the initial conditions imposed by the external stimulus. However with one invariant measure associated to a unique attractor (no matter how many wings it possesses) we would expect the system to have a non-negligible probability to explore all regions, leading to unreliable pattern identification. It is probably better to consider that, to each external stimulus, corresponds a different invariant measure, a scheme that has recently been illustrated using simple Bernoulli units[17]. Whatever the exact dynamical mechanism, schemes of this type where, from a low-intensity chaotic basal state, the system is excited into a different chaotic attractor, have a very high storage capacity due to the diversity of all possible dynamics and, on the other hand, display a very fast response time. The recognition time in the olfactory system (in the sense of the time needed to switch between two space-coherent patterns) being of the order of magnitude of the response time of individual neurons, it easy to conclude that a chaos-based mechanism is much more likely than the paradigm of attractor neural networks[18]. A recognition time of the same order of the magnitude as the cycle time is incompatible with the fixed point paradigm, but it is easy to understand in a chaos-based scheme.

The Feigenbaum attractor is not chaotic, in the Lyapunov exponent sense. Nevertheless, as an accumulation point of period-doubling bifurcations it accesses an infinite number of different periodic orbits and, in principle, is capable of processing a very large amount of information. As we have seen, a system of many such units, interacting through synaptic-type connections, is driven into periodic behavior by the coupling. The nature of the orbits reflects the pattern of synaptic connections and if these are obtained by a learning mechanism, through exposure to some external signal, the system

might act as a signal identifier. If, in addition, the external signal is made to modulate some of the inputs to the individual units, mechanisms of associative pattern recognition would be obtained.

Let, for example, the  $W_{ij}$  connections be constructed from an input signal  $x_i$  by a correlation learning process

$$\begin{aligned} W_{ij} &\rightarrow W'_{ij} = (W_{ij} + \eta x_i x_j) e^{-\gamma} \text{ for } i \neq j \\ W_{ii} &\rightarrow W'_{ii} = 1 - \sum_{j \neq i} W_{ij} \end{aligned} \quad (17)$$

The dynamical behavior of the network, at a particular time, will reflect the learning history, that is, the data regularities, in the sense that  $W_{ij}$  is being structured by the patterns that occur more frequently in the data. The decay term  $e^{-\gamma}$  insures that the off-diagonal terms remain small and that the network structure is determined by the most frequent recent patterns. Alternatively, instead of the decay term, we might use a normalization method and the connection structure would depend on the weighted effect of all the data.

In the operating mode described above the network acts as a *signal identifier*. For example if the signal patterns are random, there is little correlation established and all the units operate near the Feigenbaum point. Alternatively the learning process may be stopped at a certain time and the network then used as a *pattern recognizer*. In this latter mode, whenever the pattern  $\{x_i\}$  appears, one makes the replacement

$$\begin{aligned} W_{ij} &\rightarrow W'_{ij} = W_{ij} x_i x_j \text{ for } i \neq j \\ W_{ii} &\rightarrow W'_{ii} = 1 - \sum_{j \neq i} W_{ij} \end{aligned} \quad (18)$$

Therefore if  $W_{ij}$  was  $\neq 0$  but either  $x_i$  or  $x_j$  is  $= 0$  then  $W'_{ij} = 0$ . That is, the correlation between node  $i$  and  $j$  disappears and the effect of this connection on the lowering of the periods vanishes.

If both  $x_i$  and  $x_j$  are one then  $W'_{ij} = W_{ij}$  and the effect of this connection persists. Suppose however that for all the  $W_{ij}$ 's different from zero either  $x_i$  or  $x_j$  are equal to zero. Then the correlations are totally destroyed and the network comes back to the uncorrelated (nonperiodic behavior). This case is what is called a *novelty filter*. Conversely, by displaying periodic behavior, the network *recognizes* the patterns that are similar to those that, in the learning stage, determined its connection structure. Recognition and *association* of similar patterns is then performed.

A numerical simulation was made for a network of 100 units. During a learning phase a single binary pattern is presented many times and the network connections evolve according to the correlation learning method (Eq.(17)) and a normalization prescription. After this learning phase we display in Fig.2 the reaction of the network to several patterns which are presented to the network, in the sense of Eqs.(18). Presenting a pattern implies that the network connections change according to (17) and the system evolves from a random initial condition. As the system is sensible to the initial conditions, we choose the same initial conditions for all patterns. Each presented pattern leaves a unique signature given by the set of periods to which each unit converges. The periods of each unit are measured after a transient delay of  $2 \times 10^4$  time steps.

In Fig.2 the first pattern that is presented to the network is the one that was memorized. The others are chosen in the following way:

- the second is the complementary binary pattern to the one that was memorized. This pattern being non-correlated to the learned pattern, the network units become uncoupled and only very high periods are seen.

- for the remaining patterns some bits are swapped (from 0 to 1) in the complementary pattern, forming new signals which are weakly correlated with the memorized one. The bit swapping structure is listed in Table II. The signature of the correlations is seen from the low periods that appear in some of the network units (Fig.2). The color map is the same as used in Sect. 2.

pattern	correlated bits
memorized	all
1	none
2	10, 76
3	10, 20, 76, 89
4	5, 10, 20, 36, 52, 76, 89, 100

Table II

## 4 Final remarks

For its dynamical diversity and the selective stabilization effect of the couplings, Feigenbaum networks, both with few and with many units, seem to be a potentially interesting system for signal processing through complex dynamics and to model self-organization.

As seen in Sect.2, the size of the basins of attraction of the sinks created by the coupling is controlled by the geometry of the pre-existing unstable periodic orbits and, as a consequence, are extremely small for large periods. It might therefore be useful for practical applications to be able to control the dynamics in a more efficient way. This implies departures from the simple linear coupling scheme of Eq.(1). For example we might use

$$x_i(t+1) = 1 - \mu_* \left\{ x_i(t) + \sum_{j \neq i} W_{ij} f(x_i(t), x_j(t)) (x_j(t) - x_i(t)) \right\}^2 \quad (19)$$

with

$$f(x, y) = \theta(x)\theta(x-y) + \theta(-x)\theta(y-x)$$

or the smooth version

$$f(x, y) = \frac{1}{2} \left\{ 1 + \tanh\left(\frac{k}{2}x\right) \tanh\left(\frac{k}{2}(x-y)\right) \right\}$$

For small couplings this leads to a much larger shift to smaller values of the  $\mu_{i,eff}$  discussed in Sect.3.

Alternatively we might think of using other maps, which also possess an infinite diversity of periodic behaviors. Consider for example a piecewise-linear dynamics for the units ( $x \rightarrow \beta x + \alpha \pmod{1}$ ). For  $\beta > 1$  the uncoupled dynamics is fully chaotic. However, through interaction, it is possible to select and stabilize an infinite number of periodic orbits. Consider a fully connected network with dynamics

$$x_i(t+1) = \beta \left( \sum_j W_{ij} x_j(t) \right) + \alpha \pmod{1} \quad (20)$$

with  $x_i$  in the interval  $[0, 1)$ . For a large number of weakly correlated units a mean-field analysis as in Sect.3 leads to

$$x_i(t+1) = \beta_{i,eff} W_{ii} x_i(t) + \alpha_{i,eff} \pmod{1} \quad (21)$$

where

$$\beta_{i,eff} = \beta W_{ii} \quad (22)$$

$$\alpha_{i,eff} = \alpha + \beta \sum_{j \neq i} W_{ij} x_j \quad (23)$$

If  $\beta > 1$  the individual uncoupled units have an invariant measure absolutely continuous with respect to Lebesgue measure. However if  $\beta_{i,eff} < 1$  the dynamics of Eq.(21) has attracting periodic orbits of all periods, depending on the values of the parameters  $\alpha_{eff}$  and  $\beta_{eff}$ . The point of the periodic orbit with the smallest coordinate is obtained from[19]

$$(1 - \beta_{i,eff}^p) x_{\min} = \alpha_{i,eff} \frac{(1 - \beta_{i,eff}^p)}{(1 - \beta_{i,eff})} - P_p(\beta) \quad (24)$$

where  $p$  is the period of the orbit and  $P_p(\beta)$  is a polynomial in  $\beta$

$$P_p(\beta) = 1 + \sum_{k=2}^{p-1} \left\{ \left[ \frac{kj}{p} \right] - \left[ \frac{(k-1)j}{p} \right] \right\} \beta^{p-k} \quad (25)$$

where  $j$  (prime to  $p$ ) is the jump number of the orbit and  $[n]$  denotes the integer part of  $n$ .

## 5 Figure captions

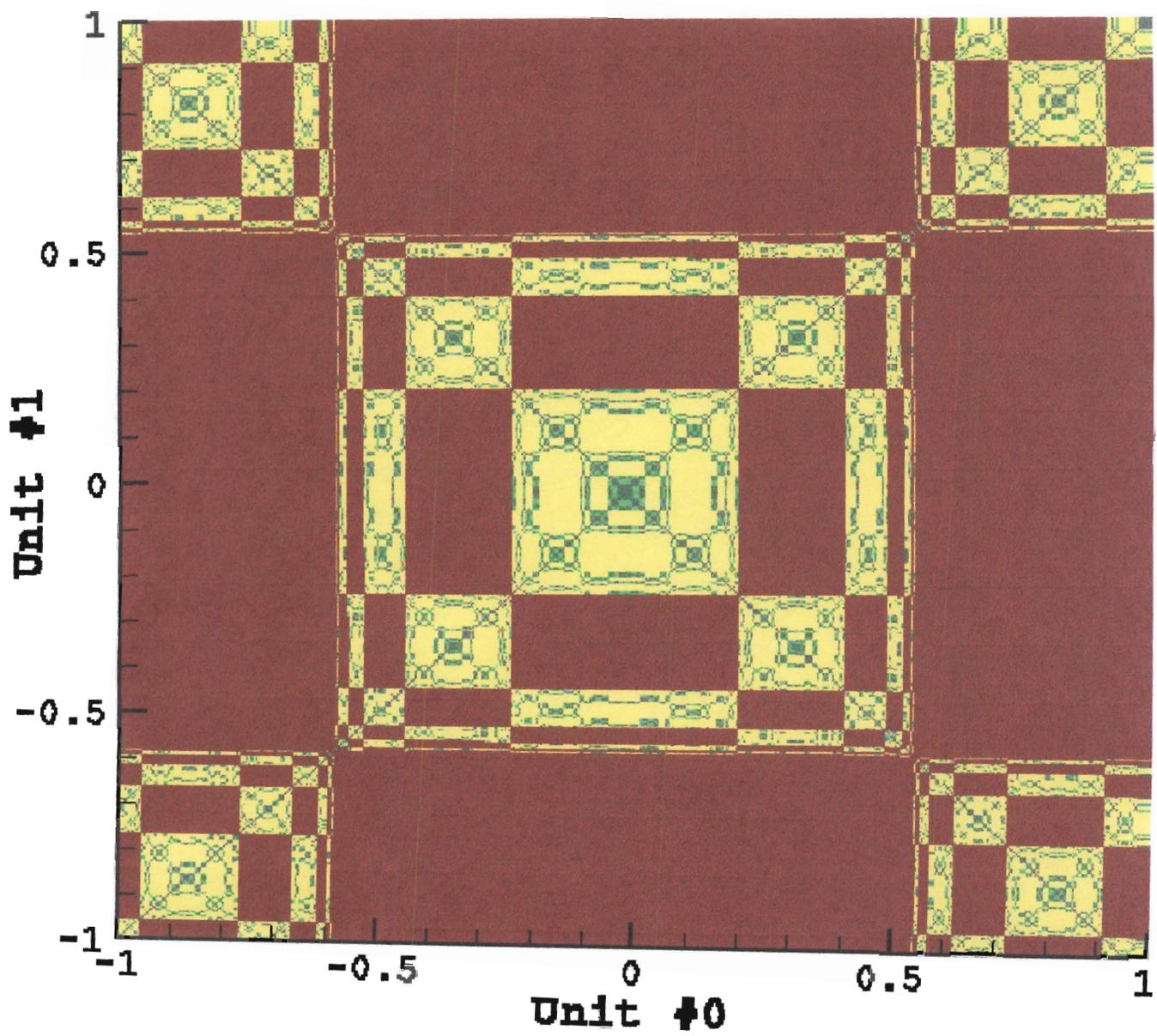
Fig.1 - Color coded basin of attraction periods for two units and  $c = 4 \times 10^{-4}$

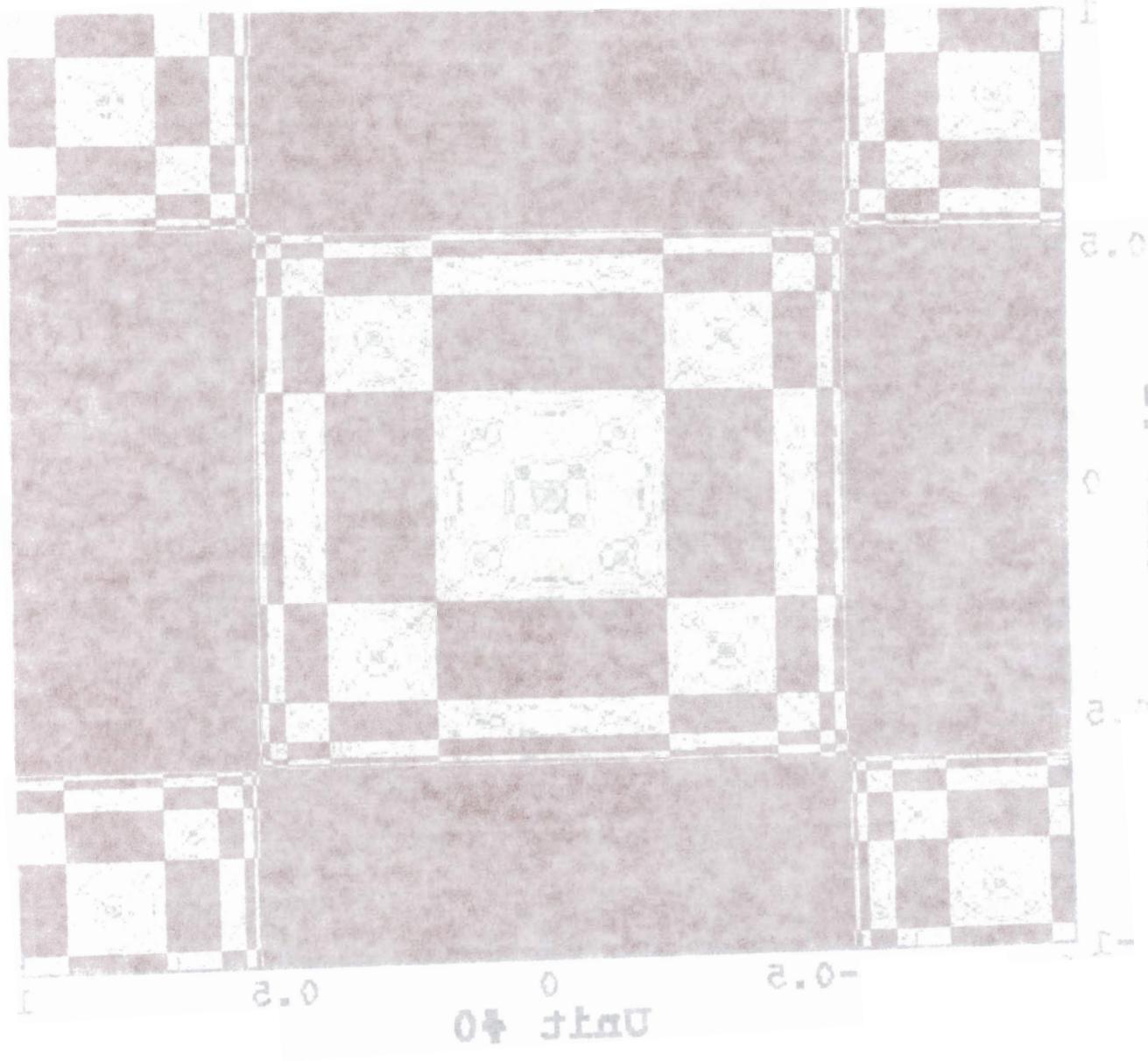
Fig.2 - Color coded periods in a network of 100 units reacting to several patterns

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Unit 40

Unit

