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INVARIANT MEASURE FOR A WAVE EQUATION ON A RIEMANNIAN MANIFOLD

A.B.CRUZEIRO* AND Z.HABA**

ABSTRACT.
We consider a wave equation on a compact Riemannian manifold in the Hamiltonian form. The measure obtained as the product of Wiener measure on path space and the one of the (flat) white noise is infinitesimally invariant under the action of the corresponding flow.

0.Introduction.

Wave equations on manifolds appear in Einstein gravity theories, where they describe scalar field self-interactions. There are also cosmological models (Dicke model) where such scalar fields in gravitational (curved) background apear. Invariant measures are then relevant to describe the corresponding statistical mechanics. The measure can be important to construct the natural Hilbert space of functionals on fields in quantum theory of gravity.

Recentely the non linear wave equation in a flat space and with a cubical potential has been studied ([F] and [McK-V]). The equation being taken in the Hamiltonian form, a measure on the phase space was constructed and shown to be invariant under the action of the flow. This measure is essentially given by the product of the white noise measure and the law of Brownian motion, together with a density (a Feynman-Kac) term which accounts for the potential.

We propose here a similar construction for the wave equation on a compact Riemannian manifold, without potential. More precisely, we propose a wave equation which is infinitesimally invariant with respect to a similar product measure in the curved setting. The actual existence of the corresponding flow is not treated here. For the construction we shall use the Wiener measure on the path space associated to the Laplace-Beltrami operator, and a basic result, the Bismut integration by parts formula [B]. We shall follow closely results as well as notations of [F-M] (cf. also the first paragraphs of [C-M]).

We refer to [D-S] for some results on invariant measures for wave type equations although their approach differs from ours in the sense that they do not take into account the momenta as variables the invariant measure depends upon.

1.The differential structure on the path space.

We shall denote by M a d-dimensional compact Riemmanian manifold. On M we consider the Levi-Civita connection and $\Gamma^j_{k,l}$ will denote the corresponding Christofell symbols. In the local chart the Laplace -Beltrami operator has the following expression:

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$$\Delta_M f = g^{i,j}\frac{\partial^2 f}{\partial m^i \partial m^j} - g^{i,j}\Gamma^k_{i,j}\frac{\partial f}{\partial m^k}$$

The Brownian motion $\varphi(\sigma)$, $\sigma \in [0,1]$, associated to this Laplacian satisfies the following Itô stochastic differential equation (cf. for example [I-W]):

$$d_\sigma \varphi^i = a^{i,k}(\varphi).dx_k - c^i(\varphi)d\sigma$$

where x is a $\mathbb{R}^d$-valued Brownian motion, a the square root of the metric g and $c^i = \frac{1}{2}\sum_{j,k} a^{j,k}\frac{\partial a^{i,k}}{\partial m^j}$.

We shall follow the frame bundle point of view for the theory of Riemannian Brownian motion ([M], exposed in [I-W]).

A frame above m_0 is an Euclidean isometry r_0 of $\mathbb{R}^d$ onto the tangent space $T_{m_0}(M)$. $O(M)$ will denote the collection of all frames of M and π the projection of $O(M)$ into M, $\pi(r_0) = m_0$. Let (θ, ω) be the canonical differential forms realizing for every $r \in O(M)$ an isomorphism between $T_r(O(M))$ and $\mathbb{R}^d \times so(d)$.

Then the horizontal Laplacian on $O(M)$, satisfying the relation

$$\Delta_{0(M)}(f o \pi) = (\Delta_M f) o \pi \ ,$$

is equal to $\sum_{k=1}^d L^2_{A_k}$, where A_k denote the horizontal vector fields, defined by

$$< \theta, A_k >= e_k$$
$$< \omega, A_k >= 0$$

(e_k the canonic basis of $\mathbb{R}^d$). The following Stratonovich stochastic differential equation

$$dr_x = \sum_{k=1}^d A_k(r_x) o dx^k \tag{1.1}$$
$$r_x(0) = r_0$$

defines a flow of diffeomorphisms on $O(M)$, the lifting to $O(M)$ of the Itô parallel transport along Brownian motion. In local coordinates, we have

$$d(r_x)^j_l(\sigma) = -\Gamma^j_{k,i}[d\varphi^k(\sigma)](r_x)^i_l(\sigma) \tag{1.2}$$
$$r_x(0) = r_0$$

We shall write

$$t^\varphi_{\sigma \leftarrow 0} r_0 = r_x(\sigma)$$

Let us denote by X the Wiener space of the Brownian motion on $\mathbb{R}^d$ and by $P_{m_0}(M)$ the space of continuous paths on M starting at m_0 at time zero.

The Itô map ([M]) $\mathcal{I} : X \to P_{m_0}(M)$ is defined by

$$\mathcal{I}(x)(\sigma) = \pi(r_x(\sigma))$$

and the Wiener measure μ on the path space $P_{m_0}(M)$, namely the law of the Brownian motion on M coincides with the measure induced by this map,i.e., it is equal to the pullback through $\mathcal{I}$ of the Wiener measure μ_0 on X.

We consider a differentiable structure on the probability space. Following [F-M], let Z be an adapted vector field along the Brownian path,i.e., an adapted map $\sigma \in [0,1] \to Z_\sigma \in T_{\varphi(\sigma)}(M)$. The stochastic Itô integral of Z is defined by

$$\int_0^1 Z.d\varphi = \int_0^1 z^k.dx_k$$

where $x = \mathcal{I}^{-1}(\varphi)$ and $z_\varphi(\sigma) = t^\varphi_{\sigma\leftarrow 0} Z_\varphi(\sigma)$.

We have (c.f.[F-M])

$$x(\sigma) = \int_0^\sigma r_x^{-1}(\xi).d\varphi(\xi) \tag{1.3}$$

Finally let us recall the definition of derivative of a functional defined on the path space. When f is a cylindrical functional, namely

$$f(\varphi) = F(\varphi(\sigma_1), \ldots, \varphi(\sigma_m))$$

for some m and some smooth F, the partial functional derivatives are defined by

$$[D_{\tau,k} f](\varphi) = \sum_i 1(\tau < \sigma_i)(t^\varphi_{0\leftarrow\sigma_i}(\partial_{\sigma_i} F) \mid e_k)$$

for $\tau \in [0,1]$ and $k = 1, \ldots, d$. With respect to the norms $||Df||^q_{L^q} = E_\mu(||Df||^q)$, where

$$||Df||(\varphi) = \left[\sum_k \int_0^1 [D_{\tau,k} f]^2 \, d\tau \right]^{\frac{1}{2}}$$

the operator D is closable in L^q. Its domain is the Sobolev space $D^q_1(P_{m_0})$. For a vector field Z such that its parallel transported to the origin along the Brownian curve, z, has a square integrable (in $[0,1]$) time derivative (z belonging to the Cameron-Martin space), one defines

$$D_Z f = \sum_k \int_0^1 \dot{z}_k(\tau) D_{\tau,k} f \, d\tau \, .$$

Notice that this operation can be also be written as $\int_0^1 \sum_k \nabla^\varphi Z(\tau) D_{\tau,k} f \, d\tau$, where $\nabla^\varphi Z(\tau) = lim_{\epsilon\to 0} \frac{1}{\epsilon}[t^\varphi_{\tau\leftarrow\tau+\epsilon} Z_\varphi(\tau+\epsilon) - Z_\varphi(\tau)]$.

The two following theorems will be used in our construction:

Theorem 1.1 ([C-M]). *A functional f defined on the path space $P_{m_0}(M)$ is differentiable along the vector field $Z \in D_1^q(P_{m_0})$ if and only if $\tilde{f} = fo\mathcal{I}$ is a Wiener funcional differentiable along the process z^**

$$dz^*(\sigma) = (\int_0^\sigma \Omega(odx, z))odx(\sigma) + \dot{z}d\sigma$$

where Ω denotes the curvature tensor in M and o Stratonovich integration. Moreover we have

$$[D_z f]o\mathcal{I} = D_{z^*}(fo\mathcal{I})$$

Derivation with respect to vector fields on the path space correspond to derivation on the path space taken along directions which do not belong to the Cameron-Martin space. Nevertheless these directions have an antisymmetric diffusion coefficient and correspond therefore to Wiener measure preserving transformations. Transfering the differentiable structure of the path space to the flat case leads to an extended differential calculus on the Wiener space (the differential calculus with respect to "tangent processes", cf. [C-M]).

Theorem 1.2 (Integration by parts formula [B] and [D]). *If f is a functional in the path space such that $f \in L^p_\mu$ and Z an adapted vector field in $D_1^q(P_{m_0})$, we have*

$$E_{\mu_0}(D_z f) \;=\; E_{\mu_0}(f \int_0^1 [\dot{z} \;+\; \frac{1}{2}R^M z]\; dx)$$

where

$$R_\tau^M = t^\varphi_{0\leftarrow\sigma} o Ricci_{\varphi(\sigma)} o t^\varphi_{\sigma\leftarrow 0}$$

and Ricci denotes the Ricci tensor on M.

The proof of this theorem can be found in [F-M].

2.The invariant measure.

Let us denote by γ_0 the measure on the space of distributions $\mathcal{S}'([0,1])$ starting at time zero at a fixed point v_0 defined by

$$E_{\gamma_0} exp\; i < h, \pi >= exp(-\frac{1}{2}\int_0^1 h^2(\sigma)d\sigma)$$

for every square integrable function in the interval $[0,1]$.

A vector field on this space will be the data of a pair $Z = (Z^1, Z^2)$, where Z^1 is a vector field along the Brownian path, $Z^1_\sigma \in T_{\varphi(\sigma)}(M)$ and $Z^2_\sigma \in L^2[0,1]$. For a functional defined on $P_{m_0}(M) \times \mathcal{S}'(\mathbb{R})$ we define $D_Z f = (D_{Z^1} f, D_{Z^2} f)$ with

$$D_{Z^2} f(\pi) = lim_{\epsilon\to 0}\frac{1}{\epsilon}[f(\pi + \epsilon Z^2) - f(\pi)]$$

the limit being taken γ_0 a.e. in π.

As a consequence of theorem 1.2, the following integration by parts formula with respect to the product measure $\mu \times \gamma_0$ holds:

Theorem 2.1. *If f is a functional defined on $P_{m_0}(M) \times \mathcal{S}'([0,1])$ such that $f \in L^p_{\mu\times\gamma_0}$ and $Z = (Z^1, Z^2)$ an adapted vector field such that $Z^1 \in D^q_1(P_{m_0})$, $Z^2 \in L^q(\gamma_0)$, we have*

$$E_{\mu_0\times\gamma_0}(D_Z f) \;=\; E_{\mu_0\times\gamma_0}(f(\int_0^1 [\dot{z}^1 \;+\; \frac{1}{2}R^M z^1]\, dx + \int_0^1 \pi Z^2 d\sigma))$$

3.The wave equation. Let us denote $t^\varphi_\sigma = t^\varphi_{\sigma\leftarrow 0}$ and consider the following wave equation on the Riemannian manifold M

$$\partial_t((t^\varphi_\sigma)^{-1}\partial_t\varphi_t(\sigma)) = \partial_\sigma((t^\varphi_\sigma)^{-1}\partial_\sigma\varphi_t(\sigma)) + \frac{1}{2}(t^\varphi_\sigma)^{-1}Ricci_\varphi o\partial_\sigma\varphi_t(\sigma) \tag{3.1}$$

subject to the initial conditions $\varphi_t(0) = m_0$ and $\partial_t\varphi_t(0) = v_0$. Here $(\Gamma(\partial_\sigma\varphi))^i_j = \Gamma^i_{k,j}[\partial_\sigma\varphi^k]$.

This equation can be written in Hamiltonian form:

$$\begin{aligned} \partial_t\varphi_t(\sigma) &= t^\varphi_\sigma\pi_t(\sigma) \\ \partial_t\pi_t(\sigma) &= \partial_\sigma((t^\varphi_\sigma)^{-1}\partial_\sigma\varphi_t(\sigma)) + \frac{1}{2}(t^\varphi_\sigma)^{-1}Ricci_\varphi(\partial_\sigma\varphi_t(\sigma)) \end{aligned} \tag{3.2}$$

Then *the measure $\mu\times\gamma_0$ is infinitesimally invariant for the wave equation (3.2).* This means that, if Z denotes the vector field corresponding to equation (3.2), we have:

$$E_{\mu\times\gamma_0}(D_Z f) \;=\; 0$$

for sufficientely regular (cylindrical) functionals f.

The rigorous proof of this result is based on its transfer to the Wiener space via the Itô map together with a finite dimensional approximation of the corresponding vector field on the Wiener space. It is a consequence of theorem 2.1. Formulae (1.2) and (1.3) are used in the proof.

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