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I: The Chaos Expansion

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The Renormalization of Self Intersection Local Times
I: The Chaos Expansion

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June 17, 1998

1. Introduction

The intersections of Brownian motion paths have been investigated since the Forties [20]. One can consider intersections of sample paths with themselves or e.g. with other, independent Brownian motions [36], one can study simple [5] or n-fold intersections [6] and one can ask all of these questions for linear, planar, spatial or - in general - d-dimensional Brownian motion: evidently self-intersections become increasingly scarce as the dimension d increases.

Intersection local times of Brownian motion were studied by many authors, see e.g. [1], [3]-[11], [15], [18]-[38]. A more systematic review than we can give here of the subject will be found e.g. in the recent reference [15].

An informal but rather suggestive definition of self intersection local time of Brownian motion B is in terms of an integral over Dirac's - or Donsker's - δ-function

$$L \equiv \int d^2t\, \delta(B(t_2) - B(t_1)),$$

intended to sum up the contributions from each pair of "times" t_1, t_2 for which the Brownian motion B is at the same point.

In Edwards' modeling of polymer molecules by Brownian motion paths, L is used to model the "excluded volume" effect: different parts of the molecule should not be located at the same point in space. As another application, Symanzik introduced L as a tool for constructive quantum field theory in [31].

A rigorous definition, such as e.g. through a sequence of Gaussians approximating the δ-function or in terms of generalized Brownian functionals [10] [30] [33], will lead to increasingly singular objects and will necessitate various "renormalizations" as the dimension d increases. For $d > 1$ the expectation will diverge in the limit and must be subtracted [18] [32], clearly L will then no more be positive. For $d > 3, 5, 7, \ldots$ further subtractions have been proposed [33] that will make L into a well defined generalized function of Brownian motion.

For $d = 3$ another renormalization has been constructed by Westwater to make the Gibbs factor $e^{-g \cdot L}$ of the polymer model well-defined [35], for yet another, recent approach see [1].

Yor, in [38], first suppresses the short time accumulation of self-intersections by the regularization

$$\delta(\overrightarrow{B}(t_2) - \overrightarrow{B}(t_1)) \to \delta(\overrightarrow{B}(t_2) - \overrightarrow{B}(t_1) + \overrightarrow{\varepsilon})$$

and shows, again for $d = 3$, that a multiplicative renormalization

$$r(\varepsilon)\,(L_\varepsilon - E(L_\varepsilon))$$

gives rise to another, independent Brownian motion as the weak limit of regularized and subtracted approximations to L.

In the present note we study similar limits, for arbitrary $d \geq 3$, using a Gaussian regularization of the δ-function for which the chaos expansion of the corresponding regularized L_ε is available [10]. For a suitably subtracted and renormalized local time, each term in this expansion converges in law to a Brownian motion.

We prepare and state these results in the following section 2, in section 3 we give their proofs. In a companion paper [3] we extend these results to the corresponding series.

2. Definitions and Main Results

2.1. White Noise Analysis and Local Times

We reproduce here some White Noise Analysis concepts as introduced in [10], referring to [14] for a systematic presentation.

Brownian motions $B_i, i = 1, ..., d$, have version in terms of white noise ω_i via

$$B_i(t) =< \omega_i, 1_{[0,t]} >= \int_0^t \omega_i(s)ds.$$

Hence we consider independent d-tuples of Gaussian white noise $\boldsymbol{\omega} = (\omega_1, ..., \omega_d)$ and correspondingly, d-tuples of test functions $\mathbf{f} = (f_1, ..., f_d) \in S(R, R^d)$, and introduce the following notation:

$$\vec{n} = (n_1, \ldots, n_d), \quad n = \sum_1^d n_i, \quad \vec{n}! = \prod_1^d n_i!$$

$$< \mathbf{f}, \mathbf{f} > = \sum_{i=1}^d \int dt \; f_i^2(t)$$

$$< F_{\vec{n}}, \mathbf{f}^{\otimes \vec{n}} >= \int d^n t \; F_{\vec{n}}(t_1, ..., t_n) \bigotimes_{i=1}^d f_i^{\otimes n_i}(t_1, ..., t_n)$$

and similarly for $<: \boldsymbol{\omega}^{\otimes \vec{n}} :, F_{\vec{n}} >$ where for d-tuples of white noise the Wick product $: \ldots :$ [14] generalizes to

$$: \boldsymbol{\omega}^{\otimes \vec{n}} := \bigotimes_{i=1}^d : \omega_i^{\otimes n_i} : .$$

The vector valued white noise $\boldsymbol{\omega}$ has the characteristic function

$$C(\mathbf{f}) = E(e^{i<\boldsymbol{\omega},\mathbf{f}>}) = \int_{S^*(R,R^d)} d\mu\,[\boldsymbol{\omega}]\, e^{i<\boldsymbol{\omega},\mathbf{f}>} = e^{-\frac{1}{2}<\mathbf{f},\mathbf{f}>}, \tag{2.1}$$

where $< \boldsymbol{\omega}, \mathbf{f} > = \sum_{i=1}^d < \omega_i, f_i >$ and $f_i \in S(R, R)$.

The Hilbert space

$$(L^2) \equiv L^2(d\mu)$$

is canonically isomorphic to the $d-$fold tensor product of Fock spaces of symmetric square integrable functions:

$$(L^2) \simeq (\bigoplus_{k=0}^{\infty} SymL^2(R^k, k!d^k t))^{\otimes d} \equiv \mathfrak{F}, \tag{2.2}$$

for the general element of (L^2) this implies the chaos expansion

$$\varphi(\omega) = \sum_{\vec{n}=0}^{\infty} <: \omega^{\otimes \vec{n}} :, F_{\vec{n}} > \tag{2.3}$$

with kernel functions F in $\mathfrak{F}$.

It is desirable to introduce regularizations for the intersection local time, with a view towards the construction of well-defined, "renormalized" intersection local times in higher dimensions where the latter do not exist without subtractions. A computationally simple regularization is, for $\varepsilon > 0$,

$$L_\varepsilon \equiv \int_0^t dt_2 \int_0^{t_2} dt_1\, \delta_\varepsilon \left(\mathbf{B}\left(t_2\right) - \mathbf{B}\left(t_1\right)\right),$$

with

$$\delta_\varepsilon \left(\mathbf{B}\left(t_2\right) - \mathbf{B}\left(t_1\right)\right) \equiv (2\pi\varepsilon)^{-d/2} e^{-\frac{|\mathbf{B}(t_2)-\mathbf{B}(t_1)|^2}{2\varepsilon}}. \tag{2.4}$$

It has the following chaos expansion, which we quote here only for $d \geq 3$:

Theorem 2.1. *[10] For any $\varepsilon > 0$, $L_\varepsilon - E\left(L_\varepsilon\right)$ has kernel functions $F \in \mathfrak{F}$ given by*

$$\begin{aligned} F_{\varepsilon,\vec{n}}(s_1, \ldots, s_n) &= (-1)^{\frac{n}{2}} \left(\varkappa(\varkappa+1)(2\pi)^{d/2} 2^{\frac{n}{2}} \ \frac{\vec{n}}{2}!\right)^{-1} \cdot \\ \Theta(u)\Theta(t-v) &\cdot \left((v-u+\varepsilon)^{-\varkappa} + (t+\varepsilon)^{-\varkappa} - (v+\varepsilon)^{-\varkappa} - (t-u+\varepsilon)^{-\varkappa}\right) \end{aligned} \tag{2.5}$$

if all n_i are even, and zero otherwise, with $v(s_1, ..., s_n) \equiv max(s_1, ..., s_n)$, $u(s_1, ..., s_n) \equiv min(s_1, ..., s_n)$, and $\varkappa \equiv (n+d)/2 - 2$. Θ is the Heaviside function.

Each kernel function is thus the sum of four terms. The first one gives rise to

Definition 2.2.

$$\begin{aligned} M_t(d, \vec{n}, \varepsilon) &\equiv \int_{[0,t]^n} d^n s (v-u+\varepsilon)^{-\varkappa} : \omega^{\otimes \vec{n}}(s) : \\ &= \int_{[0,t]^n} d^n s (v-u+\varepsilon)^{-\varkappa} \bigotimes_{i=1}^{d} : \omega_i(s_1^i) \ldots \omega_i(s_{n_i}^i) : \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
&= \sum_{i=1}^{d}\sum_{m=1}^{n_i}\int_0^t ds_m^i \int_0^{s_m^i} d^{n-1}s(v-u+\varepsilon)^{-\varkappa} \bigotimes_{i=1}^{d} : \omega_i(s_1^i)...\omega_i(s_{n_i}^i) : \\
&= \sum_{i=1}^{d} n_i \int_0^t dB_i(\tau) \int_0^{\tau} d^{n-1}s(\tau-u+\varepsilon)^{-\varkappa} : \omega^{\otimes\overrightarrow{n-\delta_i}}(s) : \\
&= \sum_{k=1}^{d}\int_0^t dB_k(v)m_k(v) \qquad (2.7)\\
&\equiv \sum_{k=1}^{d} M_{k,t} \qquad (2.8)
\end{aligned}
$$

The others give

$$
N_t(d,\overrightarrow{n},\varepsilon) \equiv \int_{[0,t]^n} d^n s\left((t+\varepsilon)^{-\varkappa}-(v+\varepsilon)^{-\varkappa}-(t-u+\varepsilon)^{-\varkappa}\right) : \omega^{\otimes\overrightarrow{n}}(s) :
$$

All of the above processes are continuous.

Definition 2.3. *We denote the $\overrightarrow{n}$ th order contribution to the regularized local time L_ε by*

$$
K_t(d,\overrightarrow{n},\varepsilon) \equiv (-1)^{\frac{n}{2}}\left(\varkappa(\varkappa+1)(2\pi)^{d/2}\,2^{\frac{n}{2}}\,\frac{\overrightarrow{n}}{2}!\right)^{-1}(M_t(d,\overrightarrow{n},\varepsilon)+N_t(d,\overrightarrow{n},\varepsilon))
$$

Remark 1. *Our key observation is that, as ε goes to zero, M is more singular than N, and that it is a Brownian martingale.*

2.2. The main theorems

Theorem 2.4. *For $d \geq 3$, the renormalized M converge in distribution to independent Brownian motions β_i:*

$$
(rM_i,;i=1,...,d) \underset{\varepsilon\to+0}{\overset{\mathfrak{L}}{\longrightarrow}} \left(\sqrt{\frac{n_i}{n}}k_n\beta_i; i=1,...,d\right)
$$

with

$$
k_n^2 = \overrightarrow{n}!\begin{cases} n(n-1) & \text{if } d=3 \\ \frac{n!(d-4)!}{(n+d-5)!} & \text{if } d>3 \end{cases} \qquad (2.9)
$$

Theorem 2.5. *For $d \geq 3$, the renormalized $\vec{n}$ th order contributions to the regularized local time L_ε converge in distribution to Brownian motions β*

$$rK(d, \vec{n}, \varepsilon) \underset{\varepsilon \to +0}{\overset{\mathcal{L}}{\to}} c_{\vec{n},d}\beta$$

with

$$c^2_{\vec{n},d} = k_n^2 \left(\varkappa(\varkappa+1)(2\pi)^{d/2} 2^{\frac{n}{2}} \frac{\vec{n}}{2}! \right)^{-2}$$

3. Proofs

Proposition 3.1. *$M_{k,t}$ are orthogonal Brownian martingales*

Proof: Orthogonality is obvious. For the martingale property see [13] and [2], it is a consequence of the fact that the kernel functions of M_t in (2.6) do not depend on t (except through the limit of integrations).■

Their limiting behavior, as $\varepsilon \to +0$, is studied in the following lemma (from now on we shall consider only the situations which require renormalisation, i.e., $d \geq 3$.)

Lemma 3.2. *As $\varepsilon \to +0$,*

$$\|M_{k,t}\|_2^2 = \frac{n_k}{n} k_n^2 \left(t + o(1)\right) \begin{cases} |ln\varepsilon| & \text{for } d = 3 \\ \varepsilon^{-(d-3)} & \text{for } d > 3 \end{cases}$$

Proof:

$$\|M_{k,t}\|_2^2 = \frac{n_k}{n} \vec{n}! \left\| (v-u+\varepsilon)^{-\varkappa} \right\|^2_{L^2([0,t]^n)}$$

For $d > 3$

$$\begin{aligned} \left\| (v-u+\varepsilon)^{-\varkappa} \right\|^2_{L^2([0,t]^n)} &= \int_0^t d^n s (v-u+\varepsilon)^{-2\varkappa} \\ &= n(n-1) \int_0^t dv \int_0^v du \frac{(v-u)^{n-2}}{(v-u+\varepsilon)^{2\varkappa}} \\ &= n(n-1)\varepsilon^{3-d} \int_0^t dv \int_0^{v/\varepsilon} dx \frac{x^{n-2}}{(x+1)^{n+d-4}} \\ &= \varepsilon^{3-d} t \left(\frac{n!(d-4)!}{(n+d-5)!} + o(1) \right) \end{aligned}$$

while for $d = 3$

$$\left\|(v-u+\varepsilon)^{-\varkappa}\right\|^2_{L^2([0,t]^n)} = n(n-1)t\,|\ln\varepsilon|\,(1+o(1))$$

■

To show convergence of these martingales by Theorem VIII.3.11 of [16] we must show convergence of characteristics. Since the processes M are continuous this reduces to showing convergence in probability of $\langle rM_i, rM_k\rangle_t$ as $\varepsilon \to +0$. This will be taken care of by proposition 3.6 which we prepare now:

$$\langle rM_i, rM_k\rangle_t = r^2\delta_{ik}\int_0^t dv\,(m_i(v))^2$$

and we need to estimate

$$\int_0^t dv\,(m_i(v))^2 \equiv \sum_{\vec{k}} \langle :\omega^{\otimes\vec{k}}:, G^{(i)}_{\vec{k}}\rangle$$

Let us note that

$$E\left(\langle rM_i, rM_k\rangle_t\right) = E\left(r^2 M_i M_k\right) = \delta_{ik} r^2 \frac{n_k}{n}\vec{n}!\,\left\|(v-u+\varepsilon)^{-\varkappa}\right\|^2_{L^2([0,t]^n)}$$

$$E\left(\langle rM_i, rM_k\rangle_t\right) = \delta_{ik}\frac{n_k}{n}k_n^2\,(t+o(1))$$

Next we intend to show that the rhs gives indeed that the rest of $\langle rM_i, rM_k\rangle_t$ goes to zero. The kernel of the highest order is

$$G^{(i)}_{2\vec{n}-2\vec{\delta}_i}(s_1, s_1'; \ldots; s_{n-1}, s_{n-1}') = \int_0^t d\tau\Theta\left(\tau - v\vee v'\right)(\tau-u+\varepsilon)^{-\varkappa}(\tau-u'+\varepsilon)^{-\varkappa} \tag{3.1}$$

$v^{(\prime)}$ and $u^{(\prime)}$ are the largest and the smallest of the $s_i^{(\prime)}$, and $\left(\vec{\delta}_i\right)_k \equiv \delta_{ik}$.

(For $n = 2$: $u^{(\prime)} = v^{(\prime)} = s^{(\prime)}$).

Remark 2. *The integral (3.1) can be calculated in closed form (using e.g. [12] nos. 2.15 and 2.263.4), but one gets a useful approximation by introducing the following auxiliary functions:*

$$\begin{aligned} & H_{2n-2}(s_1, \ldots, s_{n-1}; s_1', \ldots, s_{n-1}'; \varepsilon; d) \\ \equiv\ & (v\vee v' - u\vee u' + \varepsilon)^{-(n+d)/2+3}(v\vee v' - u\wedge u' + \varepsilon)^{-(n+d)/2+2} \end{aligned}$$

where $v \equiv max(s_i), u \equiv min(s_i)$ and $v' \equiv max(s_i'), u' \equiv min(s_i')$. These functions majorize the kernel functions

Lemma 3.3. *For $n > 2$, and $\varkappa > 1$*

$$\begin{aligned} 0 &\leq G^{(i)}_{2\vec{n}-2\vec{\delta}_i}(s_1, s_1'; , \ldots ; s_{n-1}, s_{n-1}') \\ &\leq \frac{1}{\varkappa - 1} H_{2n-2}(s_1, ..., s_{n-1}; s_1', ..., s_{n-1}'; \varepsilon; d) \end{aligned} \tag{3.2}$$

Proof:

$$\begin{aligned} 0 &\leq \int_0^t d\tau \Theta\left(\tau - v \vee v'\right)(\tau - u + \varepsilon)^{-\varkappa}(\tau - u' + \varepsilon)^{-\varkappa} \\ &\leq \int_{v \vee v'}^t d\tau (\tau - u \vee u' + \varepsilon)^{-\varkappa}(v \vee v' - u \wedge u' + \varepsilon)^{-\varkappa} \\ &\leq \frac{1}{\varkappa - 1}(v \vee v' - u \vee u' + \varepsilon)^{-\varkappa+1}(v \vee v' - u \wedge u' + \varepsilon)^{-\varkappa} \end{aligned}$$

This estimate suffices to show

Lemma 3.4. *For $2\vec{n} - \vec{2\delta}_i \neq 0$*

$$r^2 G^{(i)}_{2\vec{n}-2\vec{\delta}_i} \to 0 \ \ in\ L^2(R^{2n-2})$$

as ε goes to $+0$.

Proof: Consider first $n \geq 3$. By the above estimate it is sufficient to show that

$$\lim_{\varepsilon \to +0} r^4 \left\| H_{2n-2} \right\|^2_{L^2([0,t]^{2n-2})} = 0.$$

$$\begin{aligned} \left\| H_{2n-2} \right\|^2_{L^2(R^{2n-2})} &= \int d^{n-1}s \int d^{n-1}s'(v \vee v' - u \vee u' + \varepsilon)^{2-2\varkappa}(v \vee v' - u \wedge u' + \varepsilon)^{-2\varkappa} \\ &= c_n \int_0^t dv \int_0^v dv' \int_0^v du \int_0^{v'} du' \\ &\quad \frac{(v-u)^{n-3}(v'-u')^{n-3}}{(v \vee v' - u \vee u' + \varepsilon)^{n+d-6}(v \vee v' - u \wedge u' + \varepsilon)^{n+d-4}} \\ &\leq c_n \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dy' \int_0^y dx \int_0^{y'} dx' \\ &\quad \frac{1}{(y - x \vee x' + 1)^{d-3}(y - x \wedge x' + 1)^{d-1}} \end{aligned}$$

and then decompose the x-integration into the following two domains

$$x' < x \quad \text{and } x < x'$$

In the first case

$$\begin{aligned} I &= \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dy' \int_0^{y'} dx' \int_{x'}^y dx \frac{1}{(y-x+1)^{d-3}(y-x'+1)^{d-1}} \\ &= \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dy' \int_0^{y'} dx' \frac{1}{(y-x'+1)^{d-1}} \int_{x'}^y dx \frac{1}{(y-x+1)^{d-3}} \end{aligned}$$

We first estimate the last integral

$$\int_{x'}^y \frac{1}{(y-x+1)^{d-3}} dx \leq \begin{cases} y - x' & \text{if } d = 3 \\ \ln(y - x' + 1) & \text{if } d = 4 \\ \frac{1}{d-4} & \text{if } d > 4 \end{cases}$$

and one finds

$$I = \begin{cases} O(1) & \text{if } d = 3 \\ O(\varepsilon^{7-2d}) & \text{if } d > 3 \end{cases}$$

Hence, in both cases, $r^4\ I$ vanishes as $\varepsilon \to 0$.

The second case is

$$I = \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dx \frac{1}{(y-x+1)^{d-1}} \int_x^y dy' \int_x^{y'} dx' \frac{1}{(y-x'+1)^{d-3}} \tag{3.3}$$

Estimating the last integrand by 1 we find

$$\int_x^y dy' \int_x^{y'} dx' \frac{1}{(y-x'+1)^{d-3}} \leq \frac{(y-x)^2}{2}$$

Substitution of these estimates into the integrals over x and y gives for $d \geq 3$

$$I \leq const.\varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dx \frac{(y-x)^2}{(y-x+1)^{d-1}} = \begin{cases} O(1) & \text{if } d = 3 \\ O(\varepsilon^{-1} \ln \varepsilon) & \text{if } d = 4 \\ O(\varepsilon^{7-2d}) & \text{if } d > 4 \end{cases}$$

so in that $r^4\ I$ vanishes as $\varepsilon \to 0$. For $n = 2$ it is sufficient to use the estimate

$$\begin{aligned} 0 \ &\leq \ G_2 = \int_0^t d\tau \Theta\left(\tau - u \vee u'\right)(\tau - u + \varepsilon)^{1-d/2}(\tau - u' + \varepsilon)^{1-d/2} \\ &\leq \ \int_{u \vee u'}^t d\tau (\tau - u \vee u' + \varepsilon)^{1-d/2}(u \vee u' - u \wedge u' + \varepsilon)^{1-d/2} = H_2(u, u') \end{aligned}$$

and to verify that

$$\lim_{\varepsilon \to +0} r^4 \left\| H_2 \right\|^2_{L^2([0,t]^2)} = 0.$$

■

With this lemma we have established that the highest order term of the (renormalized) quadratic variation goes to zero in quadratic mean for any $t > 0$. The kernels G_k of the other terms are obtained by integrating over pairs of s, s' such as e.g. in

$$Sym \int_0^t ds G^{(i)}_{2(\vec{n} - \vec{\delta}_i)}(s, s; s_2, s_2'; , \ldots ; s_{n-1}, s_{n-1}'),$$

these new functions are in fact also bounded by an expression like (3.1), and hence by (3.2), so that for all $\varepsilon > 0$,

$$\|G_k\| \leq const. \|H_k\|$$

With this goal we show

Lemma 3.5. *Let $n \geq 2$ and $m > 1$, and let $F_{n,m}$ be a function symmetric in the variables s and in the variables s', with*

$$\begin{aligned} 0 \ &\leq \ F_{n,m}(s_1, \ldots, s_{n-1}, s_n, s_1', \ldots, s_{n-1}', s_n') \qquad (3.4) \\ &\leq \ c \int_0^t d\tau \Theta\left(\tau - \max_{i \leq n}(s, s')\right)(\tau - \min_{i \leq n}(s_i) + \varepsilon)^{-m}(\tau - \min_{i \leq n}(s_i') + \varepsilon)^{-m} \end{aligned}$$

Then $\exists c_m < \infty$ such that

$$\begin{aligned} 0 \ &\leq \ \int_0^t ds F_{n,m}(s_1, \ldots, s_{n-1}, s, s_1', \ldots, s_{n-1}', s) \qquad (3.5) \\ &\leq \ c_m \int_0^t d\tau \Theta\left(\tau - \max_{i < n}(s, s')\right)(\tau - \min_{i < n}(s_i) + \varepsilon)^{-m+1/2}(\tau - \min_{i < n}(s_i') + \varepsilon)^{-m+1/2} \end{aligned}$$

Proof: Under the assumption of the lemma

$$\int_0^t ds F_{n,m}\left(s_1, ..., s_{n-1}, s, s_1', ..., s_{n-1}', s\right) \leq$$

$$c\int_0^t ds \int_0^t d\tau \Theta\left(\tau - \max_{0\leq i\leq n-1}\left(s_i, s_i', s\right)\right)\left(\tau - \min_{0\leq i\leq n-1}\left(s_i, s\right) + \varepsilon\right)^{-m}\left(\tau - \min_{0\leq i\leq n-1}\left(s_i', s\right) + \varepsilon\right)^{-m}.$$

Using

$$u = \min_{0\leq i\leq n-1} s_i \quad , \quad u' = \min_{0\leq i\leq n-1} s_i'$$
$$v = \max_{0\leq i\leq n-1} s_i \quad , \quad v' = \max_{0\leq i\leq n-1} s_i'$$

(For $n = 2$: $u^{(\prime)} = v^{(\prime)} = s_1^{(\prime)}$). Assuming without loss of generality that $u < u'$ we can decompose the s-integration of our estimate as follows

$$c\int_0^t ds \int_0^t d\tau \Theta\left(\tau - \max\left(v, v', s\right)\right)\left(\tau - \min\left(u, s\right) + \varepsilon\right)^{-m}\left(\tau - \min\left(u', s\right) + \varepsilon\right)^{-m}$$

$$\begin{aligned}
&= c\int_0^t d\tau \left(\int_0^u + \int_u^{u'} + \int_{u'}^{v\vee v'} + \int_{v\vee v'}^t\right) ds \\
&\quad \cdot\Theta\left(\tau - \max\left(v, v', s\right)\right)\left(\tau - \min\left(u, s\right) + \varepsilon\right)^{-m}\left(\tau - \min\left(u', s\right) + \varepsilon\right)^{-m} \\
&= c\int_{v\vee v'}^t d\tau \left(\int_0^u ds\left(\tau - s + \varepsilon\right)^{-2m} + \int_u^{u'} ds\left(\tau - u + \varepsilon\right)^{-m}\left(\tau - s + \varepsilon\right)^{-m}\right. \\
&\quad \left. + \int_{u'}^{v\vee v'} ds\left(\tau - u + \varepsilon\right)^{-m}\left(\tau - u' + \varepsilon\right)^{-m} + \int_{v\vee v'} \tau ds\left(\tau - u + \varepsilon\right)^{-m}\left(\tau - u' + \varepsilon\right)^{-m}\right).
\end{aligned}$$

We now show that each of the four terms obeys the postulated estimate (3.5)

$$\int_{v\vee v'}^t d\tau \int_0^u ds\left(\tau - s + \varepsilon\right)^{-2m} \leq \frac{1}{2m-1}\int_{v\vee v'}^t d\tau\left(\tau - u + \varepsilon\right)^{-2m+1}$$

$$\leq \frac{1}{2m-1}\int_0^t d\tau\Theta\left(\tau - v\vee v'\right)\left(\tau - u + \varepsilon\right)^{-m+\frac{1}{2}}\left(\tau - u' + \varepsilon\right)^{-m+\frac{1}{2}}$$

since $u < u'$ and $m > 1/2$.

The second term

$$\begin{aligned}
&\int_{v\vee v'}^t d\tau \int_u^{u'} ds\left(\tau - u + \varepsilon\right)^{-m}\left(\tau - s + \varepsilon\right)^{-m} \\
\leq\ & \frac{1}{m-1}\int_{v\vee v'}^t d\tau\left(\tau - u + \varepsilon\right)^{-m}\left(\tau - u' + \varepsilon\right)^{-m+1} \\
\leq\ & \frac{1}{m-1}\int_0^t d\tau\Theta\left(\tau - v\vee v'\right)\left(\tau - u + \varepsilon\right)^{-m+\frac{1}{2}}\left(\tau - u' + \varepsilon\right)^{-m+\frac{1}{2}}.
\end{aligned}$$

using again $u < u'$. The third term

$$\begin{aligned}
&\int_{v\vee v'}^t d\tau \int_{u'}^{v\vee v'} ds\left(\tau - u + \varepsilon\right)^{-m}\left(\tau - u' + \varepsilon\right)^{-m} \\
=\ & \int_{v\vee v'}^t d\tau\left(\tau - u + \varepsilon\right)^{-m}\left(\tau - u' + \varepsilon\right)^{-m}\left(v\vee v' - u'\right) \\
\leq\ & \int_{v\vee v'}^t d\tau\left(\tau - u + \varepsilon\right)^{-m}\left(\tau - u' + \varepsilon\right)^{-m}\left(\tau - u'\right) \\
\leq\ & \int_0^t d\tau\Theta\left(\tau - v\vee v'\right)\left(\tau - u + \varepsilon\right)^{-m+\frac{1}{2}}\left(\tau - u' + \varepsilon\right)^{-m+\frac{1}{2}}.
\end{aligned}$$

Finally

$$\int_{v\vee v'}^t d\tau \int_{v\vee v'} \tau ds\left(\tau - u + \varepsilon\right)^{-m}\left(\tau - u' + \varepsilon\right)^{-m}$$

$$
\begin{aligned}
&= \int_{v\vee v'}^{t} d\tau\,(\tau-u+\varepsilon)^{-m}\,(\tau-u'+\varepsilon)^{-m}\,(\tau-v\vee v') \\
&\leq \int_{v\vee v'}^{t} d\tau\,(\tau-u+\varepsilon)^{-m}\,(\tau-u'+\varepsilon)^{-m}\,(\tau-u') \\
&\leq \int_{0}^{t} d\tau\Theta\,(\tau-v\vee v')\,(\tau-u+\varepsilon)^{-m+\frac{1}{2}}\,(\tau-u'+\varepsilon)^{-m+\frac{1}{2}}
\end{aligned}
$$

■

Combining this Lemma with the previous one we conclude that for all kernel functions G_k with $k \geq 2$ arguments

$$\lim_{\varepsilon\to+0} r^4 \left\|SymG_k\right\|^2 \leq \lim_{\varepsilon\to+0} r^4 \left\|G_k\right\|^2 \leq \lim_{\varepsilon\to+0} r^4 \left\|H_k\right\|^2 = 0$$

i.e. we have shown

Proposition 3.6.

$$ms - \lim_{\varepsilon\to+0} \langle rM_i, rM_k\rangle_t = \delta_{ik}\frac{n_k}{n}k_n^2 t$$

Proof of Theorem 2.4: The above limit is clearly (up to constants) the quadratic variation of a Brownian motion. Theorem 2.4 is then a consequence of e.g. Theorem VIII.3.11 in [16], which in the present case of continuous martingales requires convergence in probability of quadratic variations for a dense set of t.

To control the remaining terms $N_t(d, \overrightarrow{n}, \varepsilon;)$ in the chaos expansion we observe that

$$
\begin{aligned}
\left\|N_t(d, \overrightarrow{n}, \varepsilon)\right\|^2_{(L^2)} &= \overrightarrow{n}!\left\|(t+\varepsilon)^{-\varkappa} - (v+\varepsilon)^{-\varkappa} - (t-u+\varepsilon)^{-\varkappa}\right\|^2_{L^2([0,t]^n)} \\
&\leq \overrightarrow{n}!\left(\left\|(t+\varepsilon)^{-\varkappa}\right\|^2_{L^2} + \left\|(v+\varepsilon)^{-\varkappa}\right\|^2_{L^2} + \left\|(t-u+\varepsilon)^{-\varkappa}\right\|^2_{L^2}\right)
\end{aligned}
$$

The first of these three norms is equal to $t^n(t+\varepsilon)^{-2\varkappa}$, i.e. $O(1)$. The second one is

$$
\begin{aligned}
\int_{[0,t]^n} d^n s(v+\varepsilon)^{-2\varkappa} &= n\int_0^t dv\frac{v^{n-1}}{(v+\varepsilon)^{n+d-4}} = n\varepsilon^{4-d}\int_0^{t/\varepsilon} dx\frac{x^{n-1}}{(x+1)^{n+d-4}} \\
&= \begin{cases} O(1) & \text{for } d=3 \\ O(\ln\varepsilon) & \text{for } d=4 \\ O(\varepsilon^{4-d}) & \text{for } d>4 \end{cases}
\end{aligned}
$$

which are suppressed by the renormalization

$$r^2(\varepsilon) = \begin{cases} |ln\varepsilon|^{-1} & \text{for } d = 3 \\ \varepsilon^{d-3} & \text{for } d > 3 \end{cases}.$$

A similar estimate holds for the third term of N, so that we have shown

Lemma 3.7.

$$ms - \lim_{\varepsilon \to +0} r(\varepsilon) N_t(d, \overrightarrow{n}, \varepsilon) = 0.$$

In fact the convergence is uniform in any finite t-interval. Next we show

Lemma 3.8. *The processes* $\{r(\varepsilon)N.(d, \overrightarrow{n}, \varepsilon;) : \varepsilon > 0\}$, $\{r(\varepsilon)M.(d, \overrightarrow{n}, \varepsilon;) : \varepsilon > 0\}$ *and their linear combinations are tight.*

Proof: A criterion for tightness of M (following [17], p.64) is

$$\sup_{\varepsilon > 0} E\left|rM_t - rM_s\right|^{\alpha} \leq C_T (t-s)^{1+\beta}, \tag{3.6}$$

$\forall T > 0$ and $0 \leq s < t \leq T$ and for some positive constants α, β and C_T.

As a first step we show

$$\sup_{\varepsilon > 0} E\left|rM_t - rM_s\right|^2 \leq C_T (t-s) \tag{3.7}$$

by direct calculation:

$$E\left|M_t - M_s\right|^2 = \overrightarrow{n}!n \int_s^t dv \int_0^v d^{n-1}s\,(v-u+\varepsilon)^{-2\varkappa}$$

The second integral may be estimated as follows

$$\begin{aligned} \int_0^v d^{n-1}s\,(v-u+\varepsilon)^{-2\varkappa} &= (n-1)\int_0^v du \frac{(v-u)^{n-2}}{(v-u+\varepsilon)^{n+d-4}} \\ &\leq (n-1)\varepsilon^{3-d}\int_0^{v/\varepsilon} dx \frac{1}{(x+1)^{d-2}} \\ &= (n-1)\begin{cases} \ln(v+\varepsilon) - \ln\varepsilon & \text{for } d = 3 \\ O(\varepsilon^{3-d}) & \text{for } d > 3 \end{cases} \end{aligned}$$

Renormalization of this estimate by the factor r^2 makes it bounded on $[0,T]$, and the integral over v gives the desired estimate (3.7), i.e.

$$\|rM_t - rM_s\|_2^2 \leq c_T\,(t-s) \tag{3.8}$$

Note that the kernel functions of K are all dominated by those of M. Hence we have also, possibly with a larger constant c_T, the estimate

$$\|rK_t - rK_s\|_2^2 \leq c_T\,(t-s) \tag{3.9}$$

By the hypercontractivity of the Ornstein-Uhlenbeck semigroup (see e.g.[14], p.235), one has for nth order white noise monomials $\varphi \in (L^2)$, and any $\alpha > 2$

$$\|\varphi\|_\alpha^\alpha \leq c_{n,\alpha}\,\|\varphi\|_2^\alpha$$

For $\varphi = rK_t - rK_s$,and using the above estimate for the 2-norm, we get

$$E\,|rK_t - rK_s|^\alpha \leq C_T\,(t-s)^{\alpha/2}$$

as required to ensure tightness. The estimates for rN etc. are of the same kind.■

Proof of Theorem 2.5: We need to consider

$$rK = rM + rN$$

knowing that, as $\varepsilon \to +0$, the rK are tight by Lemma 3.8, the rM converge in law, and the rN_t go to zero in mean square. The latter two facts are sufficient, via the Cramér-Wold device (see e.g. [17] p.61), for finite dimensional convergence of rK; tightness then implies convergence in law.

Acknowledgements: This work has had partial support from PRAXIS XXI and FEDER. L. S. would like to express his appreciation for an inspiring discussion with L. Chen, for a very helpful remark of J. Potthoff, and for the splendid hospitality of the Grupo de Física Matemática da Universidade de Lisboa, under the auspices of a "Marie Curie" fellowship (ERBFMBICT 971949).

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