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Symplectic structure for Gaussian diffusions

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SYMPLECTIC STRUCTURE FOR GAUSSIAN DIFFUSIONS

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ABSTRACT. Using exclusively information about classical Hamiltonian boundary value problems (with quadratic time dependent potentials) we construct probability measures providing a symplectic structure associated with some inhomogeneous Gaussian diffusion processes introduced in [12].

1. INTRODUCTION

1.1. **Historical Remarks.** At the end of the 40's, Feynman introduced a new interpretation of non relativistic quantum mechanics [1]. Let us briefly discuss it, in order to introduce some notations, by considering a one particle system in $\mathbb{R}^3$ (of unit mass) described by the quantum Hamiltonian

$$H = -\frac{\hbar^2}{2}\Delta + V$$

where $\hbar$ is the Planck constant and V a regular scalar potential. Quantum dynamics is given (in any time interval e.g., $[0, T]$) by the solution of the Cauchy problem in $L^2(\mathbb{R}^3)$, and for χ in the domain of H, by

$$i\hbar\partial_t\psi = H\psi, \quad \psi(0, x) = \chi(x)$$

According to Born's interpretation of ψ, the probability of finding the particle on a certain region B (B being a Borelian in R^3) at time t is given by:

$$\int_B \psi(x, t)\bar{\psi}(x, t)dx$$

where $\bar{\psi}$ is the complex conjugate of ψ Introducing the integral kernel or "propagator", written symbolically as:

$$K(t_a, b, t_b) = \int_{\Omega_a^b} e^{\frac{i}{\hbar}S(\omega)}\mathcal{D}\omega, \ t_a < t_b$$

where S is the action of the underlying classical system i.e., here,

$$S(\omega) = \int_a^b \left(\frac{1}{2}\dot{\omega}(s)^2 - V(\omega(s), s)\right)ds$$

$\omega \in \Omega_a^b$ is a continuous path joining a and b and $D\omega$ represents a formal "measure" $D\omega = \Pi_{a<s<b}d\omega(s)$ on Ω_a^b, the solution to the Schrödinger initial value problem can be represented by

$$\psi(t, y) = \int_{\mathbb{R}^3} K(x, 0, y, t)\chi(x)dx$$

Date: march 1997.

So the probability that at time t the particle is in B should be:

$$(1)\quad P(t) = \int_B \psi(t,y)\bar{\psi}(t,y)dy = \int_B \int_{\mathbb{R}^3} K(a,t_a,y,t)\chi(a)\bar{\psi}(t,y)dady$$

$$= \int_B \int_{\mathbb{R}^3} \chi(a)\bar{\chi}(a)K(a,t_a,y,t)\frac{\bar{\psi}(t,y)}{\bar{\chi}(a)}dady = \int_B \int_{\mathbb{R}^3} \chi(a)\bar{\chi}(a)\mathbf{K}(a,t_a,y,t)\ dady$$

with

$$\mathbf{K}(a,t_a,y,t) = K(a,t_a,y,t)\frac{\bar{\psi}(t,y)}{\bar{\chi}(a)}$$

This strongly suggests that $\mathbf{K}$ is formally the density of a transition probability in the sense of the classical theory of Markov processes [2] w.r.t dy . Also observe that Feynman's path integral formula provides a informal way to describe Quantum Mechanics exclusively in terms of classical data, the action functional of the system.

Unfortunately, as it is well known today:

- $\mathcal{D}_\omega$ is not a measure (doesn't exist) .
- $\mathbf{K}$ is complex valued so it cannot be the transition probability of a real valued stochastic process. There is no probability space where to find such a stochastic process compatible with Born interpretation and the results of regular quantum mechanics.

On the other hand, Schrödinger (1932) has considered the following associated problem of classical statistical mechanics:(Cf. [3]-[6]), when the scalar potential V is zero.

Suppose that the position of a classical Brownian particle in a domain $B \in \mathbb{R}^3$ is observed at times t_a and t_b ($t_a < t_b$) and found to have strictly positive probability densities with respect to Lebesgue measure:

$$\rho_{t_a}(x)dx, \quad \rho_{t_b}(z)dz$$

What is then the most probable evolution of a probability density compatible with those two arbitrary measurements measurement?

Schrödinger shows that the probability density should be of the form:

$$\rho(y,t) = \eta(t,y)\hat{\eta}(t,y) \quad , t_a \leq t \leq t_b \ ,$$

where the pair $\{\eta, \hat{\eta}\}$ are regular positive solutions of the two adjoint heat equations:

$$\partial_t \eta(t,y) = \frac{1}{2}\Delta\eta(t,y) \quad t \geq t_a, \quad y \in B$$

$$-\partial_t \hat{\eta}(t,y) = \frac{1}{2}\Delta\hat{\eta}(t,y) \quad t \leq t_b, \quad y \in B$$

So the problem amounts to prove the existence and uniqueness of positive solutions of the following non-linear system of equations for $\eta(x,t_a) = \chi(x)$ and $\hat{\eta}(z,t_b) = \hat{\chi}(z)$,

$$(2)\qquad \eta_{t_a}(x)\int_{\mathbb{R}^3} h(x,t_a,z,t_b)\hat{\eta}_{t_b}(z)dz = \rho_{t_a}(x)$$

$$(3)\qquad \hat{\eta}_{t_b}(z)\int_{\mathbb{R}^3} \eta_{t_a}(x)h(x,t_a,z,t_b)dx = \rho_{t_b}(z)$$

where h denotes the (positive) integral kernel associated with the abovementioned heat equation.

This problem was not solved by Schrödinger but by A.Beurling, almost 30 years after in a much more general setting [4].

In particular, Beurling's result provides us with the existence and uniqueness of the pair $\{\eta, \hat{\eta}\}$ of solutions of the adjoint heat equations for any potential V in the Kato class [6]. Using this, Zambrini [5] established the existence of a class of $\mathbb{R}^3$

Markovian (" Bernestein") diffusion processes Z_t, for any V in the Kato class such that:

$$P(Z_t \in B) = \int_B \eta(t,y)\hat{\eta}(t,y)dy$$
$$\frac{d^2E(Z_t)}{dt^2} = E[\nabla V(Z_t)]$$

For such processes Z_t we can define two transition probabilities:

$$P_f(s,x,t,B) = P(Z_t \in B | Z_s = x) \longrightarrow Z_{s,x}$$
$$P_b(t,B,u,z) = P(Z_t \in B | Z_u = z) \longrightarrow Z^{u,z}$$

where $s < t < u$ in the time interval and $Z_{s,x}, Z^{u,z}$ denote the associated conditioned processes. The Euclidean analogue of Feynman's formula "trasition amplitude" [1] can be expressed as:

$$P(Z_t \in B) = \int_B \int_{\mathbb{R}^3} \eta(s,x)\hat{\eta}(s,x)\rho_{s,x}(t,y)dxdy \tag{4}$$

(5) but also as

$$P(Z_t \in B) = \int_B \int_{\mathbb{R}^3} \eta(u,z)\hat{\eta}(u,z)\rho^{u,z}(t,y)dzdy \tag{6}$$

where $\rho_{s,x}, \rho^{u,z}$ are respectively the probability densities of the conditioned processes $Z_{s,x}, Z^{u,z}$

The essential element of Feynman's path integral formula is its relation with classical Lagrangian mechanics, and, in this respect, the quadratic Lagrangians play a very special role. In this case, Feynman shows that the sum over the paths of his formal stochastic process is entirely characterised by classical mechanics. This aspect has also been essential in his use of path integrals in quantum field theory.

We are going to show in details why this remains true in the nonrelativistic version of Euclidean Quantum Mechanics whose existence was suggested by Schrödinger and realized in the mid-eighties. For a recent review of this framework and early references see, for example [18]-[7].

1.2. **Basic Terminology.** Let us first introduce the basic terminology for our use of diffusions processes in this paper . A stochastic process $Z(t,\omega)$ (or simply Z_t) on a time interval $I = [s,u]$ is defined as a random variable on a certain probability space (Ω, Σ, P), where Ω is a subset of $C(I, \mathbb{R}^n)$ (a trajectory of the process will correspond to a random choice of a path $\omega \in \Omega$), Σ denotes the sigma algebra of the Borelians of Ω (containing the history of the process on the time interval I) and, finally, P is the probability measure that determines the process. This measure will be characterised by the finite dimensional distributions of Z_t i.e. the collection

$$P[Z_{t_1} \in B_1, ..., Z_{t_n} \in B_n]$$

where $s \leq t_1 \leq ... \leq t_n \leq u$ and $B_1, ..., B_n \in \mathcal{B}$ (where $\mathcal{B}$ denotes the sigma algebra of the Borelians of $\mathbb{R}^n$).

In this paper all the processes that we consider are Markovian, i.e.

$$P(Z_t \in B \,|\mathcal{P}_{t_1}) = P(Z_t \in B), \quad s \leq t_1 \leq t \leq u$$

where $\mathcal{P}_{t_1}$ denote the sigma algebra containing the history of the process until time t_1 . Another fundamental concept is the transition function: the function $P(t_1, x, t_2, B)$ ($s \leq t_1 \leq t_2 \leq u$, $x \in \mathbb{R}$, $B \in \mathcal{B}$ is a transition function if.

- $P(t_1, x, t_2, .)$ is a probability measure on $\mathcal{B}$.
- $P(t_1, .., t_2, B)$ is $\mathcal{B}$ measurable.

- For $s \leq t_1 \leq t_2 \leq t_3 \leq u$ the following equality (Chapman Kolmogrov) is satisfied:

$$P(t_1, x, t_3, B) = \int_{\mathbb{R}^n} P(t_2, y, t_3, B) P(t_1, x, t_2, dy)$$

- If Z_t is a Markovian process, we define its transition function by the conditional expectation

$$P(t_1, Z_{t_1}, t_2, B) = P(Z_{t_2} \in B \mid Z_{t_1})$$

Using the transition functions for Markov processes we can obtain all finite dimensional distributions of the process if we know its initial distribution at time $t = s$

Theorem 1. *If Z_t is a Markovian process on the interval $I = [s, u]$, with a transition function given by $P(t_1, x, t_2, B)$ and initial distribution P_s i.e for $A \in \mathcal{B}$*

$$P_s(A) = P(Z_s \in A),$$

then for $s \leq t_0 \leq ... \leq t_n \leq u$, we have

$$\begin{aligned} &P(Z_{t_1} \in B_1, Z_{t_2} \in B_2, ..., Z_{t_n} \in B_n) \\ &= \int_{R^n} \int_{B_1} ... \int_{B_{n-1}} P(t_{n-1}, x_{n-1}, t_n, B_n) P(t_{n-2}, x_{n-2}, t_{n-1}, dx_{n-1}) ... \\ &\qquad ... P(t_0, x_0, t_1, dx_1) P_{t_0}(dx_0), \end{aligned}$$

and in particular we have, for any $t \geq s$,

$$P(Z_t \in B) = \int_{\mathbb{R}^n} P(s, x, t, B) P_s(dx)$$

Since the process we are going to work with are Gaussian we recall the following definition:

A random vector $X = (X_1, X_2, ..., X_n)$ has a Gaussian distribution if its characteristic function is of the following form:

$$\phi(u) = Ee^{i<u,X>} = e^{i<m,u> - \frac{1}{2}<Cu,u>}, \tag{7}$$

where $m = (m_1, ..., m_n)$, $u = (u_1, ..., u_n)$ are vectors in $\mathbb{R}^n$, $C = (C_{ik})$, $i, k = 1, ..., n$, is a positive definite symmetric matrix, E denotes the mathematical expectation and $< .,. >$ denotes the Euclidean inner product in $\mathbb{R}^n$. m is the mean of the random variable X and R is its covariance i.e.

$$m = E(X) \tag{8}$$

$$C_{jk} = E[(X_j - m_j)(X_k - m_k)]. \tag{9}$$

2. The Classical Model

In this paper, our basic dynamical equation will be the classical n- dimensional (Euclidean) Newtonian equation,

$$\ddot{q}(t) = K(t) q(t) \tag{10}$$

on a time interval $I \equiv [s, u]$, where $K(t)$ is a symmetric $n \times n$ matrix on I with a continuous dependence in t. . Moreover, we will assume that (10) is disconjugate [9, 10] on the time interval I, i.e. equation (10) has no trivial solution $q(t)$ such that $q(t_1) = q(t_2) = 0$, where t_1 and t_2 are distinct points of I. Equation (10) can

be regarded as the classical Jacobi equation for the Euler-Lagrange equations with (Euclidean) Lagrangian of the form

$$L(q,\dot{q},t) = \frac{1}{2}|\dot{q}|^2 + V(q,t) \tag{11}$$

where $|.|$ denote the Euclidean norm. Then

$$K(t) = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\bigg|_{\bar{q}(t)} \right), \quad i,j = 1,2,3,...n$$

for $\bar{q}(t)$ an extremal of the action functional with Lagrangian L. This is the point of view of the semiclassical (or WKB) approximation in Feynman's approach to quantum mechanics, [1]. Equation (10) can be transformed into an Hamiltonian form on $\mathbb{R}^{2n}$ for the Hamiltonian function given by

$$H(q,p,t) = \frac{1}{2}|p|^2 - \frac{1}{2} < K(t)q, q > \tag{12}$$

The correspondent Hamilton's equation of motion will be given by the system:

$$\begin{cases} \dot{q}_i & = p_i \\ \dot{p}_i & = \sum_{1\leq j\leq n} K_{ij}(t)q_j \end{cases} \quad i = 1...n \tag{13}$$

We also consider the matrix form of (13) i.e

$$\begin{cases} \dot{C}_{ij} & = D_{ij} \\ \dot{D}_{ij} & = \sum_{1\leq l\leq n} K_{il}(t)C_{lj} \end{cases} \tag{14}$$

with the correspondent matrix second order differential system:

$$\ddot{C}_{ij}(t) = \sum_{1\leq l\leq n} K_{il}(t)C_{lj}(t). \tag{15}$$

We will denote by $\mathbf{C}_t$ the matrix solution $(C_t, D_t) = (C_t, \dot{C}_t)$ of (14). The disconjugacy assumption for (10) on the time interval I implies that for any solution (C_t, D_t) of (14) , C_t can only be non invertible at one point of I. In particular, for $t_0 \in I$, if one defines, $E_{t_0}(t)$ as the solution on I of (15) that satisfies the Cauchy data:

$$E_{t_0}(t_0) = 0, \quad \dot{E}_{t_0}(t_0) = I_{n\times n} \tag{16}$$

then $E_{t_0}(t)$ is a non singular matrix for $t \neq t_0$.

The linearity of the Hamiltonian system ensures that the phase space has a natural vector space structure that we can identify with $\mathbb{R}^{2n}$. Hence the canonical symplectic structure provided by the 2 form [8]

$$\omega = dq \wedge dp = \sum_{i=1}^{n} dq_i \wedge dp_i$$

can be used to define an antisymmetric inner product between elements of the phase space. Given two phase space vectors $X_1 \equiv (q_1^1, ..., q_n^1, p_1^1, ...p_n^1)$, $X_2 \equiv (q_1^2, ..., q_n^2, p_1^2, ...p_n^2)$ their *symplectic inner product* is defined as

$$\omega(X_1, X_2) = \sum_{i=1}^{2n} q_i^1 p_i^2 - q_i^2 p_i^1 \tag{17}$$

It is easy to prove that if $(X_1(t), X_2(t))$ are solutions of (13) on a time interval I then $\omega(X_1(t), X_2(t))$ is constant.

We will need a generalisation of the symplectic inner product to matrix functions. In order to simplify the notation if $A = (A_{\alpha\beta})$ and $B = (B_{\alpha\beta})$ are $n \times r$ matrices, then the $2n \times r$ matrix $C = (C_{\alpha,\beta})$, $(\alpha = 1, 2, ..., n;\ \beta = 1, 2, ..., r)$, with $C_{\alpha\beta} = A_{\alpha\beta}$,

$C_{n+\alpha,\beta} = B_{\alpha\beta}$ will be denoted as $(A;B)$. Hence if $C = (C_1;C_2)$ and $D = (D_1;D_2)$ are $2n \times n$ matrices and ^{T} denotes the transpose we will call the matrix function

$$\omega(C,D) = C_1^T D_2 - C_2^T D_1 \tag{18}$$

the symplectic inner product of C and D. If $X \in \mathbb{R}^{2n}$ and $C = (C_1;C_2)$ then $\omega(X,C)$ or $\omega(C,X)$ can be defined as $\omega(I_X,C)$ or $\omega(C,I_X)$ where I_X is the diagonal matrix with $(I_X)_{ii} = X_i$ for $i = 1,\ldots,2n$.

Although it is true that if $\mathbf{C}_1$ and $\mathbf{C}_2$ are two solutions of (14) on an time interval I then $\omega(\mathbf{C}_t^1,\mathbf{C}_t^2)$ is a constant matrix on I, but, in contrast with (18) ω is not antisymmetric, hence in general we do not have $\omega(\mathbf{C},\mathbf{C}) = 0$. The solutions $\mathbf{C}_t$ of (14) (or the solutions C_t of (15)) such that $\omega(\mathbf{C},\mathbf{C}) = 0$, are sometime called *isotropic* solutions of (15) on I.

Definition 1. A Lagrangian manifold M is defined as an n manifold in a $2n$ dimensional phase space such that for all points (x,p) on M and for all vectors X,Y tangent to M at (x,p) we have:

$$\omega(X,Y) = 0 \tag{19}$$

Some obvious consequences of this definition are:

1. In $\mathbb{R}^2$ it is clear that a Lagrangian manifold must be a 1- dimensional surface i.e. a curve.
2. Any manifold of the form $q =$ constant or $p =$ constant on a phase space is a Lagrangian manifold. We will call *singular Lagrangian manifold associated with a configuration* $x \in \mathbb{R}^n$ a Lagrangian manifold of the form

$$M^x = \{(q,p)|q = x\} \tag{20}$$

3. Any graph of a curl-free momentum field $p = p(q)$ is a Lagrangian manifold. If the Lagrangian manifold M is of the form

$$M = \{(q,p) \in \mathbb{R}^{2n} \,|p = \frac{\partial S(q)}{\partial q}\} \tag{21}$$

for some regular function $S(q)$, this function is called the *generating function* of the Lagrangian manifold. Given an arbitrary Lagrangian manifold, its generating function may not be defined, a typical example of this is the singular Lagrangian manifold associated with a configuration $x \in \mathbb{R}^n$:

In this paper we only consider linear Lagrangian manifolds i.e n-planes on $\mathbb{R}^{2n}$.

2.1. **Evolution of Lagrangian manifolds.** Consider the Lagrangian manifolds M_s and M_s^x defined by

$$M_s = \{(q,p) \in \mathbb{R}^{2n} \,|p = \alpha q - \beta\} \tag{22}$$

$$M_s^x = \{(q,p) \in \mathbb{R}^{2n} \,|q = x\} \tag{23}$$

where α is a non negative definite invertible $n \times n$ matrix and $\beta, x \in \mathbb{R}^n$.

Our first aim is to explicitly compute the images under the Hamiltonian flow associated with (12) of these two Lagrangian manifolds in a time interval $I = [s,u]$.

Definition 2. Given an initial Lagrangian manifold we define its associated (matrix) function F_t on an interval $I = [s,u]$, as an isotropic matrix solution of (15) such that:

1. If the Lagrangian manifold is defined by (22), F_t is a non singular matrix for $t \in I$ such that

$$\dot{F}_s F_s^{-1} = \alpha \tag{24}$$

2. If the Lagrangian manifold is singular (i.e defined by (23)) then F_t is a non singular matrix for $t \in (s, u]$ and satisfies $F(s) = 0$. A example of such a function is $E_s(t)$ the solution of (15) on I that satisfies the Cauchy data $E_s(s) = 0$. $\dot{E}_s(s) = I_{n\times n}$

Lemma 1. *If F_t and H_t are two associated functions to a Lagrangian manifold M_s or M_s^x on the time interval I then $\omega(\mathbf{F}_t, \mathbf{H}_t) = 0$.*

Proof. If the Lagrangian manifold is defined by (22) then both F_t and H_t satisfies (24), hence:

$$\begin{aligned} 0 &= \dot{F}_s F_s^{-1} - \dot{H}_s H_s^{-1} \\ &= F_s^{-T}(\dot{F}_s^T H_s - F_s^T \dot{H}_s) H_s^{-1} \\ &= -F_s^{-T}\omega(\mathbf{F}_t, \mathbf{H}_t) H_s^{-1} \end{aligned}$$

which implies that:

$$\omega(\mathbf{F}_t, \mathbf{H}_t) = 0 \tag{25}$$

If the Lagrangian manifold is defined by (23) then F_t, and H_t satisfies $F(s) = H(s) = 0$. Since both matrix functions are solutions of (14) their symplectic product is time independent which implies that

$$\omega(\mathbf{F}_t, \mathbf{H}_t) = F_s^T \dot{H}_s - \dot{F}_s H_s = 0$$

□

Proposition 1. *The image of the Lagrangian manifolds defined by (22) or (23) under the Hamiltonian flow defined by (12), for $t \in I$, is given by*

$$M_t = \{(q,p) \mid \dot{F}^T(t)q - F^T p = \delta\} \tag{26}$$

where $F(t)$ is the associated function to M_s (respc. M_s^x) and $\delta = F_s^T\beta$ (respc. $\delta = x$) Moreover the expression (26) is independent of the choice of the associated function to M_s i.e if H_t is another associated function to M_s on I then

$$\{(q,p) \mid \dot{F}^T(t)q - F^T p = \delta\} = \{(q,p) \mid \dot{H}^T(t)q - H^T p = \delta^H\}$$

where $\delta^H = H_s^T\beta$

For the proof we will need the following Lemma:

Lemma 2. *On the time interval $I = [s, u]$ and for any two configurations $x, y \in \mathbb{R}^n$ consider Hamilton's principal function $R(s, x, t, y)$ defined [8] as the line integral*

$$R(s, x, t, y) = \int_\gamma L dt. \tag{27}$$

where L denotes the Lagrangian defined by (11) along the solution of (10), $\gamma_\tau = (q(\tau), \dot{q}(\tau))$ connecting the points (s, x) and (t, y). We have:

$$\begin{aligned} R(s, x, t, y) = \frac{1}{2} < \dot{E}_s E_s^{-1}(t)y, y > - \\ - \frac{1}{2} < \dot{E}_t(s)E_t^{-1}(s)x, x > - < E_s^{-1}(t)y, x > \end{aligned} \tag{28}$$

where $E_t(\tau)$ and $E_s(\tau)$ are solutions of (14) on the interval I satisfying the Cauchy data

$$E_t(t) = 0 \quad \dot{E}_t(t) = I_{n\times n}; \tag{29}$$

$$E_s(s) = 0 \quad \dot{E}_s(s) = I_{n\times n}. \tag{30}$$

Proof. By (27),

$$R(s,x,t,y) = \int_s^t \Big[\frac{|\dot{q}_\lambda|^2}{2} + \frac{1}{2} < K(\lambda)q_\lambda, q_\lambda >\Big] d\lambda$$

where q_λ is the solution of (10) satisfying $q(s) = x$ and $q(t) = y$ (we will assume for the moment that this solution is well defined on I and is unique). Hence

$$\begin{aligned} R(s,x,t,y) &= \int_s^t \Big[\frac{|q_\lambda|^2}{2} + \frac{1}{2} < K(\lambda)q_\lambda, q_\lambda >\Big] d\lambda \\ &= \int_s^t \frac{|\dot{q}_\lambda|^2}{2} d\lambda + \frac{1}{2}\int_s^t < \ddot{q}_\lambda, q_\lambda > d\lambda \\ &= \frac{1}{2} < \dot{q}_t, y > - \frac{1}{2} < \dot{q}_s, x > \end{aligned} \tag{31}$$

Using the ansatz

$$q_\tau = E_s(\tau)\lambda_1 + E_t(\tau)\lambda_2$$

where $\lambda_i \in \mathbb{R}^n$, $i = 1, 2$ and taking into account the boundary conditions, ($q(s) = x$, $q(t) = y$) we will get

$$q_\tau = E_s(\tau)E_s^{-1}(t)y + E_t(\tau)E_t^{-1}(s)x \tag{32}$$

Substituting in (31),

$$\begin{aligned} R(s,x,t,y) = \frac{1}{2} < \dot{E}_s E_s^{-1}(t)y, y > - \frac{1}{2} < \dot{E}_t(s)E_t^{-1}(s)x, x > \\ + \frac{1}{2} < E_s^{-1}(t)y, x > - \frac{1}{2} < E_s^{-1}(t)x, y > \end{aligned} \tag{33}$$

As $E_s(\tau)$ $E_t(\tau)$ are both solutions of (14) the symplectic inner product $\omega(\mathbf{E}_s, \mathbf{E}_t)$ will be constant on the interval I, hence using (29) and (30)

$$\begin{aligned} \omega(\mathbf{E}_s(s), \mathbf{E}_t(s)) &= \omega(\mathbf{E}_s(t), \mathbf{E}_t(t)) \\ E_s^T(s)\dot{E}_t(s) - \dot{E}_s^T(s)E_t(s) &= E_s^T(t)\dot{E}_t(t) - \dot{E}_s^T(t)E_t(t) \\ -E_t(s) &= E_s(t) \end{aligned}$$

Using this equality in (33), we obtain the desired result. Observe also that as we are assuming the disconjugacy of (13) , $E_s(\tau)$ and $E_t(\tau)$ will be invertible respectively on the time interval $(s, u]$ and $[s, t)$; in particular $E_s(t)$ is invertible which proves that (32) is the unique solution of (10) that satisfies the required boundary conditions. □

Proof of Proposition 1. Let us assume that the initial Lagrangian manifold is given by (22) i.e

$$M_s = \{(q,p) | p = \alpha q - \beta\}$$

and let us take the associated function to the Lagrangian manifold. F_t which, by definition, satisfies the following relation at time s:

$$F_s F_s^{-1} = \alpha$$

Then we can write the generating function for the initial Lagrangian manifold as :

$$S(x) = \frac{1}{2}\langle F_s F_s^{-1}x, x\rangle - \beta y + \gamma \tag{34}$$

where γ is an arbitrary real constant.

The idea of the proof is to explicitly compute $\nabla_y S(t,y)$, where $S(t,y)$ will be the generating function for the Lagrangian manifold M_t, since, then, we will have [11] :

$$M_t = \left\{(q,p) \ | p = \frac{\partial S}{\partial q}(t,q)\right\}. \tag{35}$$

for $S(t,y)$ the solution of the Hamilton-Jacobi equation on I:

$$\frac{\partial S}{\partial y} + H\left(t, y, \frac{\partial S}{\partial y}\right) = 0, \tag{36}$$

subject to the Cauchy data:

$$S(s,x) = S(x) \tag{37}$$

The solution of the initial value problem defined by (36) ,(37) is given by (cf. [11]):

$$S(t,y) = S_s(x) + R(s,x,t,y) \tag{38}$$

where $R(s,x,t,y)$ denotes Hamilton's principal function defined as the line integral of Poincaré-Cartan form $pdq - Hdt$ evaluated along the solution of Hamilton's equation that connects x at time s a to y at time t. The starting configuration x may itself be regarded as a function of y and t, i.e. as the starting point of a trajectory that at time t is in the configuration y. A simple computation will give us (cf. [14]) the following expression for x:

$$x = F^{-1}(t)y + F^{-1}(t)E_s(t)\beta. \tag{39}$$

where $F(t)$ is an associated function to M_s. Observing that (38) implies (cf. [11])

$$\nabla_y S(t,y) = \nabla_y R(s,x,t,y) \tag{40}$$

we obtain, using Lemma 2,

$$\nabla_y R = \frac{1}{2}(\dot{E}_s E_s^{-1}(t) + E_s^{-T}\dot{E}_s^T(t))y - [E_s^{-1}(t)]^T x \tag{41}$$

As $E_s(t)$ is an isotropic solution of (15), $\dot{E}_s(t)E_s^{-1}(t)$, in particular, is symmetric on the interval $[s,u]$. Hence:

$$\nabla_y R = \dot{E}_s E_s^{-1}(t)y - [E_s^{-1}(t)]^T x$$

Using (39) we obtain:

$$\nabla_y S(t,y) = f(t)y - \beta(t) \tag{42}$$

where $f(t)$ and $\beta(t)$ are defined by

$$\begin{aligned} f(t) &= \dot{E}_s(t)E_s^{-1}(t) - [E_s^{-1}(t)]^T F(s)F^{-1}(t) \\ \beta(t) &= [E_s^{-1}(t)]^T F^{-1}(t)E_s(t)\beta, \end{aligned}$$

As $-F(s) = \omega(\mathbf{E}_s(\tau), \mathbf{F}(\tau))$ we can simplify the expression for $f(t)$:

$$\begin{aligned} f(t) &= \dot{E}_s(t)E_s^{-1}(t) - [E_s^{-1}(t)]^T F(s)F^{-1}(t) \\ &= [E_s^{-1}(t)]^T[\dot{E}_s^T(t) - F(s)F^{-1}(t)] \\ &= [E_s^{-1}(t)]^T[\dot{E}_s^T(t)F(t) - F(s)]F^{-1}(t) \\ &= [E_s^{-1}(t)]^T E_s^T(t)\dot{F}(t)F^{-1}(t) \\ &= \dot{F}F^{-1}(t) \end{aligned}$$

As F_t is the associated function for M_s on I, F_t is an isotropic solution of (15) which implies that $\dot{F}F^{-1}(t)$ is a symmetric matrix in I. Using this property we can simplify as well the expression for $\beta(t)$;

$$
\begin{aligned}
\beta(t) &= [E_s^{-1}(t)]^T F(s) F^{-1}(t) E_s(t)\beta \\
&= [\dot{E}_s(t) - f(t)E_s(t)]\beta \\
&= [\dot{E}_s(t) - \dot{F}(t)F^{-1}(t)E_s(t)]\beta \\
&= [\dot{E}_s(t) - [F^{-1}(t)]^T \dot{F}^T(t)E_s(t)]\beta \\
&= [F^{-1}(t)]^T [F^T \dot{E}_s(t) - \dot{F}^T(t)E_s(t)]\beta \\
&= [F^{-1}(t)]^T \omega(F, E_s)\beta \\
&= [F^{-1}(t)]^T F^T(s)\beta \\
&= [F^{-1}(t)]^T \delta
\end{aligned}
$$

Hence using (35) and (42) we get

$$
M_t = \left\{ (q,p) \mid p = \dot{F}_t F_t^{-1}(t) q - [F^{-1}(t)]^T \delta \right\} \tag{43}
$$

Using again the fact that F_t is an associated function for M_s on I we will get for $(q,p) \in M_t$.

$$
F_t^T p - \dot{F}_t^T q = -\delta \tag{44}
$$

If the initial manifold is a singular one associated with a configuration $x \in \mathbb{R}^n$ i.e:

$$
M_s = \{(q,p) \mid q = x\},
$$

then we do not have a generating function for this manifold, however for $t > s$ the corresponding momentum field is given by

$$
M_t = \{(q,p) \mid p = \nabla R(s,x,t,q)\}
$$

using Lemma 2 directly we have, for $t > s$,

$$
\nabla R(s,x,t,y) = \dot{E}_s(t) E_s^{-1}(t) q - E_s^{-T}(t) x
$$

hence as E_s is invertible and isotropic on the interval $(s,u]$, we obtain (26), for the singular Lagrangian manifold case with $\delta = x$:

$$
M_t = \left\{ (q,p) \mid E_s^T(t) p - \dot{E}_s^T(t) q = -x \right\}
$$

Suppose now that we choose another associated function H_t to M_s on I. In particular H_t will be another solution of (15) on I, hence we will have the following relation between F_t and H_t:

$$
H_t = F_t [F_s^{-1} H_s + G(t)\omega(\mathbf{F}_t, \mathbf{H}_t)] \tag{45}
$$

where

$$
G(t) = \int_s^t F_\tau^{-1} F_\tau^{-T} d\tau.
$$

applying Lemma 2, $\omega(\mathbf{F}_t, H_t) = 0$ which implies:

$$
H_t = F_t F_s^{-1} H_s \tag{46}
$$

and consequently for $(q,p) \in M_t$

$$
\begin{aligned}
\dot{H}_t^T p - H_t^T q &= [F_s^{-1} H_s]^T [\dot{F}_t^T p - F_t^T q] \\
&= [F_s^{-1} H_s]^T F_s^T \beta \\
&= H_s^T \beta
\end{aligned}
$$

Which proves that $M_t \subset M_t^H$ where

$$M_t^H = \{(q,p) \mid \omega(\mathbf{H}_t, \mathbf{I}_{q,p}) = -H_s^T \beta\}$$

In a similar way we could prove the reverse statement. Hence

$$M_t = M_t^H$$

□

Now instead of giving an initial Lagrangian manifold or an initial momenta field we are given a final one. Consider the following Lagrangian manifolds

$$R_u = \{(q,p) \mid -p = \hat{\alpha} q - \hat{\beta} x\}, \tag{47}$$

$$R_u^z = \{(q,p) \mid q = z\}, \tag{48}$$

where $\hat{\alpha}$ is a symmetric invertible $n \times n$ matrix and $\hat{\beta}, z \in \mathbb{R}^n$. The trajectories consistent to these final manifolds will be solutions, $\gamma(t) = (q(t), p(t))$ of Hamilton's equation of motion on I such that $\gamma(u) \in R_u$ (or $q(u) = z$) . We want to compute for $t \in I$ the time reversed image, R_t of these manifolds.

Definition 3. Given an final Lagrangian manifold we define its *associated function* on an interval $I = [s, u]$, $\hat{F}_t$, as an isotropic matrix solution of (15) such that:

1. If the Lagrangian manifold is defined by (47), $\hat{F}_t$ is a non singular matrix for $t \in I$ such that

$$-\dot{\hat{F}}_u \hat{F}_u^{-1} = \hat{\alpha} \tag{49}$$

2. If the Lagrangian manifold is a singular i.e defined by (48), then $\hat{F}_t$ is a non singular matrix for $t \in [s, u)$ and satisfies $F(u) = 0$

As in Lemma 1:

Lemma 3. *If $\hat{F}_t$ and $\hat{H}_t$ are two associated functions on the time interval I, to a Lagrangian manifold R_u then $\omega(\hat{\mathbf{F}}_t, \hat{\mathbf{H}}_t) = 0$.*

Proposition 2. *The image of the Lagrangian manifolds defined by (47) (or (48)) under the time reversed Hamiltonian flow defined by (12) is, for $t \in I$, given by*

$$R_t = \{(q,p) \mid \dot{\hat{F}}^T(t) q - \hat{F}^T p = \hat{\delta}\} \tag{50}$$

where $\hat{F}(t)$ is the associated function to R_u (or R_z^u) and $\hat{\delta} = \hat{F}_u^T \hat{\beta}$ (or $\hat{\delta} = z$). Moreover the expression (50) is independent of the choice of the associated function to M_s i.e if $\hat{H}_t$ is another associated function to R_u on I then

$$\{(q,p) \mid \dot{\hat{F}}^T(t) q - \hat{F}^T p = \hat{\delta}\} = \{(q,p) \mid \dot{\hat{H}}(t) q - \hat{H}^T p = \hat{\delta}^H\}$$

where $\hat{\delta}^H = H_u^T \hat{\beta}$

Proof. The proof follows the same lines as the proof of the last proposition i.e R_t will be now given by

$$R_t = \left\{(q,p) \mid -p = \frac{\partial \hat{S}(t,y)}{\partial y}\right\}$$

where $\hat{S}(t,y)$ is the solution for the time reversed Hamilton-Jacobi equation

$$-\frac{\partial \hat{S}(t,y)}{\partial t} + H(t, y, -\frac{\partial \hat{S}(t,y)}{\partial y}) = 0, \ y \in \mathbb{R}^n, \ t \in I \tag{51}$$

satisfying:

$$\hat{S}(u,z) = \hat{S}(z) \tag{52}$$

with $\hat{S}(z)$ denoting the generating function of $\hat{R}_u = \mathcal{T}(R_u)$, where $\mathcal{T} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is defined as $\mathcal{T}(q,p) = (q,-p)$.

□

Remark 1. It follows from propositions 1 and 2 that the images of the Lagrangian manifolds M_s and R_u can be written in terms of the symplectic product i.e

$$M_t = \left\{(q,p) \mid \omega(\mathbf{F}_t, \mathbf{I}_{q,p}) = F_s^T \beta\right\} \tag{53}$$

$$R_t = \left\{(q,p) \mid \omega(\hat{\mathbf{F}}_t, \mathbf{I}_{q,p}) = \hat{\beta}\right\} \tag{54}$$

where $\mathbf{I}_{q,p} = (\delta_{ij} q_i \ ; \ \delta_{ij} p_i)_{1 \le i,j \le n}$, $\mathbf{F}_t = (F_t, \dot{F}_t)$ and $\hat{\mathbf{F}}_t = (\hat{F}_t, \dot{\hat{F}}_t)$, with F_t and $\hat{F}_t$ denoting the solutions associated with M_s and R_u.

Remark 2. From the proof of Proposition 1 we can obtain explicit expressions for the solution of the Hamilton Jacobi equation initial value problem given by (36) and (37). Using (43) we obtain

$$S(t,y) = \langle \dot{F}_t F_t^{-1}, y \rangle - F_t^{-T} \beta y + \gamma_t. \tag{55}$$

As the above expression is a solution of the boundary value problem (36) and (37) we can easily show that γ_t is a solution on I of the first order differential equation:

$$\dot{\gamma}_t + \left|F_t^{-T}\beta\right|^2 = 0, \ \gamma(s) = \gamma_s. \tag{56}$$

Using proposition 2 we obtain in the same way a solution of the boundary value problem (51) and (52):

$$\hat{S}(t,y) = -\langle \dot{\hat{F}}_t \hat{F}_t^{-1}, y \rangle - \hat{F}_t^{-T} \hat{\beta} y + \hat{\gamma}_t. \tag{57}$$

In this case, $\hat{\gamma}_t$, solves the time reversed equation on the time interval I, (56) i.e

$$\dot{\hat{\gamma}}_t + \left|\hat{F}_t^{-T}\hat{\beta}\right|^2 = 0, \ \hat{\gamma}(u) = \hat{\gamma}_u. \tag{58}$$

2.2. **Consistent Solutions.** Consider the pair of Lagrangian manifolds M_s and R_u defined by the associated functions F_t and $\hat{F}_t$ on the an interval $I = [s,u]$ and the two $\mathbb{R}^n$ vectors $\delta, \hat{\delta}$

Consider the problem of finding regular trajectories $\mathbf{m}_t \equiv (m(t), \dot{m}(t)) \in \mathbb{R}^{2n}$ such that $\mathbf{m}(t)$ is a solution of Hamilton's equations of motion (13) on I satisfying the following boundary conditions:

$$\mathbf{m}(s) \in M_s \tag{59}$$

$$\mathbf{m}(u) \in R_u \tag{60}$$

it turns out that this boundary value problem is equivalent to find, for each $t \in I$, the intersection . $L_t \cap R_t$:

Proposition 3. *Consider a pair of Lagrangian manifolds M_s and R_u defined by the associated functions $F_t, \hat{F}_t$ on the time interval I, satisfying $\omega(\mathbf{F}_t, \hat{\mathbf{F}}_t) \neq 0$. Then, for each $t \in I$, the set $M_t \cap R_t$ is a single element set given by*

$$L_t \bigcap R_t = \{(\mathbf{m}_t)\} \tag{61}$$

with $\mathbf{m}_t = (m_t, \dot{m}_t)$, where $\mathbf{m}_t$ is the solution of Hamilton's equations of motion (13) on I satisfying the boundary conditions given by (59) and (60). Moreover we can write $\mathbf{m}_t$ as

$$\mathbf{m}_t = \mathbf{F}_t[\omega(\hat{\mathbf{F}}, \mathbf{F})]^{-1}\hat{\delta} + \hat{\mathbf{F}}_t\left[[\omega(\hat{\mathbf{F}}, \mathbf{F})]^T\right]^{-1}\delta \tag{62}$$

where $\hat{\mathbf{F}} \equiv (\hat{F}_t, \dot{\hat{F}}_t)$, and $\mathbf{F}_t \equiv (F_t, \dot{F}_t)$ This solution $\mathbf{m}_t$ will be referred as the "$M_s R_u$ consistent solution " .

Proof. Consider (q,p) in the intersection set. Then, by Propositions 1 and 2,

$$\begin{cases} \omega(\mathbf{F}_t, \mathbf{I}_{q,p}) & = -\delta \\ \omega(\hat{\mathbf{F}}_t, \mathbf{I}_{q,p}) & = -\hat{\delta} \end{cases}$$

hence using the fact that $\omega(\mathbf{F}_t, \hat{\mathbf{F}}_t) \neq 0$ and solving the system we will get the unique solution:

$$\mathbf{m}_t = \mathbf{F}_t[\omega(\hat{\mathbf{F}}, \mathbf{F})]^{-1}\hat{\delta} + \hat{\mathbf{F}}_t\left[[\omega(\hat{\mathbf{F}}_t, \mathbf{F}_t)]^T\right]^{-1}\delta$$

That it is obviously a solution of Hamilton's equations of motion that satisfies the required boundary conditions.

□

Remark 3. In the conditions of the above proposition we can also consider for any $t \in I$ and $y \in \mathbb{R}^n$ the pairs of Lagrangian manifolds M_s, R_t^y and M_t^y, R_u , where the Lagrangian manifolds R_t^y and M_t^y both denote the singular Lagrangian manifold at time t associated with the configuration y (we adopt different notation since R_t^y is viewed as a final condition with respect to the Lagrangian manifold M_s on the time interval $[s,t]$ whereas M_t^y is an initial manifold with respect with the manifold R_u on the time interval $[t,u]$). We can take $\hat{E}_t(\tau)$ as the associated function to R_t^y on the interval $[s,u]$ and $E_t(\tau)$ as the associated function to the manifold M_t^y. Denoting by $\gamma_{M_s}^{R_t^y}(\tau)$ and $\gamma_{M_t^y}^{R_u}(\tau)$ the consistent solutions to the pairs of Lagrangian manifolds M_t^y, R_u and M_s, R_t^y respectively, we will have, applying proposition 3, the following expressions:

$$\gamma_{M_s,R_t^y}(\tau) = \mathbf{F}_\tau[\omega(\hat{\mathbf{E}}_t(\tau), \mathbf{F})]^{-1}y + \hat{\mathbf{E}}_t(\tau)\left[[\omega(\hat{\mathbf{E}}_t(\tau), F)]^T\right]^{-1}\delta$$
$$\gamma^{M_t^y,R_u}(\tau) = \mathbf{E}_t(\tau)[\omega(\hat{\mathbf{F}}(\tau), \mathbf{E}_t)]^{-1}\hat{\delta} + \hat{\mathbf{F}}(\tau)\left[[\omega(\hat{\mathbf{F}}(\tau), \mathbf{E}_t)]^T\right]^{-1}y$$

but as

$$\omega(\hat{\mathbf{E}}_t(\tau), \mathbf{F}(\tau)) = \hat{E}_t(t)^T\dot{F}(t) - \dot{\hat{E}}_t(t)^T F(t) = F(t)$$
$$\omega(\hat{\mathbf{F}}(\tau), \mathbf{E}_t) = \hat{F}(t)^T\dot{E}_t(t) - \dot{\hat{F}}(t)^T E_t(t) = \hat{F}(t)^T$$

we get:

$$\gamma_{M_s,R_t^y}(\tau) = \mathbf{F}_\tau F(t)^{-1}y + \hat{\mathbf{E}}_t(\tau)\left[F(t)^T\right]^{-1}\delta \tag{63}$$

$$\gamma^{M_t^y,R_u}(\tau) = \mathbf{E}_t(\tau)[\hat{F}^T(t)]^{-1}\hat{\delta} + \hat{\mathbf{F}}(\tau)\left[\hat{F}(t)\right]^{-1}y \tag{64}$$

3. Euclidean (or probabilistic) Quantization

We will show that there is a natural way to associate to a pair of Lagrangian manifolds a Gaussian probability density function defining a $\mathbb{R}^n$-valued diffusion process.

3.1. Associating a Gaussian probability density to a pair of Lagrangian manifolds. Proposition 3 allow us to consider a regular solution $m(t)$ of $\ddot{q}(t) = K(t)q(t)$ on $I = [s,u]$ with boundary values given by the pair of Lagrangians manifolds L_s and R_u Any configuration $y \neq m(t)$ will determine two points, respectively $(y,p_1) \in M_t$ and $(y,p_2) \in R_t$, to which can be associated a pair of trajectories $\gamma_{M_s}^{R_t^y}(\tau) = (q^{t,y}(\tau), \dot{q}^{t,y}(\tau))$ and $\gamma_{M_t^y}^{R_u}(\tau) = (q_{t,y}(\tau), \dot{q}_{t,y}(\tau))$ where $q^{t,y}(\tau)$ and $q_{t,y}(\tau)$ $\tau \in [s,u] = I$ denote, respectively, solutions of Hamilton's equation of motion satisfying the relations:

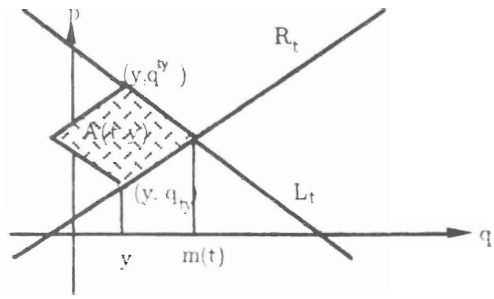
$$\begin{cases} \gamma_{M_s}^{R_t^y}(s) & \in M_s \\ \gamma_{M_s}^{R_t^y}(t) & = (y,p_1) \end{cases} \quad \begin{cases} \gamma_{M_t^y}^{R_u}(u) & \in R_u \\ \gamma_{M_t^y}^{R_u}(t) & = (y,p_2) \end{cases}$$

Definition 4. Given a pair of Lagrangian manifolds M_s, R_u with associated functions defined on a time interval $I = [s, u]$, we will call the area associated to the pair of manifolds the symplectic inner product defined by:

$$A_{M_s}^{R_u}(t, y) = \omega(\mathbf{X}_{t,y}(\tau), \mathbf{Y}^{t,y}(\tau)) \tag{65}$$

where $\mathbf{X}_{t,y}(\tau)$ and $\mathbf{Y}^{t,y}(\tau)$ are respectively the solutions of Hamilton's equations (13), defined by:

$$\mathbf{X}_{t,y}(\tau) \equiv \gamma_{M_s}^{R_t^y}(\tau) - \mathbf{m}(\tau)$$
$$\mathbf{Y}^{t,y}(\tau) \equiv \gamma_{M_t^y}^{R_u}(\tau) - \mathbf{m}(\tau)$$



Lemma 4. *Consider a pair of Lagrangian manifolds M_s, R_u defined by its associated functions $F, \hat{F}$ on $I = [s, u]$, and by the two vectors $\delta, \hat{\delta} \in \mathbb{R}^n$. Then we can write the associated area $A_{M_s}^{R_u}(t, y)$ as*

$$A_{M_s}^{R_u}(t, y) = (y - m(t))^T C^{-1}(t)(y - m(t)) \tag{66}$$

where $m(t)$ and $C(t)$ are respectively the projection onto the configuration space of $\mathbf{m}_t$, the consistent trajectory to the pair of Lagrangian manifold, a symmetric matrix defined by:

$$m_t = F_t[\omega(\hat{\mathbf{F}}, \mathbf{F})]^{-1}\hat{\delta} + \hat{F}_t\left[[\omega(\hat{\mathbf{F}}, \mathbf{F})]^T\right]^{-1}\delta \tag{67}$$

$$C(t) = F(t)[\omega(\hat{\mathbf{F}}, \mathbf{F})]^{-1}\hat{F}^T(t) \tag{68}$$

Moreover $C(t)$ is independent of the choice of the associated functions $F_t, \hat{F}_t$ to the Lagrangian manifolds M_s, R_u.

Proof. As associated functions to the pair of Lagrangian manifolds on the time interval I, they are isotropic, which implies that $\dot{F}_t F_t^{-1}$ and $\dot{\hat{F}}_t \hat{F}_t^{-1}$ are symmetric matrices on the interval I. Hence $\dot{F}_t F_t^{-1} - \dot{\hat{F}}_t \hat{F}_t^{-1}$ is also a symmetric matrix on I. But

$$\begin{aligned}
\dot{F}_t F_t^{-1} - \dot{\hat{F}}_t \hat{F}_t^{-1} &= \hat{F}_t^{-T}\hat{F}_t^T \dot{F}_t F_t^{-1} - \hat{F}_t^{-T}\dot{\hat{F}}_t^T F_t F_t^{-1} \\
&= \hat{F}_t^{-T}[\hat{F}_t^T \dot{F}_t - \dot{\hat{F}}_t^T F_t]F_t^{-1} \\
&= C^{-1}(t)
\end{aligned}$$

which proves that for all $t \in I$, $C(t)$ is a symmetric matrix.

Now we prove (66). By (65) we can write:

$$A_{M_s}^{R_u}(t, y) = \omega(\gamma_{M_s}^{R_t^y}(\tau), \gamma_{M_t^y}^{R_u}(\tau)) - \omega(\gamma_{M_s}^{R_t^y}(\tau), \mathbf{m}(\tau)) - \omega(\mathbf{m}(\tau), \gamma_{M_t^y}^{R_u}(\tau)) \tag{69}$$

Using (63) and (64), we compute the first term of (69):

$$\omega(\gamma_{M_s}^{R_t^y}(\tau), \gamma_{M_t^y}^{R_u}(\tau)) = y^T\hat{F}^{-T}\hat{\delta} + y^T F^{-T}\omega(\mathbf{F}, \hat{\mathbf{F}})\hat{F}^{-1}y + \delta^T F^{-1}y \tag{70}$$

but,

$$
\begin{aligned}
y^T\hat{F}^{-T}\hat{\delta} &= y^T\hat{F}^{-T}\left[F\omega^{-1}(\hat{\mathbf{F}},\mathbf{F})\right]^{-1}\left[F\omega^{-1}(\hat{\mathbf{F}},\mathbf{F})\right]\hat{\delta} \\
&= y^T\hat{F}^{-T}\omega(\hat{\mathbf{F}},\mathbf{F})F^{-1}\left[F\omega^{-1}(\hat{\mathbf{F}},\mathbf{F})\right]\hat{\delta} \\
&= y^T c^{-1}(t)F\omega^{-1}(\hat{\mathbf{F}},\mathbf{F})\hat{\delta} \qquad (71)
\end{aligned}
$$

and

$$
\begin{aligned}
\delta^T F^{-T}y &= \delta^T F^{-T}\left[\omega^T(\hat{\mathbf{F}},\mathbf{F})\hat{F}^{-1}\right]\left[\omega^T(\hat{\mathbf{F}},\mathbf{F})\hat{F}^{-1}\right]^{-1}y \\
&= \delta^T c^{-T}\hat{F}\omega^{-T}(\hat{\mathbf{F}},\mathbf{F})y \\
&= y^T c^{-1}\hat{F}\omega^{-T}(\hat{\mathbf{F}},\mathbf{F})\delta \qquad (72)
\end{aligned}
$$

hence

$$
\begin{aligned}
\omega(\gamma_{M_s}^{R_t^y}(\tau),\gamma_{M_t^y}^{R_u}(\tau)) &= y^T c^{-1}\left[F\omega^{-1}(\hat{\mathbf{F}},\mathbf{F})\hat{\delta} + \hat{F}\omega^{-T}(\hat{\mathbf{F}},\mathbf{F})\delta\right] + y^T F^{-T}\omega(\mathbf{F},\hat{\mathbf{F}})\hat{F}^{-1}y \\
&= y^T c^{-1}m(t) + y^T c^{-1}(t)y \\
&= y^T c^{-1}(t)(m(t)-y) \qquad (73)
\end{aligned}
$$

By similar computations, we also verify that:

$$
\omega(\gamma_{M_s}^{R_t^y}(\tau),\mathbf{m}(\tau)) = \delta^T F^{-1}(m(t)-y) \qquad (74)
$$

$$
\omega(\gamma_{M_t^y}^{R_u}(\tau),\mathbf{m}(\tau)) = (y-m(t))^T\hat{F}^{-T}\hat{\delta} \qquad (75)
$$

Using (69), (73), (74) and (75) we will get

$$
\begin{aligned}
A_{M_s}^{R_u}(t,y) &= \omega(\gamma_{M_s}^{R_t^y}(\tau),\gamma_{M_t^y}^{R_u}(\tau)) - \omega(\gamma_{M_s}^{R_t^y}(\tau),\mathbf{m}(t)) - \omega(\mathbf{m}(t),\gamma_{M_t^y}^{R_u}(\tau)) \\
&= y^T c^{-1}(t)(m(t)-y) - \delta^T F^{-1}(m(t)-y) - (y-m(t))^T\hat{F}^{-T}\hat{\delta} \\
&= (y-m_t)C^{-1}(t)(y-m_t)
\end{aligned}
$$

Now we prove that $C(t)$ is independent of the choice of the associated functions to the Lagrangian manifolds. Let us denote by H_t another function associated to the Lagrangian manifold L_s. Using the argument of the proof of Proposition 2, we have the following relationship between F_t and H_t for $t \in I$:

$$
H_t = F_t F_s^{-1} H_s
$$

Therefore

$$
\begin{aligned}
H_t\left[\omega(\hat{\mathbf{F}}_t,\mathbf{H}_t)\right]^{-1}\hat{F}_t^T &= F_t F_s^{-1}H_s\left[\omega(\hat{\mathbf{F}}_t,\mathbf{F}_t F_s^{-1}H_s)\right]^{-1}\hat{F}_t^T \\
&= F_t F_s^{-1}H_s\left[\omega(\hat{\mathbf{F}}_t,\mathbf{F}_t)F_s^{-1}H_s\right]^{-1}\hat{F}_t^T \\
&= F_t\left[\omega(\hat{\mathbf{F}}_t,\mathbf{F}_t)\right]^{-1}\hat{F}_t^T \\
&= C(t)
\end{aligned}
$$

A similar argument is valid if one replaces $\hat{F}_t$ by another associated function to the Lagrangian manifold R_u. □

Definition 5. A pair of Lagrangian manifolds M_s, R_u is called a compatible pair in the time interval $I = [s,u]$ if:

- Their associated functions $F_t, \hat{F}_t$ are defined on I.
- $C(t)$, defined by (68), is positive definite on the interior of I and non negative definite for $t = s$ and $t = u$.

Proposition 4. *Given an arbitrary final (respectively. initial) Lagrangian manifold with an associated function $\hat{F}_t$ (respectively. F_t) defined on the interval $I = [s, u]$ there exists a family of compatible initial (respect. final) Lagrangian manifolds*

Proof. Consider a final Lagrangian manifold R_u and $\hat{F}_t$ its associated function defined on the interval I. For any $n \times n$ non singular matrix σ, let us denote by P the positive definite matrix

$$P \equiv \sigma\sigma^T \tag{76}$$

We now define a family F_t^P of solutions of the matrix differential equation on I.

$$\ddot{F}_t = K(t)F_t$$

(where, as usual. $K(t)$ denotes a symmetric non singular matrix on I) by

$$F_t^P = \hat{F}_t P + \hat{F}_t \int_s^t \hat{F}_\lambda^{-1} F_\lambda^{-T} d\lambda \tag{77}$$

which, in turn, will allow us to define a family of initial Lagrangian manifolds. $M_s^{P,\delta}$:

$$M_s^{P,\delta} = \{(q,p) : p = \dot{F}_s^P [F_s^P]^{-1} q - [F_s^P]^{-T}\delta\},$$

compatible with R_u. Observe that for any P fulfilling (76) we have

$$\omega(\hat{F}_t, F_t^P) = I_{n\times n} \tag{78}$$

An analogous argument is valid if one starts from a fixed initial Lagrangian manifold M_s.

□

Remark 4. In fact we can always choose for two compatible Lagrangian manifolds on I, associated functions such that (78) holds.

Indeed consider F_t and $\hat{F}_t$ any associated functions for the pair of Lagrangian manifolds on the time interval I. If (78) is not verified we can replace $\hat{F}_t$ by $H \equiv \hat{F}_t \omega(\hat{\mathbf{F}}_t, \mathbf{F}_t)^{-T}$ (which is also an function on I associated to the Lagrangian manifold R_u). Then

$$\begin{aligned}
\omega(\mathbf{H}_t, \mathbf{F}_t) &= \omega(\omega(\hat{\mathbf{F}}_t, \mathbf{F}_t)^{-T}\hat{\mathbf{F}}, \mathbf{F}_t) \\
&= (\hat{F}_t\omega(\hat{\mathbf{F}}_t, \mathbf{F}_t)^{-T})^T \dot{F}_t - (\dot{\hat{F}}\ \omega(\hat{\mathbf{F}}_t, \mathbf{F}_t)^{-T})^T F_t \\
&= \omega(\hat{\mathbf{F}}_t, \mathbf{F}_t)^{-1}\omega(\hat{\mathbf{F}}_t, \mathbf{F}_t) = I_{n\times n}
\end{aligned}$$

Proposition 5. *If R_u (respectively. M_s) is a final (respect. initial) Lagrangian manifold with associated function $\hat{F}_t$ (respect. F_t) on I, then the pair $M_{t_0}^y, R_u$ (respect. $M_s, R_{t_0}^y$)is compatible in the interval $[t_0, u]$ (respect. $[s, t_0[$) and for all configurations $y \in \mathbb{R}^n$.*

Proof. A solution associated to the singular manifold $M_{t_0}^x$ on the time interval $[t_0, u]$ is given by $E_{t_0}(t)$ (i.e. the solution of the matrix equation (15) satisfying the Cauchy data $E_{t_0}(t_0) = 0$, $\dot{E}_{t_0}(t_0) = I_{n\times n}$). The corresponding matrix (68) relative to this case is:

$$C(t) = E_{t_0}(t)\left[\omega(\hat{\mathbf{F}}_t, E_{t_0}(t))\right]^{-1}\hat{F}(t).$$

To prove that the pair of Lagrangian manifolds are compatible we need to prove that the above expression is non negative definite. For this, we write $E_{t_0}(t)$ on $[t_0, u]$ as:

$$E_{t_0}(t) = \hat{F}(t)\left[\int_{t_0}^t \hat{F}^{-1}(\lambda)\hat{F}^{-T}(\lambda)d\lambda\right]F^T(t_0)$$

Then we obtain for $C(t)$:

$$\begin{aligned} E_{t_0}(t)\left[\omega(\hat{\mathbf{F}}_t, E_{t_0}(t))\right]^{-1}\hat{F}^T(t) &= E_{t_0}(t)\left[\hat{F}^T(t_0)\right]^{-1}\hat{F}^T(t) \\ &= \hat{F}(t)\left[\int_{t_0}^{t}\hat{F}^{-1}(\lambda)\hat{F}^{-T}(\lambda)d\lambda\right]F^T(t_0)\left[\hat{F}^T(t_0)\right]^{-1}\hat{F}^T(t) \\ &= \hat{F}(t)\left[\int_{t_0}^{t}\hat{F}^{-1}(\lambda)\hat{F}^{-T}(\lambda)d\lambda\right]\hat{F}^T(t), \end{aligned}$$

which is clearly a non negative definite matrix. In the same way we can prove that given an initial Lagrangian manifold L_s such that the associated solution $F(t)$ is invertible on the time interval $[s, u]$, then, for all configurations $z \in \mathbb{R}^n$ and all $t_0 \in I - \{s\}$, the pair of Lagrangian manifolds M_s, $R^z_{t_0}$ are compatible.

Finally, for $s \leq t_1 < t_2 \leq u$ and for all configurations $x, z \in \mathbb{R}^n$, the pairs of singular Lagrangian manifolds $M^x_{t_1}$, $R^z_{t_2}$ are compatible. In this case the associated functions are $E_{t_1}(t)$ and $\hat{E}_{t_2}(t) \equiv -E_{t_2}(t)$, hence the corresponding $C(t)$ is given by:

$$E_{t_1}(t)\left[\omega(\hat{\mathbf{E}}_{t_2}(t), \mathbf{E}_{t_1}(t))\right]^{-1}\hat{E}_{t_2}(t)$$

which is clearly a positive definite matrix.

□

In a two dimensional phase space there is no restriction on the pairs of boundary Lagrangian manifolds i.e all pairs of Lagrangian manifolds with associated functions on an interval I are compatible.

Let us consider the following pairs of Lagrangian manifolds in $\mathbb{R}^2$:

$$M_s = \{(q,p) \in \mathbb{R}^2 \mid p = \alpha q - \beta\}$$
$$R_u = \{(q,p) \in \mathbb{R}^2 \mid -p = \hat{\alpha} q - \hat{\beta}\}$$

where $\alpha, \hat{\alpha}, \beta, \hat{\beta} \in \mathbb{R}$. Consider associated functions $F_t, \hat{F}_t$ on the time interval I. By definition of associated function, both functions have a definite sign on the interior of the interval I. Without loss of generality we can assume also that $\omega(\hat{\mathbf{F}}_t, \mathbf{F}_t) = 1$. Hence two cases are possible:

1. $F_t\hat{F}_t > 0$
2. $F_t\hat{F}_t < 0$

In the first case, we have $C(t) > 0$ on I so the Lagrangians manifolds are compatible. For the second case, we consider the following two Lagrangians manifolds:

$$M'_s = R_u \tag{79}$$

$$R'_u = M_u \tag{80}$$

As it can easily be verified, the correspondent functions for the pair M'_s, R'_u will be given by $H_t, \hat{H}_t$ defined as

$$H_t = \hat{F}_t \tag{81}$$

$$\hat{H}_t = F_t \tag{82}$$

Therefore we will have $H_t\hat{H}_t < 0$ and:

$$\omega(\hat{\mathbf{H}}_t, \mathbf{H}_t) = \omega(\mathbf{F}_t, \hat{\mathbf{F}}_t) = -1$$

which implies that $C(t) > 0$

Returning to the general case, we state the main result of this section i.e. we can associate to any compatible pair of Lagrangian manifolds a Gaussian probability density:

Theorem 2. *A pair of compatible Lagrangian manifolds M_s, R_u define on the time interval I a Gaussian probability density $\rho(t,y)$ whose mean is the consistent trajectory associated with M_s R_u and variance $C(t)$ given by*

$$C(t) = F_t[\omega[\hat{\mathbf{F}}_t, \mathbf{F}_t)]^{-1}\hat{F}_t^T \tag{83}$$

where F_t and $\hat{F}_t$ are the functions associated with the Lagrangian manifolds M_s and R_u.

Proof. By Lemma 4, a pair of Lagrangian manifolds defines a symplectic area $A(t,y)$ defined by

$$A(t,y) = (y - m(t))^T C^{-1}(t)(y - m(t))$$

As we are assuming that the pair of Lagrangians manifolds are compatible $C(t)$ in the above expression is a non negative definite symmetric matrix which implies that

$$\rho(y,t)dy = \frac{1}{\sqrt{2\pi c(t)}}\exp(-A(t,y))dy$$

is a Gaussian density function having as mean the consistent trajectories and variance (83)

□

3.2. **The associated probability density and a system of adjoint heat equations.** Given a pair of compatible Lagrangians manifolds L_s, R_u we know, using last section results, that we have a Gaussian probability density function associated to the manifolds. We now show that this density can be expressed as a product of positive solutions of the adjoint heat equations.

$$\frac{\partial \eta}{\partial t} = H\eta \tag{84}$$

$$-\frac{\partial \hat{\eta}}{\partial t} = H\hat{\eta} \tag{85}$$

where $y \in \mathbb{R}^n$ and $t \in I$ and

$$H = -\frac{1}{2}\Delta + \langle K(t)y, y\rangle$$

with $K(t)$ denoting a symmetric $n \times n$ positive matrix with a continuos dependence in t.

Proposition 6. *Consider a pair of compatible Lagrangian manifolds M_s, R_u on an time interval I. Then (up to a normalising factor) we can write the associated density as a product of two positive solutions $\eta(t,y)$, $\hat{\eta}(t,y)$ of the adjoint heat equations (84) and (85), satisfying boundary conditions on the interval $I = [s,u[$ which are specified by the pair of Lagrangian manifolds.*

Proof. Let us first consider the compatible Lagrangian manifolds on a time interval $I \equiv [s,u]$:

$$M_s = \{(q,p) \mid p = \alpha q - \beta\} \tag{86}$$

$$R_u = \{(q,p) \mid -p = \hat{\alpha} q - \hat{\beta}\}, \tag{87}$$

where, as usual, $\alpha, \hat{\alpha}$ are invertible symmetric matrix and $\beta, \hat{\beta} \in \mathbb{R}^n$. This pair will determine a pair of generating functions (up to additive constants γ_s and γ_u) $S(s,x)$ and $-\hat{S}(u,z)$, namely

$$S(s,x) = \frac{1}{2}\langle \alpha x, x\rangle - \beta x + \gamma_s$$

$$-\hat{S}(u,z) = -\frac{1}{2}\langle \hat{\alpha} z, z\rangle + \hat{\beta} z + \hat{\gamma}_u$$

For any $t \in I$, the solutions $S(t,y)$ and $\hat{S}(t,y)$ of the Hamilton Jacobi equations (36) and (51) with boundary conditions given, respectively, by $S(s,x)$ and $S(u,z)$ are given according to propositions 1 and 2, by:

$$S(t,y) = \frac{1}{2}\langle \dot{F}_t F_t^{-1} y, y\rangle - F_t^{-T}\beta y + \gamma_t \tag{55 bis}$$

$$\hat{S}(t,y) = \frac{1}{2}\langle \dot{\hat{F}}_t \hat{F}_t^{-1} y, y\rangle - \hat{F}_t^{-T}\hat{\beta} y + \hat{\gamma}_t \tag{57 bis}$$

where γ_t and $\hat{\gamma}_t$ are, respectively, solutions on I of the first order differential equations (56) and (58). Adding (55) with (57) we will obtain:

$$S(t,y) + \hat{S}(t,y) = \frac{A(t,y)}{2} + g \tag{88}$$

where $A(t,y)$ is the symplectic area associated to the pair of manifolds and g denotes a reminder, to be determined. Let us take m_t, the consistent solution to the manifolds M_s, R_u. Obviously $A(t,m_t) = 0$, which implies

$$g = S(t,m_t) + \hat{S}(t,m_t).$$

Using (38),

$$\begin{aligned} g &= S_s(m_s) + R(s,m_s,t,m_t) + R(t,m_t,u,m_u) + \hat{S}(u,m_u) \\ &= S_s(m_s) + R(s,m_s,u,m_u) + \hat{S}(u,m_u) \end{aligned}$$

Hence g does not depend on the variables (t,y), $t \in I$. Using Theorem 2,we can express the associated probability density, $\rho(y,t)$ as

$$\begin{aligned} \rho(t,y) &= \frac{1}{\sqrt{(2\pi)^n \det C(t)}} \exp(-\frac{A(t,y)}{2}) \\ &= \frac{1}{\sqrt{(2\pi)^n \det C(t)}} \exp(-S(t,y) - \hat{S}(t,y) + g) \\ &= \frac{1}{\sqrt{(2\pi)^n \det[F_t\omega(\hat{\mathbf{F}},\mathbf{F})\hat{F}_t]}} \exp(-S(t,y) - \hat{S}(t,y) + g) \\ &= \frac{\exp(-S(t,y))}{\sqrt{(2\pi)^n|\det[F_t]|}} \frac{\exp(-\hat{S}(t,y))}{\sqrt{(2\pi)^n|\det[\hat{F}_t]|}} \sqrt{\frac{(2\pi)^n}{|\omega(\hat{\mathbf{F}},\mathbf{F})|}} \exp(g) \end{aligned} \tag{89}$$

Which implies that this density can be written as

$$\rho(y,t)dy = \mathbf{N}^{-1}\eta(t,y)\hat{\eta}(t,y)dy \tag{90}$$

where $\mathbf{N}$ is a normalising constant defined by

$$\mathbf{N} = \sqrt{\frac{(2\pi)^n}{|\omega(\hat{\mathbf{F}},\mathbf{F})|}} \exp(g) \tag{91}$$

and $\eta(t,y)$ and $\hat{\eta}(t,y)$ are defined by:

$$\eta(t,y) = \frac{\exp(-S(t,y))}{\sqrt{(2\pi)^n|\det[F_t]|}}$$

$$\hat{\eta}(t,y) = \frac{\exp(-\hat{S}(t,y))}{\sqrt{(2\pi)^n|\det[\hat{F}_t]|}}$$

which, as we can easily verify, are positive solutions of the adjoint heat equations (84) and (85). Also the boundary conditions satisfied by η and $\hat{\eta}$ depend on the pair of Lagrangian manifolds M_s, R_u.

Although (90) was obtained assuming that both Lagrangian manifolds were non singular, a similar expression follows in the case where one or both of them are

singular. If one considers the pair M_s^x, R_u (were, as before M_s^x denotes the singular Lagrangian manifold associated with an initial configuration $x \in \mathbb{R}^n$) then using (28), changing $S(t,y)$ for Hamilton's principal function $R(s,x,t,y)$, we obtain :

$$R(s,x,t,y) + \hat{S}(t,y) = \frac{1}{2}A_{s,x}(t,y) + g_1 \tag{92}$$

where $A_{s,x}(t,y)$ is the symplectic area defined by (65) and associated with the pair of Lagrangian manifolds M_s^x, R_u . Substituting, in (92), y by $m_{s,x}(t)$ (the solution consistent to the Lagrangian manifolds $M_{s,x}$ and R_u), we find that:

$$g_1 = R(s,x,u,m_{s,x}(u)) + \hat{S}(u,m_u) = \hat{S}(s,x).$$

Adapting to this case the calculations used in (89),

(93)

$$\begin{aligned}
\rho_{s,x}(t,y) &= \frac{\exp(-R_{s,x}(t,y))}{\sqrt{(2\pi)^n \det[E_s(t)]}} \frac{\exp(-\hat{S}(t,y))}{\sqrt{(2\pi)^n |\det[\hat{F}_i]|}} \sqrt{\frac{(2\pi)^n}{|\det[\omega(\hat{\mathbf{F}}, \mathbf{E}_s(t))]|}} \exp(g_1) \\
&= h(s,x,t,y)\hat{\eta}(t,y) \sqrt{\frac{(2\pi)^n}{|\det[\hat{F}(s)]|}} \exp(\hat{S}(s,x)) \\
&= \frac{h(s,x,t,y)\hat{\eta}(t,y)}{\exp\sqrt{\frac{(2\pi)^n}{|\det[\hat{F}(s)]|}}(-\hat{S}(s,x))} \\
&= \frac{h(s,x,t,y)\hat{\eta}(t,y)}{\hat{\eta}(s,x)}
\end{aligned}$$

Here $h(s,x,t,y)$, $s \leq t$ is the fundamental solution of (57) which, as can be easily verified, can also be written as

$$\frac{\exp(-R_{s,x}(t,y))}{\sqrt{(2\pi)^n \det[E_s(t)]}}.$$

The case where one considers the pair of Lagrangian manifolds M_s, R_u^z or M_s^x, R_u^z are treated similarly. We would obtain respectively:

$$\rho^{u,z}(t,y) = \frac{\eta(t,y)h(t,y,u,z)}{\eta(u,z)} \tag{94}$$

$$\rho_{s,x}^{u,z}(t,y) = h(s,x,t,y)h(t,y,u,z) \tag{95}$$

□

3.3. **Diffusion Processes.** In section 3.1, we saw how a pair of compatible Lagrangian manifolds determine a probability density and how this density is written in terms of a pair of solutions of adjoint heat equations with boundary values specified by the pair of Lagrangian manifolds. In this section, we will show that a pair of compatible Lagrangian manifolds determine a Gaussian diffusion process. Our setting will be given by the pair of compatible manifolds M_s, R_u. We will begin by the definition of a $\mathbb{R}^m$- valued a diffusion process Z_t on the time interval $[s,u]$ by means of the value of the density associated with the pair M_s, R_u at time s, $\rho(s,x)$ and a transition function . This process will verify $E[Z_t] = m_t$, where E denotes the mathematical expectation and m_t denotes the consistent solution associated with the pair of Lagrangian manifolds. In this sense, we call Z_t the Euclidean quantization of m_t. It is crucial to stress that, as suggested by Feynman's path integral approach [1], the probabilistic framework is extracted exclusively from the classical model.

Let us first define a transition probability function: Given any configuration $x \in \mathbb{R}^n$ and a Borelian $B \in \mathcal{B}$ we define for t_1, t such that $s \leq t_1 < t \leq u$ the function:

$$Q^{R_u}_{L^x_{t_1}}(t, B) = \int_B \rho^{R_u}_{L^x_{t_1}}(t, y)dy \tag{96}$$

where $\rho^{R_u}_{L^x_{t_1}}(t, y)$ is the density associated to the pair of Lagrangian manifolds $L^x_{t_1}, R_u$ which was given (cf. Theorem 6) by

$$\rho^{R_u}_{L^x_{t_1}}(t, y) = \frac{h(t_1, x, t, y)\hat{\eta}(t, y)}{\hat{\eta}(t_1, x)}$$

We can easily verify that $Q^{R_u}_{L^x_{t_1}}(t, y)$ is a transition probability using the semigroup properties of the solution of the heat equation or using the additivity of the solutions of the Hamilton Jacobi equation. Together with the initial probability density provided by the density associated with the pair of manifolds M_s, R_u at time $t = s$, i.e $\rho^{R_u}_{M_s}(s, x)$ and the transition probability defined by (96), we satisfy the conditions of existence of a Gaussian diffusion process Z_t (cf. [12]) called the Euclidean quantization of the classical solution consistent with M_s, R_u. This process will verify, by construction,

$$P(Z_t \in B) = \int_B \rho(t, y)dy \tag{97}$$

But we could as well start from the " final " information i.e. the value of the probability density associated with the pair of Lagrangian manifolds M_s, R_u at time u, $\rho(u, z)$ and use a time reversed version of the probability function $Q^{R_u}_{L^x_{t_1}}(t, y)$ constructed as follows:

Consider for $t < t_2 \leq u$ and $y_1 \in \mathbb{R}^n$ the density $\rho^{R^z_{t_2}}_{M_s}(t, y)$ associated with the pair $M_s R^z_{t_2}$:

$$\rho^{R^z_{t_2}}_{M_s}(t, y) = \frac{\eta(t, y)h(t, y, t_2, z)}{\eta(t_2, z)}$$

and define for $x \in \mathbb{R}^n$, $B \in \mathcal{B}$ and t_1, t_2 such that $s \leq t_1 < t_2 \leq u$ the function:

$$Q^{R^z_{t_2}}_{M_s}(t_1, B) = \int_B \rho^{R^z_{t_2}}_{M_s}(t_1, y)dy \tag{98}$$

which is, as easily verified a (backward) transition probability function. This will enable us to construct a $\mathbb{R}^n$ valued Gaussian diffusion process Y_t with a backward transition function given by (98) and final probability given by the probability density associated with the pair of manifolds M_s, R_u at time $t = u$, $\rho^{R_u}_{M_s}(u, z)$. This process satisfies (cf. [12]) as well

$$P(Y_t \in B) = \int_B \rho(t, y)dy \tag{99}$$

Y_t and Z_t, as diffusion processes having the same probability density for all t in $[s, u]$. We will call this process the Euclidean quantization of the $M_s R_u$ consistent trajectory. Summarising we have:

Theorem 3. *Given any pair of compatible manifolds in an interval $I = [s, u]$, there exists a Gaussian diffusion process Z_t such that:*

- *$E[Z_t] = m_t$, where m_t is the solution consistent with the pair of Lagrangian manifolds, hence in particular we have*

$$\frac{d^2}{dt^2}E[Z_t] = K(t)E[Z_t]$$

where $K(t)$ is a symmetric $n \times n$ matrix non singular on I.

- *The variance of this process is completely described by the functions F_t and $\hat{F}_t$ associated to both Lagrangian manifolds i.e*

$$C(t) = F_t[\omega(\hat{F}_t, F_t)]^{-1}\hat{F}_t^T$$

3.4. **Uniqueness.** One can see that the Euclidean Quantization, Z_t associated with a pair of compatible Lagrangian manifolds M_s and R_u in a time interval I solves the following Itô's sde's

$$dZ_t = (A(t)Z_t + a(t))dt + dW_t \tag{100}$$

$$d_*Z_t = (A^*(t)Z_t - a^*(t))dt + d_*W_{*\,t} \tag{101}$$

with respect to the past and future filtration $(\mathcal{P}_t)$ and $(\mathcal{F}_t)$ respectively. Here d and d_* denote forward and backward differentials and W and W_* are $\mathbb{R}^n$-valued Wiener processes w.r.t these filtration [12]. The drift terms are given in terms of the associated functions of the Lagrangians manifolds i.e

$$f(t) = \dot{\hat{F}}_t\hat{F}_t^{-1}, \quad a(t) = \hat{F}_t^{-T}\hat{F}_u^T\hat{\beta}$$
$$f^*(t) = \dot{F}_tF_t^{-1}, \quad a^*(t) = F_t^{-T}F_s^T\beta$$

Now we will prove the reverse statement. Consider a $\mathbb{R}^n$-valued Gaussian diffusion process Z_t with a fixed identity diffusion matrix and satisfying the following hypothesis:

(H1) The expectation of the process $m_t = E(Z_t)$ is a solution of the ordinary differential equation on I

$$\ddot{m}_t = K(t)m_t. \tag{102}$$

This assumption establish the link between the diffusion process Z_t and the classical Hamiltonian system.

(H2) The variance, $C(t)$ of the process is non zero on the interior of the time interval. Also let us denote by C_s and C_u the nonnegative values of the variance at the endpoints of the interval I, i.e $C_s = C(s)$ and $C_u = C(u)$.

(H3) Z_t satisfies in I the following forward and backward Itô's sde's

$$dZ_t = (f(t)Z_t + \beta(t))dt + dW_t \tag{103}$$

$$d_*Z_t = (f^*(t)Z_t - \beta(t)^*)dt + d_*W_{*\,t} \tag{104}$$

where $f(t), f^*(t)$ are two non singular symmetric matrices on the time interval I.

Lemma 5. *Consider a $\mathbb{R}^n$-valued Gaussian diffusion process Z_t defined on the interval I, satisfying the assumptions (H2) and (H3). Then its variance is given by*

$$C(t) = M_t^*\omega^{-1}(\mathbf{M}_t, \mathbf{M}_t^*)M_t^T \tag{105}$$

where M_t and M_t^* satisfy the (deterministic) matrix equations on I

$$\dot{M}_t = f(t)M_t, \; M_s = I_{n\times n} \tag{106}$$

$$\dot{M}_t^* = f^*(t)M_t^*, \; M_u^* = I_{n\times n} \tag{107}$$

Moreover, if assumption (H1) is satisfied, then the expectation m_t of the process can be written as

$$m_t = M_t^*\omega^{-1}(\mathbf{M}_t, \mathbf{M}_t^*)\delta + M_t\omega^{-T}(\mathbf{M}_t, \mathbf{M}_t^*)\delta^* \tag{108}$$

Proof. The general solutions of (103) and (104) are given respectively by

$$Z_t = M_t(Z_s + \int_s^t M_\lambda^{-1} f(\lambda) d\lambda + \int_s^t M_\lambda^{-1} dW_\lambda) \tag{109}$$

$$Z_t = M_t^*(Z_u + \int_t^u [M_\lambda^*]^{-1} \beta^*(\lambda) d\lambda + \int_t^u [M_\lambda^*]^{-1} dW_{*,\lambda}) \tag{110}$$

where M_t and M_t^* satisfy (106) and the last integrals are Itô's forward and backward integrals [17]. Using (109), we can conclude (cf.[2]) that the variance $C(t)$ of the process is the only positive definite matrix solution of the following ordinary differential equation

$$\dot{C}(t) = f(t)C(t) + C(t)f(t)^T + I, \ C(s) = C_s$$

But $C(t)$ is also the variance of the backward process. Hence the solution of the above ordinary differential equation must be also the solution of

$$\dot{C}(t) = f^*(t)C(t) + C(t)f^*(t)^T - I, \ C(u) = C_u.$$

As f and f^* are both symmetric matrices we conclude that $C(t)$ verifies:

$$2f(t)C(t) + I = 2f^*(t)C(t) - I$$

hence

$$C(t) = (f^*(t) - f(t))^{-1} \tag{111}$$

Using (106) and (107) we obtain

$$\begin{aligned} C(t) &= (\dot{M}_t^*[M_t^*]^{-1} - \dot{M}_t M_t)^{-1} \\ &= (M^{-T}\omega(\mathbf{M}_t, \mathbf{M}_t^*)M^*)^{-1} \\ &= M_t^*\omega^{-1}(\mathbf{M}_t, \mathbf{M}_t^*)M_t^T \end{aligned}$$

Taking expectations in both sides of (109) and (110) and differentiating with respect to t we will get

$$\dot{m}_t = f(t)m_t + \beta(t) \tag{112}$$

$$\dot{m}_t = f^*(t)m_t - \beta^*(t), \tag{113}$$

obtaining

$$f(t)m_t + \beta(t) = f^*(t)m_t - \beta^*(t).$$

which implies that

$$m_t = (f^*(t) - f(t))^{-1}$$

And by (111) we get:

$$m_t = C(t)[\beta(t) + \beta^*(t)] \tag{114}$$

Since m_t is, by assumption, a solution of Hamilton's version of equations (10) of motion we obtain that $\beta(t), \beta^*(t)$ are solutions of the differential equations:

$$\dot{\beta}(t) + f(t)\beta(t) = 0$$

$$\dot{\beta}^*(t) - f^*(t)\beta^*(t) = 0$$

Using (106) and (107) the solutions are

$$\beta(t) = M_t^{-T}\delta \tag{115}$$

$$\beta^*(t) = [M_t^*]^{-T}\delta^* \tag{116}$$

where δ and δ^* are two arbitrary vectors of $\mathbb{R}^n$. Substituting in (114), the following formula holds for the expectation:

$$\begin{aligned} m_t &= C(t)[\beta(t) + \beta^*(t)] \\ &= M_t^* \omega^{-1}(\mathbf{M}_t, \mathbf{M}_t^*) M_t^T [M_t^{-T}\delta + [M_t^*]^{-T}\delta^* \\ &= M_t^* \omega^{-1}(\mathbf{M}_t, \mathbf{M}_t^*)\delta + M_t \omega^{-T}(\mathbf{M}_t, \mathbf{M}_t^*)\delta^* \end{aligned}$$

□

Theorem 4. *Consider a $\mathbb{R}^n$- valued Gaussian diffusion Z_t on an interval I, satisfying the assumptions (H1), (H2) and (H3). Then Z_t is the Euclidean quantization of the pair of Lagrangian manifolds M_s R_u given by :*

$$M_s = \{(q,p) \mid p = f_s^* q - \beta^*(s)\} \tag{117}$$

$$R_u = \{(q,p) \mid -p = f_u q + \beta_u\} \tag{118}$$

Proof. Consider $F_t, \hat{F}_t$ defined as

$$F_t = M_t^* \tag{119}$$

$$\hat{F}_t = M_t \tag{120}$$

where M_t, M_t^* are defined by (106) and (107). As $f(t)$ and $f^*(t)$ are symmetric matrices non singular on the interval I, F_t and $\hat{F}_t$ are disconjugate solutions of (102). Using these functions together with δ and $\hat{\delta}$ defined by (115) and (116) we can introduce a pair of Lagrangian manifolds :

$$M_s = \{(q,p) \mid p = \dot{F}_s F_s^{-1} q - F_s^{-T}\delta\} \tag{121}$$

$$R_u = \{(q,p) \mid -p = \dot{\hat{F}}_u \hat{F}_u^{-1} q - \hat{F}_u^{-T}\hat{\delta}\} \tag{122}$$

They define a Gaussian process whose mean is given by:

$$m_t = F_t[\omega(\hat{\mathbf{F}}, \mathbf{F})]^{-1}\delta + \hat{F}_t\left[[\omega(\hat{\mathbf{F}}, \mathbf{F})]^T\right]^{-1}\delta^* \tag{123}$$

$$= M_t^*[\omega(\mathbf{M}_t, \mathbf{M}_t^*)]^{-1}\delta + M_t\left[[\omega(\mathbf{M}_t, \mathbf{M}_t^*)]^T\right]^{-1}\delta^* \tag{124}$$

and variance given by

$$C(t) = M_t^* \omega^{-1}(\mathbf{M}_t, \mathbf{M}_t^*) M_t^T$$

□

Remark 5. In [13] the authors consider diffusions satisfying the sdes (100) and (101) as " Gaussian Bernstein processes" associated with the quantum Hamiltonian

$$H = -\frac{1}{2}\Delta + \frac{1}{2} < K(t)x, x >$$

in the following way: The forward and backward drifts are related to a pair of positive solutions η, η^* of the adjoint heat equations (84) and (85):

$$\frac{\nabla \eta_t}{\eta_t} = f(t)x + \beta(t)$$

$$-\frac{\nabla \eta_t^*}{\eta_t^*} = f^*(t)x - \beta_t^*$$

From these two relations we can easily obtain (102). In the present paper, however, we insist on starting exclusively from the classical Hamiltonian system. Now we see that these Bernstein processes are , in fact, associated to a pair of Lagrangian manifolds. It should be noted that in [13] the necessary assumption of symmetry on the matrices f, f^* is missing.).

4. Examples

4.1. **The Bernstein bridge.** Let H be the time dependent quadratic Hamiltonian on $\mathbb{R}^{2n}$

$$H = \frac{1}{2}|p|^2 - < K(t)q, q >$$

where $K(t)$ denotes a symmetric non singular matrix on the interval $[s, u]$. The Bernstein bridge associated with H is characterised as a Bernstein diffusion Z_t with boundary data given by $Z_s = x$, $Z_u = z$, where x and z are two configurations in $\mathbb{R}^n$. This implies that the variance of Z_t satisfies $C(s) = C(u) = 0$. By Lemma 5, using in particular (106) and (107), the matrices M_t and M_t^* are determined by the boundary conditions:

$$\begin{cases} M_s &= I_{n\times n} \\ M_u &= 0 \end{cases} \begin{cases} M_s^* &= 0 \\ M_u^* &= I_{n\times n} \end{cases}$$

which implies that :

$$M_t = E_u(t)E_u^{-1}(s), \;\; M_t^* = E_s(t)E_s^{-1}(u).$$

Using (107) we can write the variance of the process as:

$$C(t) = E_s(t)\omega^{-1}(\hat{\mathbf{E}}_u, \mathbf{E}_s)\hat{E}_u^T(t) \;\; t \in [s, u]$$

where $\hat{E}_u(t) = -E_u(t)$. Using now (107) the mean of the process is:

$$m_t = E_s(t)\omega^{-1}(\hat{\mathbf{E}}_u, \mathbf{E}_s)z + \hat{E}_u(t)\omega^{-T}(\hat{\mathbf{E}}_u, \mathbf{E}_s)x$$

The formulas for the mean and variance suggest that we can rescale (M_t and M_t^*) so that

$$M_t = \hat{E}_u(t), \;\; M_t^* = E_s(t)$$

This shows that the Bernstein bridge is the Euclidean quantization associated with the pair of Lagrangians manifolds:

$$M_s^x = \{(q, p) \mid q = x\} \tag{125}$$

$$R_u^z = \{(q, p) \mid q = z\} \tag{126}$$

The stochastic differential equations satisfied by the process Z_t are, respectively

$$dZ_t = [\frac{\dot{\hat{E}}_u(t)}{\hat{E}_u(t)}Z_t + \frac{z}{\hat{E}_u^{-1}(t)}]dt + dW_t \tag{127}$$

$$dZ_t = [\frac{\dot{E}_s(t)}{E_s(t)}Z_t - \frac{x}{E_u^{-1}(t)}]dt + dW_t \tag{128}$$

with respect to the past and future filtrations.

When $K = 0$ this is the rigorous version of Feynman's example (cf. p179-180 of [1])

4.2. **Brownian motion starting at $x_0 \in \mathbb{R}$.** The Brownian motion can be characterised as a Bernstein process associated with the free Hamiltonian

$$H = \frac{p^2}{2}$$

on $I = [0, T]$ that satisfies the forward sde

$$dZ_t = dW_t, \;\; Z_0 = x$$

As the forward drift is zero we must have $\dot{M}_t \equiv 0$ on the time interval and as $M_T = 1 \;\forall t \in I$ we get $M_t \equiv 1$ and $\delta = 0$. Using the same reasoning as in the

previous example (since $Z_0 = x$ we must have $C(0) = 0$) we get $M_t^* = E_0(t) = t$. With these values, the expectation will be

$$m_t = \frac{\delta^* M_t + \delta M_t^* \delta}{\omega(\mathbf{M}_t, \mathbf{M}_t^*)} = \delta^*$$

but as $m_0 = x$ this will imply that $\delta^* = x$ and $m_t \equiv x \ \forall t \in I$.

As for the variance of Z_t,

$$C(t) = \frac{M_t M_t^*}{\omega(\mathbf{M}_t, \mathbf{M}_t^*)} = t$$

Hence the associated Lagrangian manifolds will be, in this case

$$L_0 = \{(q,p) \ | q = x\} \tag{129}$$

$$R_T = \{(q,p) \ | p = 0\} \tag{130}$$

The backward stochastic differential equations satisfied by this process is

$$d^* Z_t = [\frac{1}{t} Z_t - \frac{x}{T-t}]dt + d_* W_t$$

4.3. **The Ornstein-Uhlenbeck process on the time interval** $[0,1]$. The process is defined as a solution of:

$$dZ_t = -\lambda Z_t + dW_t, \quad \lambda > 0 \tag{131}$$

$$Z_0 = x \tag{132}$$

Equation (131) shows that we should choose $f(t) \equiv \lambda$ and $\beta(t) \equiv 0$ on the whole interval. This is equivalent to choose a final Lagrangian manifold (i.e at time $t = 1$) of the form

$$R_1 = \{(q,p) \ | p = \lambda q\}$$

the missing Lagrangian manifold will be determined by the initial condition, (132) . Hence we should choose

$$L_0 = \{(q,p) | \ q = x\}$$

this will correspond to choose $M^*(t) = e_0(t)$ and $\delta^* = x$, where $e_0(t)$ is the solution of

$$\ddot{q}(t) = \lambda^2 q(t), \quad q(0) = 0, \ \dot{q}(0) = 1$$

i.e

$$e_0(t) = \frac{1}{a} sinh(at), a = sqrt\lambda$$

Hence we can conclude that this Ornstein-Uhlenbeck on the interval $[0,1]$ can be considered as the EQ under the Hamiltonian

$$H = \frac{1}{2}p^2 - \lambda^2 \frac{q^2}{2}$$

of the pair of Lagrangians manifolds L_0, R_1. Moreover the expectation of the process will be

$$m(t) = \frac{\delta M_t^* + \delta^* M_t}{\omega(M_t, M_t^*)} = \frac{xe^{-\lambda t}}{M(0)} = xe^{-\lambda t}$$

and its variance

$$c(t) = \frac{M_t M_t^*}{\omega(M_t, M_t^*)} = \frac{1}{2a}(1 - e^{-2at})$$

The backward differential equation will be

$$d_* Z_t = [\lambda coth(at) Z_t - \frac{x}{sinh(\lambda t)}]dt + dW_t$$

If Z_0 is $N(x, c_0)$ then one can easily find that

$$M_t = \frac{1}{\lambda} sinh(\lambda t) + c_0 e^{-at}$$

and

$$\delta^* = \frac{x}{c_0}$$

which will give us mean and variance

$$m_t = [\frac{x}{c_0}\frac{1}{\lambda}sinh(\lambda t) + c_0 e^{-at}] \tag{133}$$

$$c_t = \frac{1}{\lambda}e^{-\lambda t}sinh(\lambda t) + c_0 e^{-2\lambda t} \tag{134}$$

The backward differential equation will be

$$d_* Z_t = [\lambda\frac{1 - 2\lambda c_0 + e^{2\lambda t}}{-1 + 2\lambda c_0 + e^{2\lambda t}}Z_t - \frac{2\lambda e^{\lambda t}x}{c_0(-1 + \lambda c_0 + e^{2\lambda t})}]dt + dW_t$$

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