IF-UFRJ-94-12

INSTITUTO DE FISICA - UFRJ

Fermilab Library
0 1160 0037953 1
FERMILAB
AUG 1 199
LIBRARY

IF/UFRJ/12-94

GREEN FUNCTION APPROACH FOR COMPUTING

NON-RELATIVISTIC DETERMINANTS

C. Farina, H. Boschi-Filho and A. de Souza Dutra

Rio de Janeiro U.

Mar/94

UNIVERSIDADE FEDERAL DO RIO DE JANEIRO
INSTITUTO DE FÍSICA
Cx. P 68.528
21.944 - RIO DE JANEIRO
BRASIL

GREEN FUNCTION APPROACH FOR COMPUTING NON-RELATIVISTIC DETERMINANTS

C.Farina, H. Boschi-Filho

Instituto de Física, Universidade Federal do Rio de Janeiro

Cidade Universitária, Ilha do fundão, Cx.P. 68528

21945-970, Rio de Janeiro, RJ, Brazil Rio de Janeiro U.

A. de Souza Dutra

Departamento de Física e Química,

Universidade Estadual Paulista - Campus de Guaratinguetá,

Av. Ariberto Pereira da Cunha 333, Cx.P. 205,

12500-000, Guaratinguetá, SP, Brazil

Sao Paulo U., Guaratingueta

Abstract

We show how to compute determinants that appear very often in non-relativistic problems using the Green function technique, a method which has been successfully applied in the computation of one-loop effective actions of Quantum Field Theories (QFT). We illustrate this method by computing determinants of functional operators of the harmonic oscillator type with different boundary conditions. The physical meaning associated to each boundary condition is also discussed.

1. Introduction

It is well known that to compute one-loop effective actions means basically to evaluate a determinant of a functional operator (for a detailed discussion on effective actions see W. Dittrich and M. Reuter[1]). However, there are many approaches to make such a calculation. For instance, in 1951, J. Schwinger[2] used the proper time method to compute the one-loop effective action for a constant electromagnetic field as well as for the external field of a plane wave. This method consists in writing an integral representation for the one-loop effective action which, apart from the integration in the proper-time, needs basically the computation of a quantum mechanical propagator in a space with an extra dimension, where the proper-time plays the role of the usual time.

The so called ζ-function method, which was firstly applied by A. Salam and J. Strathdee[3], J. S. Dowker and R. Critchley[4] and S. Hawking[5], consists in the following steps: one constructs the "generalized ζ- function" associated with some operator $\mathcal{M}$ ($\det \mathcal{M}$ is the desired determinant), that is, $\zeta(s,\mathcal{M}) = \sum_n \lambda_n^{-s}$, where $\mathcal{M}\phi_n = \lambda_n \phi_n$. Then, one makes an analytical extension of this series to a meromorphic function in the whole complex plane, and finally one computes $\det \mathcal{M} = \exp[-\frac{d}{ds}\zeta(0,\mathcal{M})]$. When the eigenvalues are known, this is an interesting way of evaluating one-loop effective actions. For instance, for QED with constant fields this procedure can be found in ref(s)**[1,6-8]**. If one does not want to use the eigenvalues (or does not know them), an alternative procedure can be made, and the ζ-function can be computed through a Mellin transformation from the corresponding Heat Kernel (see for instance ref(s)**[3,4,9]**).

Finally, we would like to make some comments about the Green function method, which is the approach we shall be concerned with in this paper. In this approach, the one-loop effective action is written in terms of the appropriate Green function in the presence of an external field. So, we could say that, among all methods, this one is the technique that best establishes the connection between QFT calculations (second quantization) and

the old potential theory (first quantization). Some details are given in the next section (this technique can be found in ref.(s) [1,10-12].

As a final remark about the Green function method, we should say that it is very convenient for solving two-dimensional models (remember that in d=2 the whole contribution to the effective action comes from the one-loop term), since for these cases the Green function can be exactly computed†. For the Schwinger and Thirring models these type of calculations are discussed by H.M. Fried[10], while the Axial model was discussed by Rothe and Stamatescu[15]. Recently, the Generalized Schwinger model was solved through this technique by S. A. Alves and C. A. Linhares[16].

The purpose of this paper is to apply the Green function method for non-relativistic examples trying to keep as much as possible a procedure analogous to the relativistic case. In the next section we establish the basic formulas to be used later in section **3** where some examples will be worked out explicitly.

2. Basic formulas and notation

Consider, for instance, the one-loop effective action $W^{(1)}(A)$ for massless QED. It can be shown that

$$iW^{(1)}(A) = Tr\log\left[D(A)\right] - Tr\log\left[\gamma^\mu\partial_\mu\right], \tag{1}$$

where $D(A) \equiv \gamma^\mu\left(\partial^x_\mu + ieA_\mu\right)$ and Tr means trace on both matrix indices and configuration space variables.

Taking the derivative with respect to the charge e in both sides of (1) we get

$$\frac{\partial W^{(1)}}{\partial e}(A) = tr\int d^4x G(x,x;e|A)\gamma^\mu A_\mu(x), \tag{2}$$

† We should also mention that for two-dimensional models, the path integral approach based in the decoupling method introduced by R. Roskies and F. A. Schaposnik[13] has been largely used (see ref.[14] for a good review and examples).

where tr here means only matrix trace and the Green function $G(x,y;e|A)$ was defined such that

$$\left[\gamma^\mu\left(\partial^x_\mu + ieA_\mu\right)\right]G(x,y;e|A) = \delta^{(4)}(x-y). \tag{3}$$

Integrating eq. (2) we finally obtain

$$W^{(1)}(A) = tr\int_0^e de'\int d^4x G(x,x;e'|A)\gamma^\mu A_\mu(x), \tag{4}$$

where we used the condition that $W^{(1)}(A)$ must vanish for $A_\mu = 0$. Last equation establishes the already mentioned connection between QFT calculations and Green functions of the old potential theory.

An alternative way of deducing eq. (4), but now in the path integral framework, is the following: using that $\det\mathcal{M} = \exp Tr\log\mathcal{M}$, it follows from (1) that $W^{(1)}(A)$ can be cast into the form

$$\begin{aligned} e^{iW^{(1)}(A)} &= \frac{\det\left[D(A)\right]}{\det\left[\gamma^\mu\partial_\mu\right]} \\ &= \frac{\int D\psi D\bar\psi\exp\left\{\int\bar\psi\left[\gamma^\mu\left(\partial^x_\mu + ieA_\mu\right)\right]\psi d^4x\right\}}{\int D\psi D\bar\psi\exp\left\{\int\bar\psi\left[\gamma^\mu\partial^x_\mu\right]\psi d^4x\right\}}. \end{aligned} \tag{5}$$

Taking the logarithm of both sides of (5) and then the derivative with respect to e, we get

$$\begin{aligned} \frac{\partial W^{(1)}}{\partial e} &= tr\left\{\frac{\int D\psi D\bar\psi\left[\int d^4x(\psi\bar\psi)\gamma^\mu A_\mu(x)\right]\exp\left\{\int\bar\psi D(A)\psi d^4x\right\}}{\int D\psi D\bar\psi\exp\left[\int\bar\psi D(A)\psi d^4x\right]}\right\} \\ &= tr\left\{\int d^4x G(x,x;e|A)\gamma^\mu A_\mu(x)\right\}, \end{aligned} \tag{6}$$

where we used that $\bar\psi\gamma^\mu A_\mu\psi = tr[\psi\bar\psi\gamma^\mu A_\mu]$, and

$$\begin{aligned} G(x,y;e|A) &\equiv <\psi(x)\bar\psi(y)> \\ &= \frac{\int D\psi D\bar\psi[\psi(x)\bar\psi(y)]\exp\left\{i\int\bar\psi D(A)\psi d^4x\right\}}{\int D\psi D\bar\psi\exp\left\{i\int\bar\psi D(A)\psi d^4x\right\}}. \end{aligned} \tag{7}$$

Equation (6) coincides precisely with eq.(2), so that eq.(4) follows immediately as before.

Of course, eq. (4) is a formal expression for $W^{(1)}(A)$ due to the coincident points appearing in the Green function. In order to make eq.(4) useful some kind of regularization prescription must be adopted.

However, as we go to lower space-time dimensions (in some sense Quantum Mechanics can be thought as a QFT in 1 dimension), the ultraviolet behaviour gets better. As we will see in the next section, the corresponding Green function in our one-dimensional problems $G_\omega(t,t')$ will not diverge for $t = t'$.

Let us then see the analogue of eq.(4) for some non-relativistic problems. For instance, suppose we want to compute the following determinant

$$\exp[\Gamma(\omega)] = \det{}^s\left(\omega^2 + \partial_t^2\right) \equiv \det{}^s\left(L\right), \tag{8}$$

where L acts on functions that satisfy some specific boundary conditions. In the next section we shall make explicit examples; depending on the boundary conditions and the value of the power s, which can be thought as a "statistical" parameter, the physical meaning of this determinant will be different.

From eq.(8) we can write

$$\Gamma(\omega) = sTr\log\left(L\right), \tag{9}$$

so that

$$\begin{aligned}\frac{\partial}{\partial\omega}\Gamma(\omega) &= 2s\omega Tr\left(L^{-1}\right)\\ &= 2s\omega\int_{t_a}^{t_b} G_\omega(t,t)dt,\end{aligned} \tag{10}$$

where we defined the Green function through

$$\left(\omega^2 + \partial_t^2\right)G_\omega(t,t') = \delta(t-t'), \tag{11}$$

which must satisfy some specific boundary conditions on $t = t_a$ and $t = t_b$.

Integrating eq.(10), and adopting the normalization $\Gamma(0) = 0$ (which means that we are assuming that $\det\partial_t^2 = 1$, as we will see later), we obtain

$$\Gamma(\omega) = 2s\int_0^\omega d\omega'\omega'\int_{t_a}^{t_b} dt G_{\omega'}(t,t), \tag{12}$$

which is the analogue of eq.(4) we were looking for.

In the next section we shall use this last equation repeatedly in many physical examples.

3. Explicit examples

As our first example, we shall evaluate $\Gamma(\omega)$ using Dirichlet boundary conditions, that is,

$$G_\omega(t_a,t') = 0 = G_\omega(t_b,t'). \tag{13}$$

For these conditions and taking $s = -1/2$, $\Gamma(\omega)$ will coincide (apart from a constant independent of ω) with the pre-exponential factor of the Feynman propagator $K(x_b,x_a;\tau \equiv t_b - t_a)$ for the one-dimensional harmonic oscillator. This can be easily seen in the following way: taking the path integral expression for $K(x_b,x_a;\tau)$, and putting $x(t) = x_{cl}(t) + \eta(t)$, with $x_{cl}(t)$ being the classical solution and $\eta(t_a) = 0 = \eta(t_b)$, we get

$$K(x_b,x_a;\tau) = e^{iS_{cl}}\int_0^0 [D\eta]\, e^{-\frac{1}{2}\int_{t_a}^{t_b}\eta(t)(\omega^2+\partial_t^2)\eta(t)dt}, \tag{14}$$

where we used the classical equation of motion, made an integration by parts and the limits of integration mean that the Dirichlet boundary conditions are assumed.

Identifying the pre-exponential factor $F(\tau)$ as the remaining path integral in the r.h.s. of (14), we conclude that

$$\begin{aligned}F(\tau) &= \int_0^0 [D\eta]\, e^{-\frac{1}{2}\int_{t_a}^{t_b}\eta(t)(\omega^2+\partial_t^2)\eta(t)dt}\\ &\propto \det{}^{-\frac{1}{2}}(\omega^2+\partial_t^2)|_{Dirich.}\\ &= \exp\left[\Gamma(\omega)\right]|_{Dirich.}.\end{aligned} \tag{15}$$

It is shown in the Appendix A that the Green function that satisfies eq.(11) with the Dirichlet boundary conditions (13) is given by

$$G_\omega(t,t') = -\frac{1}{\omega\sin(\omega\tau)}\sin\omega(t_< - t_a).\sin\omega(t_b - t_>), \tag{16}$$

where $t_>$ and $t_<$ mean respectively the greatest and smallest value between t and t'.

Taking $t = t'$ and using that $\sin a.\sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$, we obtain

$$G_\omega(t,t) = -\frac{1}{2\omega}\left[\frac{\cos\omega\tau - \cos\omega(2t - t_a - t_b)}{\sin\omega\tau}\right], \tag{17}$$

a finite expression as we had already anticipated. Inserting (17) into (12) (with $s = -1/2$), we get

$$\begin{aligned}
\exp[\Gamma(\omega)] &= \exp\left\{-\frac{1}{2}\int_{t_a}^{t_b}\int_0^\omega d\omega'\left[\frac{\cos\omega'\tau}{\sin\omega'\tau} - \frac{\cos\omega'(2t - t_a - t_b)}{\sin\omega'\tau}\right]\right\} \\
&= \lim_{\epsilon\longrightarrow 0}\exp\left\{-\frac{1}{2}\int_{t_a}^{t_b} dt\left[\frac{\log(\sin\omega\tau) - \log\epsilon}{\tau}\right]\right\}\times \\
&\times \lim_{\epsilon\longrightarrow 0}\exp\left\{\frac{1}{2}\int_\epsilon^\omega \frac{d\omega'}{\sin(\omega'\tau)}\int_{t_a}^{t_b}\cos\omega'(2t - t_a - t_b)dt\right\} \\
&= \frac{1}{\sqrt{\sin(\omega\tau)}}\times\lim_{\epsilon\longrightarrow 0}\exp\left(\frac{1}{2}\int_\epsilon^\omega\frac{d\omega'}{\omega'}\right) \\
&= \sqrt{\frac{\omega}{\sin(\omega\tau)}},
\end{aligned} \tag{18}$$

which, apart from a normalization constant, coincides with the well known result of the pre-exponential factor for the propagator of a harmonic oscillator.

As our second example, we shall compute $\exp[\Gamma(\omega)]$ using periodic boundary conditions. In this case this quantity will coincide with the partition function of the (bosonic, $s = -1/2$) harmonic oscillator, after we make a Wick rotation to the Euclidean space and identify $\tau = -i\hbar\beta$, with $\beta = \frac{1}{KT}$. This can be seen as follows: the partition function is defined as

$$Z(\beta) = Tre^{-\beta\mathcal{H}} = \int dx < x|e^{-\beta\mathcal{H}}|x > . \tag{19}$$

However, $< x|e^{-\beta\mathcal{H}}|x >$ can be seen as the Feynman propagator $K(x_b, x_a; \tau)$ if we identify $x_b = x_a = x$ and $\tau = -i\hbar\beta$, so that $Z(\beta)$ will admit the following path integral representation

$$\begin{aligned}
Z(\beta) &= \int_{x(\beta)=x(0)}[Dx]e^{-\frac{1}{2}\int_0^\beta x(t)(\omega^2 - \partial_t^2)x(t)dt} \\
&\propto \det{}^{-\frac{1}{2}}(\omega^2 - \partial_t^2)|_{period.}.
\end{aligned} \tag{20}$$

However, instead of working in the Euclidean space, we shall compute $\det^{-\frac{1}{2}}(\omega^2 + \partial_t^2)$ with periodic conditions and only at the end of the calculations we shall put $\tau = -i\beta$ ($\hbar = 1$).

The Green function that satisfies eq.(11) with the periodic boundary condition

$$G_\omega^p(t+\tau, t') = G_\omega^p(t, t'), \tag{21}$$

is constructed in the Appendix B, and it is given by

$$G_\omega^p(t,t') = \frac{1}{2\omega}\frac{\cos[\omega(t - t' - \frac{\tau}{2})]}{\sin(\omega\frac{\tau}{2})} \quad ; \quad t - t'\epsilon\,[0,\tau). \tag{22}$$

Outside the interval $[0,\tau)$, the result is obtained by periodicity.

Inserting (22) into (12), we have

$$\begin{aligned}
\exp[\Gamma(\omega)]\,|_{periodic} &= \exp\left\{-\frac{1}{2}\int_{t_a}^{t_b}dt\int_0^\omega\frac{\cos(\omega'\tau/2)}{\sin(\omega'\tau/2)}d\omega'\right\} \\
&= \exp\left[-\log\sin(\frac{\omega\tau}{2})\right] \\
&\propto \frac{1}{\sinh(\frac{\omega\beta}{2})},
\end{aligned} \tag{23}$$

where we made the identification $\tau = -i\beta$ ($\hbar = 1$). Hence, apart from a numerical normalization factor, we have that $Z^{Bosonic}(\beta) = \frac{1}{\sinh(\frac{\omega\beta}{2})}$.

As our next example, we shall compute $\exp[\Gamma(\omega)]$ with anti-periodic conditions. In this case, after making the identification $\tau = -i\beta$, this quantity can be thought as the partition function of the fermionic harmonic oscillator[17].

The Green function that satisfies (11) with the anti-periodic boundary conditions

$$G^a_\omega(t+\tau,t') = -G^a_\omega(t,t') \tag{24}$$

is given by (its construction is given in Appendix B)

$$G^a_\omega(t,t') = \frac{1}{2\omega}\frac{\sin[\omega(t-t'-\frac{\tau}{2})]}{\cos(\frac{\omega\tau}{2})} \quad ; \quad t-t' \epsilon\, [0,\tau) \tag{25}$$

In analogy with the previous case, outside the interval $[0,\tau)$ the result is obtained by anti-periodicity.

Inserting (25) into (12), and following steps completely analogous to the previous case, we obtain ($s=+1$)

$$\begin{aligned}\exp[\Gamma(\omega)]\,|_{Anti} &= \exp\left\{\int_{t_a}^{t_b} dt \int_0^\omega d\omega' \frac{\sin(\omega'\tau/2)}{\cos(\omega'\tau/2)}\right\} \\ &\propto \cosh^2(\frac{\omega\beta}{2}),\end{aligned} \tag{26}$$

where we made the substitution $\tau = -i\beta$, so that, apart from a numerical normalization constant we have that $Z^{Fermions}(\beta) = \cosh^2(\frac{\omega\beta}{2})$. Note that this result differs from the one given by Gibbons[17] in the quadratic power of $\cosh(\frac{\omega\beta}{2})$, once he considered an equivalent linear Grasmann oscillator opposed to the quadratic case discussed here.

In fact, the above two examples can be presented in an unified way starting from a generalized boundary condition

$$G^\theta_\omega(t+\tau,t') = e^{-i\theta}G^\theta_\omega(t,t') \tag{27}$$

which reproduce the previous periodic ($\theta=0$) and antiperiodic ($\theta=\pi$) cases. Note also that this condition of general periodicity may be related to particles which statistics interpolate bosons and fermions, *i. e.*, anyons [21].

Taking the constructed Green function satisfying Eq. (27) from Appendix B, we have:

$$\begin{aligned}\Gamma^\theta(\omega) &= 2s\int_{t_a}^{t_b} dt \int_0^\omega \exp\left\{\frac{e^{-i\theta/2}}{4\omega'}\left[\frac{e^{-i\omega'\tau/2}}{\sin(\frac{\omega'\tau+\theta}{2})} + \frac{e^{i\omega'\tau/2}}{\sin(\frac{\omega'\tau-\theta}{2})}\right]\right\} \\ &= \log\left\{\left[1-e^{-i(\theta+\omega\tau)}\right]\left[1-e^{-i(\theta-\omega\tau)}\right]\right\}^s,\end{aligned} \tag{28}$$

up to a constant normalization factor independent of $\omega\tau$. It is straightforward to show that Eq. (28) reduces to its particular cases (23) and (26) when $\theta=0$ and $\theta=\pi$, respectively. Note that in this example, we left the statistics parameter s free. In fact, it must be a function of the periodicity parameter θ, interpolating between $s(\theta=0) = -1/2$ and $s(\theta=\pi) = +1$, as for example $s(\theta) = -1/2 + 3/2 f(\theta)$, where $f(\theta)$ may be any function which satisfies $f(0)=0$ and $f(\pi)=+1$. This factor should be obtained from the functional integration of a generalized variable, with arbitrary commutation relation, interpolating the cases of bosonic (c-number) and Grasmanian variables, as a kind of a q-deformed calculation [22].

As our next example, we shall evaluate $\det^{-\frac{1}{2}}(\omega^2+\partial_t^2)$, but without imposing any boundary condition, that is, we will make $t_a = -\frac{T}{2}$ and $t_b = +\frac{T}{2}$ and will take T so large (but finite) that the boundary conditions will become irrelevant. The corresponding Green function for this case reads (see Appendix C for a non-usual deduction using the proper-time method)

$$G(t,t') = \frac{\sqrt{\pi}}{2\omega}\, e^{-\omega|t-t'|}. \tag{29}$$

Hence, using (12) we get

$$\Gamma(\omega) = -\int_{-T/2}^{+T/2}\int_0^\omega d\omega'\omega'\frac{\sqrt{\pi}}{2\omega'} \propto \omega T, \tag{30}$$

as expected (this result is the very particular case of $-\frac{1}{2}\log\det(\partial^2+m^2)$ considered in only one dimension, where ω^2 plays the role of m^2).

As our last example, we shall discuss a kind of fermionic determinant in Quantum Mechanics, that mimics in some sense the fermionic determinant for the first order Dirac operator. This problem was discussed by R. Jackiw[18], who considered the following Lagrangian

$$L = a^\dagger(t)\left(i\frac{d}{dt} - eA(t)\right)a(t), \tag{31}$$

where $a(t)$ and $a^\dagger(t)$ satisfy an anti-commutation relation. This Lagrangian is invariant

under the gauge transformations

$$\delta a^\dagger = i\omega(t) \quad ; \quad \delta a = -i\omega(t) \quad ; \quad \delta A((t) = -\frac{1}{e}\dot{\omega}(t). \tag{32}$$

Here, using the Green function approach we shall compute the following fermionic determinant

$$\begin{aligned} e^{i\Gamma(A)} &\equiv \det\left(i\frac{d}{dt} - eA(t)\right) \\ &= expTr\log\left(i\frac{d}{dt} - eA(t)\right), \end{aligned} \tag{33}$$

but without imposing any boundary conditions (a ζ-function computation for the case of periodic and anti-periodic boundary conditions can be found in ref.[18]).

As before, deriving with respect to e the logarithm of both sides of eq.(33), we get

$$\frac{\partial}{\partial e}[i\Gamma(A)] = -\int_{\infty}^{+\infty} dt\; G(t,t;eA)A(t)\;, \tag{34}$$

where we defined the Green function

$$\left(i\frac{d}{dt} - eA(t)\right) G(t,t';eA) = \delta(t-t'). \tag{35}$$

Hence, we have (this is the analogue of (4))

$$i\Gamma(A) = -\int_0^e de' \int_{-\infty}^{\infty} dt\; G(t,t;eA)A(t). \tag{36}$$

The computation of $\Gamma(A)$ depends, basically, on the evaluation of the complete Green function $G(t,t';eA)$. A convenient procedure is to consider its integral equation,

$$\begin{aligned} G(t,t';eA) &=< t|\frac{1}{(I+ieGA)}|t'> \\ &= G(t-t') - ie\int_{-\infty}^{\infty} G(t-t')A(t')G(t'-t)dt' + ..., \end{aligned} \tag{37}$$

where we defined the free Green function†

$$i\frac{d}{dt}G(t,t') = \delta(t-t') \quad \longrightarrow \quad G(t-t') = -i\theta(t-t') \tag{38}$$

† The expression for $G(t-t')$ could have been anticipated from the definition $< 0|T\left(a(t)a^\dagger(t')\right)|0>$ and the fact that for this problem there are only two states, the zero fermion state and the one fermion state.

Substituting expansion (37) into (36), and using (38) (as well as the assumption $\theta(0) = \frac{1}{2}$), we get

$$\det\left(i\frac{d}{dt} - eA(t)\right) = e^{i\Gamma(A)} = \exp\left\{\frac{i}{2}e\int_{-\infty}^{\infty} dtA(t)\right\}, \tag{39}$$

in agreement with R. Jackiw[18].

4. Final Remarks

We have shown how to use the Green function approach for computing determinants that usually appear in non-relativistic problems. Since the corresponding Green functions can be evaluated without a great effort, this method seems to be a convenient one.

However, the correct answer for $\Gamma(\omega)$ is obtained only up to an additive constant of integration which is independent of ω, but which can, in principle, depend on τ (or β depending on the example). In any case, the ω-dependence (ω characterizes the interaction for these cases) is completely obtained by this method.

Another interesting remark is to establish a connection with an unusual way of using the ζ-function. Observing that $\zeta(s,L) = Tr(L^{-s})$, we can write

$$\begin{aligned} \int_{t_a}^{t_b} dtG_\omega(t,t) &= Tr(L^{-1}) \\ &= \zeta(s=1,L), \end{aligned}$$

so that

$$\frac{\partial}{\partial\omega}\Gamma(\omega) = -\omega\zeta(s=1,L).$$

The ζ-function associated to the operator $(\omega^2 + \partial_t^2)$ can be constructed explicitly and $\zeta(1,L)$ can be computed without any problem (see G.W.Gibbons[17] for explicit results). Of course the result must coincide with the one given here by the Green function approach.

As a final remark, we would like to observe that the infinite present in $\det(\omega^2 + \partial_t^2)$ is factorizable. In fact, it is already contained in $\det(\partial_t^2)$, so that, $\frac{\det(\omega^2+\partial_t^2)}{\det(\partial_t^2)}$ is finite (the

same can not be said in four dimensional QED; for instance, even for constant $F_{\mu\nu}$, the ratio $\frac{\det[\gamma^\mu(\partial_\mu+ieA_\mu)]}{\det(\gamma^\mu\partial_\mu)}$ is still divergent and needs a renormalization procedure. In our approach, this infinite was absorbed when we made the normalization $\Gamma(\omega=0)=0$ $(\det(\partial_t^2)=1)$.

One of the authors (C.F.) would like to thank to M.Asorey and A.J.Seguí- Santonja for enlightening discussions and to the Theoretical Physics Department of the University of Zaragoza for hospitality during his stay in Zaragoza, where part of this work was done. This work was partially supported by MEC-CAICYT and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

Appendix A

In this Appendix we shall briefly construct the Green function $G_\omega(t,t')$ that satisfies

$$(\omega^2+\partial_t^2)G_\omega(t,t')=\delta(t-t') \qquad (A.1)$$

as well as the Dirichlet boundary conditions $G_\omega(t_a,t')=0=G_\omega(t_b,t')$. Starting with $t\neq t'$, it is easy to show that

$$G_\omega(t,t')=A(t')\sin\omega(t_b-t)\ ;\ for\ t>t'$$
$$G_\omega(t,t')=B(t')\sin\omega(t-t_a)\ ;\ for\ t<t' \qquad (A.2)$$

Using symmetry arguments $(G_\omega(t,t')=G_\omega(t',t))$, and imposing continuity in $t=t'$, we get

$$\frac{A(t)}{\sin\omega(t-t_a)}=\frac{B(t')}{\sin\omega(t_b-t')}=\lambda=const., \qquad (A.3)$$

so that $G_\omega(t,t')$ takes the form

$$G_\omega(t,t')=\lambda\sin\omega(t_<-t_a).\sin\omega(t_b-t_>), \qquad (A.4)$$

where $t_>$, $t_<$ are respectively the greatest and smallest values of t and t'.

In order to compute λ, we shall use the discontinuity of $\frac{\partial}{\partial t}G_\omega(t,t')$ in $t=t'$, that is, integrating both sides of (A.1) from $t'-\epsilon$ to $t'+\epsilon$ we have

$$\partial_t G_\omega|_{t'+\epsilon}-\partial_t G_\omega|_{t'-\epsilon}=1. \qquad (A.5)$$

Substituting in (A.5) the corresponding solutions for $t>t'$ and $t<t'$, we obtain

$$\lambda=-\frac{1}{\omega\sin(\omega\tau)}\ ;\ \tau\equiv t_b-t_a, \qquad (A.6)$$

so that

$$G_\omega(t,t')=-\frac{1}{\omega\sin(\omega\tau)}\sin\omega(t_<-t_a).\sin\omega(t_b-t_>). \qquad (A.7)$$

Appendix B

In this Appendix we shall construct the Green function associated with the same differential operator $(\omega^2+\partial_t^2)$, but now using generalized periodic boundary conditions

$$G_\omega^\theta(t,t')=G_\omega^\theta(t-t')=e^{i\theta}G_\omega^\theta(t-t'+\tau), \qquad (B.1)$$

where $G_\omega^\theta(t,t')$ is the desired Green function, which satisfies

$$G_\omega^\theta(t,t')=(\omega^2+\partial_t^2)^{-1}\delta(t-t')\ ;\ t-t'\epsilon\,[t_a,t_b). \qquad (B.2)$$

It is easy to see that this reduce to the periodic and anti-periodic cases $G_\omega(t_a,t')=G_\omega(t,t')$ and $G_\omega(t_a,t')=-G_\omega(t,t')$, for $\theta=0$ and $\theta=\pi$, respectively. The following discussion is very similar to that found in H.Kleinert[19], when restricted to these particular cases, so that we shall omit non-important mathematical steps.

The spectral representation for $G_\omega^\theta(t)$ is then given by

$$\begin{aligned}G_\omega^\theta(t)&=\frac{1}{\tau}\sum_{m=-\infty}^{\infty}\frac{e^{-i\omega_m t}}{\omega^2-\omega_m^2}\\&=\frac{1}{\tau}\sum_{m=-\infty}^{\infty}\left\{\frac{e^{-i\omega_m t}}{2i\omega}\left[\frac{i}{\omega-\omega_m}+\frac{i}{\omega+\omega_m}\right]\right\}\\&=\frac{1}{2\omega i}\left\{G_-^\theta(t)+G_+^\theta(t)\right\},\end{aligned} \qquad (B.3)$$

where $\omega_m = (2\pi m + \theta)/\tau$ and we identified the spectral representation of $G^\theta_\pm(t)$, which are respectively the Green functions associated to the first order operators $(i\partial_t \pm \omega)$ with generalized periodic boundary conditions $(G^\theta_\pm(t+\tau) = e^{-i\theta}G^\theta_\pm(t))$.

Using the Poisson summation formula

$$\sum_{m=-\infty}^{\infty} f(m) = \int_{-\infty}^{\infty} d\mu \sum_{n=-\infty}^{\infty} e^{2\pi i\mu n} f(\mu), \tag{B.4}$$

we can write for $G^\theta_\pm(t)$

$$\begin{aligned} G^\theta_\pm(t) &= \frac{i}{\tau}\sum_{m=-\infty}^{\infty} \frac{e^{-i\omega_m t}}{\omega \pm \omega_m} \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i(\omega' t - n\omega'\tau + n\theta)} \left(\frac{i}{\omega \pm \omega'}\right). \end{aligned} \tag{B.5}$$

Identifying

$$G_\pm(t) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i\omega' t}\left(\frac{i}{\omega \pm \omega'}\right) \tag{B.6}$$

as the Green functions associated with the first order operators $(i\partial_t \pm \omega)$, but this time valid for an infinite time interval, we may cast (B.5) into the form

$$G^\theta_\pm(t) = \sum_{n=-\infty}^{\infty} e^{-in\theta} G_\pm(t - n\tau). \tag{B.7}$$

Hence, to obtain $G^\theta_\pm(t)$, we need first to compute $G_\pm(t)$. By residue calculations this can be made after setting $\omega \to \omega - i\eta$. With this prescription, it can be easily shown that

$$G_\pm(t) = -\theta(\mp t)e^{\pm i\omega t}. \tag{B.8}$$

Substituting (B.8) into (B.7), we get

$$G^\theta_\pm(t) = -\sum_{n=-\infty}^{\infty} e^{-in\theta}\theta(\mp t \pm n\tau)e^{\pm i\omega(t-n\tau)}. \tag{B.9}$$

Since this expression has a period τ, we can restrict ourselves to the interval $t\epsilon[0,\tau)$. Hence, the sum appearing in the r.h.s. of (B.9) can be easily done, yielding for $G^\theta_+(t)$

$$\begin{aligned} G^\theta_+(t) &= -e^{i\omega t}\sum_{n=1}^{+\infty} e^{-in(\omega\tau+\theta)} \\ &= -e^{i\omega t}\left\{e^{-i(\omega\tau+\theta)} + e^{-2i(\omega\tau+\theta)} + \ldots\right\} \\ &= -\frac{e^{-i\omega t}e^{-i(\omega\tau+\theta)}}{1 - e^{-i(\omega\tau+\theta)}}. \end{aligned} \tag{B.10a}$$

Analogously, for $G^\theta_-(t)$ we have

$$\begin{aligned} G^\theta_-(t) &= -e^{i\omega t}\sum_{n=0}^{-\infty} e^{-in(\omega\tau+\theta)} \\ &= -e^{i\omega t}\left\{1 + e^{-i(\omega\tau-\theta)} + e^{-2i(\omega\tau-\theta)} + \ldots\right\} \\ &= -\frac{e^{-i\omega t}}{1 - e^{-i(\omega\tau-\theta)}}. \end{aligned} \tag{B.10b}$$

Of course, outside this interval, the result can be obtained by periodicity. Substituting (B.10a,b) into (B.3) and substituting t by $t - t'$, we finally obtain

$$G^\theta_\omega(t-t') = \frac{e^{-i\theta/2}}{4\omega}\left[\frac{e^{i\omega(t-t'-\tau/2)}}{\sin(\frac{\omega\tau+\theta}{2})} + \frac{e^{-i\omega(t-t'-\tau/2)}}{\sin(\frac{\omega\tau-\theta}{2})}\right] \;;\; t - t' \,\epsilon\, [0,\tau). \tag{B.11}$$

For the particular case of periodic boundary condition $(\theta = 0)$ we have

$$G^{\theta=0}_\omega(t-t') \equiv G^p_\omega(t-t') = \frac{1}{2\omega}\frac{\cos[\omega(t-t'-\tau/2)]}{\sin(\omega\tau/2)} \;;\; t - t' \,\epsilon\, [0,\tau) \tag{B.12}$$

and for the anti-periodic boundary conditions $(\theta = \pi)$

$$G^{\theta=\pi}_\omega(t-t') \equiv G^a_\omega(t-t') = \frac{1}{2\omega}\frac{\sin[\omega(t-t'-\tau/2)]}{\cos(\omega\tau/2)} \;;\; t - t' \epsilon\, [0,\tau). \tag{B.13}$$

Appendix C

In this Appendix we shall construct the Green function that satisfies $(\omega^2 + \partial_t^2)G_\omega(t,t') = \delta(t-t')$, but which is valid for an infinite time interval. There are many standard ways of obtaining this result (by Fourier transform, etc.), but we shall present a non-usual one.

Defining the operator P_t so that $P_t = -i\partial_t$ in "coordinate representation", we can write $G_\omega(t,t')$ as the matrix element

$$\begin{aligned} G_\omega(t,t') &= < t|\frac{1}{\omega^2 - P_t^2}|t' > \\ &= i\int_0^\infty ds e^{-is(\omega^2 - i\varepsilon)} < t|e^{-iP_t^2 s}|t' > . \end{aligned} \qquad (C.1)$$

However, $< t|e^{-iP_t^2 s}|t' >$ is like a "free particle propagator", with $m = \frac{1}{2}$, and t and t' playing the role of x andx' respectively, so that

$$\begin{aligned} G_\omega(t,t') &= i\int_0^\infty ds e^{-is\omega^2}\frac{1}{\sqrt{4\pi i s}}e^{\frac{i(t-t')^2}{4s}} \\ &= \frac{1}{\sqrt{4\pi}}\int_0^\infty \frac{d\sigma}{\sqrt{\sigma}}\exp\left\{-\sigma\omega^2 - \frac{(t-t')^2}{4\sigma}\right\}, \end{aligned} \qquad (C.2)$$

where instead of using the prescription $\omega^2 \to \omega^2 - i\varepsilon$ we preferred to use the equivalent prescription $s = -i\sigma$.

Using the well known result for this integral[20] we obtain

$$G_\omega(t,t') = \frac{\sqrt{\pi}}{2\omega}e^{-\omega|t-t'|} \qquad (C.3)$$

References

1. Walter Dittrich and Martin Reuter, *Lecture Notes in Physics: Effective Lagrangians in Quantum Electrodynamics* (Springer-Verlag, Berlin, 1984), Vol.**220**.
2. J. Schwinger, Phys. Rev. **82**, 664 (1951). PHRVA
3. Abdus. Salam and J. Strathdee, Nucl.Phys. **B90**, 203 (1975) NUPHA
4. J. S. Dowker and Raymond Critchley, Phys. Rev. **D13**, 3224 (1976). PHRVA
5. S.W.Hawking, comm. Math. Phys. **55**, 133 (1977). CMPHA
6. S.K.Blau, M.Visser and A. Wipf, Int. J. Mod. Phys. **A6**, 5409 (1991). IMPAE
7. A.Basseto, L.Griguolo and R.Soldati, Phys. Rev. **D44**, 1343 (1991). PHRVA
8. A.Romeo, *New Aspects of Zeta Function Regularization Procedures with Incidence on QFT Vacuum Effects*, Ph.D. Thesis, Universitat de Barcelona (April, 1990).
9. M. Asorey, C. Farina and Antonio J. Seguí-Santonja, in preparation.
10. H.M.Fried, *Functional Methods and Models in Quantum Field Theory*, MIT Press, Cambridge (1972).
11. W. Dittrich, J. Phys. **A9**, 1171 (1976). JPAGB
12. R.J.Rivers, *Cambridge Monographs on Mathematical Physics: Path Integral Methods in Quantum Field Theory*, Cambridge University Press, Cambridge (1987).
13. R.Roskies and F.A.Schaposnik, Phys. Rev. **D23**, 558 (1981). PHRVA
14. R.E.Gamboa-Saravi, M.A.Muschietti, F.A.Schaposnik and J.E.Solomin, Ann. Phys. (N.Y.) **157**, 360 (1984). APNYA
15. K.Rothe and I.O.Stamatescu, Ann. Phys. (N.Y.) **95**, 202 (1975). APNYA
16. S.A.Dias and C.A.Linhares, Phys. Rev. **D45**, 2162 (1992). PHRVA
17. G.W.Gibbons, Phys. Lett. **A60**, 385 (1977). PHLTA

18. R.Jackiw, in *current Algebra and Anomalies: Topological Investigations of Quantized gauge Theories*, World Scientific Publishing, Singapore (1985).

19. Hagen Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics*, World Scientific Publishing, Singapore (1990).

20. I.S.Gradshteyn and I.M.Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).

21. F.Wilczek, *Fractional statistics and anyon superconductivity*, (ed.) World Scientific, Singapore, 1990.

22. J.L. Matheus-Valle and M.A.R. Monteiro, *Anyonic construction of the SL-q, s(2) algebra*, preprint CBPF - HEPTH 9311053.

NAME | I.D.No. | M.S.

RETURN TO FERMILAB LIBRARY