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M.A.Almeida, I.C.Moreira, M.E.Magalhães
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Transcendental Lie Symmetries and the Lotka -Volterra System

M.A.Almeida, I.C.Moreira*, M.E.Magalhães and O.M.Ritter
Instituto de Física, Caixa Postal 68528 .UFRJ.
21945-970 . Rio de Janeiro. Brazil.

*E-mail: IFT10018@UFRJ.BR

Abstract. In this paper we use the Lie symmetry method for finding rational or transcendental symmetry transfomations and invariants for the 3D Lotka-Volterra system.

Transcendental Lie Symmetries and the Lotka -Volterra System

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The identification of integrable systems and the study of the relationships between integrability and the symmetry structure of dynamical equations is an important problem in applied mathematics. One of the few systematic procedures for the analysis of this question is the Lie symmetry method. The practical implementation of this method depends on the supposition of a specific polynomial form for the symmetry vectors fields, as considered by several authors in the last years [1-5]. This kind of restriction conduces, in the general case, only to polynomial first integrals. Here we show that the Lie method can be used also for the obtention of rational and transcendental invariants for first-order differential equations. In this case we will find transcendental expressions for the symmety transformations. We take as examples the so-called Lotka-Volterra system in the cases with two and three dimensions.

The Lie method permits us the determination of the symmetry transformations of a set of differential equations [6]. By using these symmetries we can, in many cases, find first integrals in a straightforward fashion and identify integrable cases. The symmetry vector fields can be obtained from the invariance of the system of first-order differential equations

$$\Delta_i(x_j, x'_j) = 0 \qquad (1)$$

under infinitesimal transformations with the form

$$x_i \Rightarrow x_i + \varepsilon\eta_i(t, x_i)$$
$$t \Rightarrow t + \varepsilon\xi(t, x_i) \quad . \qquad (2)$$

We will take $\xi = 0$. The symmetry evolutionary vector field has the form

$$\mathbf{U} = \eta_i \partial_{x_i} . \qquad (3)$$

The Lie conditions for invariance of the system (1) are

$$pr^{(1)}\mathbf{U} \ (\Delta_i)\Big|_{\Delta_i=0} = \left[\eta_j \partial_{x_j} + (D_t \eta_j)\partial_{x_j}\right](\Delta_i)\Big|_{\Delta_i=0} = 0. \qquad (4)$$

If I is a time-dependent first integral for the system (1) then

$$D_t I = \partial_t I + \mathbf{X}(I) = 0 \qquad (5)$$

where X is the dynamical vector field.

A result that permits the obtention of first integrals, starting from the symmetries of the equations, is the following: given a set of first-order differential equations, if $\mathbf{U}_i$ is a symmetry vector field of Δ_i then $\mathbf{U}_2 = \mathrm{I}\mathbf{U}_1$ is also a symmetry vector field of Δ_i, if and only if I is a first integral of the system Δ_i. In particular, if a symmetry vector field $\mathbf{U}_1$ is not functionally independent of the dynamical vector field **X**,

$$\mathbf{U}_1 = F(t, x_i)\mathbf{X} \qquad (6)$$

then F is a first integral of the system.

We will look first for the symmetries of the 2D L-V system:

$$\begin{aligned} \dot{x} &= ax - xy \\ \dot{y} &= -by + xy \end{aligned} \qquad (7)$$

where a e b are constants.

If we apply the Lie conditions (4) to the system (7) we get the following equations to be solved:

$$\begin{aligned} (y-a)\eta_1 + x\eta_2 + \frac{\partial \eta_1}{\partial t} + (ax - xy)\frac{\partial \eta_1}{\partial x} + (xy - by)\frac{\partial \eta_1}{\partial y} &= 0 \\ -y\eta_1 + (b-x)\eta_2 + \frac{\partial \eta_2}{\partial t} + (ax - xy)\frac{\partial \eta_2}{\partial x} + (xy - by)\frac{\partial \eta_2}{\partial y} &= 0 \end{aligned} \qquad (8)$$

Solving this system, with the supposition that η_1 and η_2 are time-independent functions, we get, for the general case, from (8) and (3)

$$\begin{aligned} \mathbf{U}_1 &= (ax - xy)\partial_x + (xy - by)\partial_y \\ \mathbf{U}_2 &= x^b y^a e^{-(x+y)}\mathbf{U}_1 \end{aligned} \qquad (9)$$

The well-know first integral $\mathrm{I} = x^b y^a e^{-(x+y)}$ is found immediatly. We note that our results generalize the analysis of the symmetries of 2D L-V system made two years ago by Baumann and Freyberg [7]. They get the symmetry vector fields only for the case where a = - b. Their supposition of a polynomial dependence of the symmetry vector fields on the variables x and y establishes a restriction on the form of the symmetry vector fields they get. They considered also time-dependent symmetry transformations for this particular case. We observe that the Hamiltonian obtained in [7] is the logarithm of our invariant I.

We will take now some cases, which are particular cases of a more complete analysis [8], with transcendental symmetries transformations and invariants for the 3D L-V

system. A thorough examination of the first integrals for this system was made in [9] trough systematic application of the Painlevé analysis, the linear compatibily analysis method and the Jacobi Last Multiplier method.

The 3D Lotka-Volterra system is described by the following equations:

$$\dot{x} = x(cy + z + d_1)$$

$$\dot{y} = y(x + az + d_2) \qquad (10)$$

$$\dot{z} = z(bx + y + d_3)$$

We applied the Lie conditions (4) on these equations, with the ansätz that the symmetry vector fields can be time-dependent and have quadratic dependence on z. This restrition is only a partial one because we admit that they are general functions of x and y. The system of equations to be solved, in the general case, is very complicated. The calculations of the symmetry vector fields were made by algebraic computation; we found 19 cases where the system has non trivial symmetries [8].

We give here three of these particular cases:

a) $c = -\frac{1}{ab}$. From the equations (4) and (10) the following symmetry vector fields are found :

$$\mathbf{U}_1 = (d_1 x + xz - 1/(ab)xy)\partial_x + (xy + d_2 y + ayz)\partial_y + (zy + zd_3 + zbx)\partial_z$$

$$\mathbf{U}_2 = (zx^{ab} y^{-b} e^{-t(d_2 b + d_3 + d_1 ab)})\,\mathbf{U}_1 . \qquad (11)$$

The invariant is

$$I = zx^{ab} y^{-b} e^{-t(d_2 b + d_3 + d_1 ab)}. \qquad (12)$$

b) a=b=1, $d_1 = d_2$, $d_3 = 0$. We get

$$\mathbf{U}_1 = x(cy + z + d_1)\partial_x + y(x + z + d_1)\partial_y + z(x + y)\partial_z$$

$$\mathbf{U}_2 = (-d_1 \ln(|x|)) + d_1 c + d_1 \ln(|y|) + d_1 \ln(|c|) - x + (1/y)xz + cy - zc)\,\mathbf{U}_1 \qquad \mathbf{(13)}$$

$$\mathbf{U}_3 = ((1/(y(x^{(1/c)})))z(-cy + x)^{(1/c)} x + d_1 c + d_1 - (1/(x^{(1/c)}))zc(-cy + x)^{(1/c)})\,\mathbf{U}_1 ,$$

and the invariants

$$\mathrm{I}_1 = (-d_1 \ln(|x|)) + d_1 c + d_1 \ln(|y|) + d_1 \ln(|c|) - x + (1/y)xz + cy - zc) \qquad (14)$$

$$\mathrm{I}_2 = ((1/(y(x^{(1/c)})))z(-cy + x)^{(1/c)} x + d_1 c + d_1 - (1/(x^{(1/c)}))zc(-cy + x)^{(1/c)}). \qquad (15)$$

c) a=1, b=2, c=$-\frac{1}{b}$, $d_1 = d_2 = d_3$. The symmetry vector fields are

$$\mathbf{U}_1 = -(1/2)x(-2z + y - 2d_1)\partial_x + y(x + z + d_1)\partial_y + z(2x + y + d_1)\partial_z$$

$$\mathbf{U}_2 = ((y + 2x)(y^2 + 2xz - yz)/(2zx^2))\mathbf{U}_3$$

$$\mathbf{U}_3 = e^{-2d_1t}(z^2x^5/(y^3(y + 2x))\partial_x - (2z^2x^5/(y^3(y + 2x))\partial_y)$$

$$\begin{aligned}\mathbf{U}_4 = {} & e^{-d_1t}((1/4)x(y + 2x)(-6zyx + 4xz^2 + zy^2 - y^3)/y^2)\partial_x + \\ & +((1/2)(y + 2x)(6zx^2 + y^2x - 3zyx - zy^2 + z^2y)/y)\partial_y + \\ & +(1/2)z(-4xy^3 + zy^3 + 2zy^2x + 4zyx^2 - 4y^2x^2 - y^4 + 8zx^3)/y^2)\partial_z)\end{aligned}$$

$$\begin{aligned}\mathbf{U}_5 = {} & e^{-d_1t}((-1/2)x(-y + 2z)(2x - z + y)\partial_x - \\ & -y(z + x)(2x - z + y)\partial_y - z(y + 2x)(2x - z + y)\partial_z)\end{aligned}$$

$$\begin{aligned}\mathbf{U}_6 = {} & e^{-2d_1t}((1/2)y(y + 2x)(2x - z + y)^2/x)\partial_x - \\ & -(y(y + 2x)(-y + x)(2x - z + y)^2/x^2)\partial_y + (y^2(y + 2x)(2x - z + y)^2/x^2)\partial_z)\end{aligned}$$

$$\mathbf{U}_7 = (ze^{(-d_1t)}x^2/y^2)\mathbf{U}_1 \qquad \textbf{(16)}$$

$$\mathbf{U}_8 = (-e^{d_1t}/(2x - z + y))\mathbf{U}_5$$

$$\mathbf{U}_9 = e^{-d_1t}((x(y^2 + 4xz)/(4y))\partial_x - ((y^2x + 4zx^2 - zy^2 - 2zyx)/(2y))\partial_2 + (\frac{yz}{2} + xz)\partial_3)$$

$$\mathbf{U}_{10} = (-2e^{d_1t}/(2x - z + y))\mathbf{U}_6$$

$$\mathbf{U}_{11} = (e^{2d_1t}/(2x - z + y)^2)\mathbf{U}_6 .$$

The first integrals are

$$I_1 = e^{-d_1t}(2x - z + y) \qquad (17)$$

$$I_2 = (y + 2x)(y^2 + 2xz - yz)/(zx^2) \qquad (18)$$

$$I_3 = ze^{-d_1t}x^2/y^2. \qquad (19)$$

The purpose of this letter was to show, taking the 2D and the 3D Lotka-Volterra systems as examples, that the determination of rational or transcendental Lie symmetry

transformations can be useful for the identification of nonpolynomial first integrals of ordinary differential equations.

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