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A TRANSITION BETWEEN THE TENT MAP AND THE BERNOULLI SHIFT

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I. Introduction

One-dimensional maps of the interval are interesting from a mathematical viewpoint and are also important as simple models of dynamical systems in physics, chemistry and biology. The simplest example of a continuous "one-hump" map is the tent map

$$X_{i+1} = 2\mu X_i, \quad \text{if } 0 \le X_i \le 1/2$$

$$= 2\mu(1 - X_i), \quad \text{if } 1/2 < X_i \le 1,$$

introduced by Lorenz [1] as an approximation for the cusp-shaped Poincaré first return map in the strange attractor of his model. Another important and similar map, the Bernoulli shift

$$X_i = 2X_i \pmod 1,$$

is frequently discussed as a clear example of a fully developed chaotic discontinuous map [2]. If $\mu = 1$, these maps can be directly related and have the same Lyapounov exponent $\lambda = \ln(2)$.

Several examples of one-dimensional maps with discontinuities at the maximum have been analyzed in the last years, specially for small discontinuities [3]. Sousa Vieira, Lazo and Tsallis [4] studied the following modified form of the logistic map :

$$X_{i+1} = 1 - \varepsilon_1 - a_1|X_i|^z, \quad \text{if } X_i > 0$$

$$= 1 - (\varepsilon_1 + \varepsilon_2)/2, \quad \text{if } X_i = 0$$

$$= 1 - \varepsilon_2 - a_2|X_i|^z, \quad \text{if } X_i < 0,$$

and discussed the modifications in the transition from the regular to the chaotic domain.

Here we consider a one-dimensional discontinuous map (the TB-map) which can be seen as a transition between the tent map, with $\mu = 1$, and the Bernoulli shift. It is given by the following expression, cf. fig. 1:

$$X_{i+1} = 2X_i, \quad \text{if } 0 \le X_i \le 1/2$$

$$= (4b - 2)X_i + (2 - 3b), \quad \text{if } 1/2 < X_i \le 1.$$

The parameter b is a measure of the gap in the point $X_i = 1/2$. If b = 0 we get the tent map and, if b = 1, the Bernoulli shift. When b is changed from 0 to 1 the behavior of the map changes from chaotic to regular and to chaotic again. We will show that this map is a simple example of a system with two different attractors (for a determined range of values of the parameter b), one with period 1 and the other with period 2, with well defined basins of attraction. The transition of regular to chaotic behavior shows also some interesting features. We analyze first the asymptotic behavior of the iterated points and, in the following, we discuss its Lyapounov exponent.

II. The TB-map

The asymptotic behavior of the TB-map is shown in figure 2a, for the initial condition $X_o = 0.30$, and in figure 2b, for the initial condition $X_o = 0.45$. If $b < 1/4$, the map is chaotic. In this case the fixed points are $X_1^{*(1)} = 0$ and $X_2^{*(1)} = (3b-2)/(4b-3)$ and are both unstable. If $1/4 < b < 1/2$, the point $X_2^{*(1)}$ becomes stable and the map has an attractor with period 1. However, in the domain $3/8 < b < 1/2$, there are two different attractors: the previous one, with period 1, and a period 2 orbit. This can be confirmed if we examine the second order graph, X_{i+2} versus X_i, in this range, cf. fig. 3,

$$X_{i+2} = 4X_i, \quad \text{if } X_i \le 1/4$$

$$= (4b - 2)2X_i + (2 - 3b), \quad \text{if } 1/4 \le X_i \le 1/2$$

$$= (4b - 2)^2X_i + (4b - 2)(2 - 3b) + (2 - 3b), \quad \text{if } 1/2 \le X_i < 3/4$$

$$= 2[(4b - 2)X_i + (2 - 3b)], \quad \text{if } 3/4 \le X_i \le 1.$$

Note that, if $b > 1/2$, the last two domains are exchanged.

In the domain $3/8 < b < 1/2$ there are three stable fixed points for the second order map, given by

$$X_1^{*(2)} = (3b-2)/(8b-5); \quad X_2^{*(2)} = X_2^{*(1)} = (3b-2)/(4b-3);$$

$$X_3^{*(2)} = 2(3b-2)/(8b-5).$$

The basins of attraction of these attractors (regions I and II) are given in figure 4. The points on the boundary of these domains are $X = 1/2^n$ and $X = (1/2^n - 1/2^{n+2})$. We can understand this behavior if we observe that, for the initial conditions in the domain II, the iterated points go through a 2-cycle , with the sequence LR (left-right).

In the range $1/2 < b < 5/8$ there are no stable fixed points for X_{i+1} and X_{i+2} (there are unstable ones, for X_{i+2}, if $b > 3/4$); however, there is a period 3 attractor for all the initial conditions. The stable fixed points are

$$X_1^{*(3)} = (4b-1)A; \quad X_2^{*(3)} = (8b-3)A; \quad X_3^{*(3)} = (8b-2)A,$$

where

$$A = [3b-2]/[2(4b-2)^2 - 1].$$

When b increases, from $b = 5/8$ to $b \cong 0.666$, the slope of the right side of the TB-map remains lesser than 1 but there are no fixed points for X_{i+3} (except $X^*_{i+3} = 0$) and we have a multiperiodic behavior. The situation, when b is near 5/8, has some similarity with the tangent bifurcation for continuous maps. Two windows, with period 11 and 19, can be easily identified in the ranges $0.640 \lesssim b \lesssim 0.654$ and $0.663 \lesssim b \lesssim 0.665$, respectively. If $b \gtrsim 0.666$ the map has a chaotic behavior.

III. The Lyapounov exponent

By using the definition of Lyapounov exponent for one-dimensional maps [5], we obtain figures 5a and 5b. The first one corresponds to initial conditions in region I, and the second to initial conditions in region II. The only difference is the behavior in the range $3/8 < b < 1/2$. In the second case, for $b = 3/8$, there is a discontinuity in the Lyapounov exponent. It becomes positive when $b \gtrsim 0.666$.

We can get a reasonable approximation to these graphs by counting approximately the number of hits that the iterated points do with the left and the right side of the TB-map (fig. 1) and by using $\varepsilon_n = \varepsilon_o e^{n\lambda}$:

Region A ($0 \le b < 1/4$) :

For $b \cong 0$ the iterated points do approximately the same number of hits on the left side and on the right side of the TB-map (sequence LR). However, if b increases the numerical experiments show that the best approximation is given by the sequence LRRR. Then,

$$\varepsilon_n = (2)^{n/4} a^{3n/4} \varepsilon_o, \quad \text{where } a = (4b-2),$$

and $\lambda_A \cong \ln[2(4b-2)^3]/4$.

Region B ($1/4 < b < 3/8$) :

In this case, in the long time behavior, the points hit only the right side of the map. Therefore,

$$\lambda_B \cong \ln |4b-2|.$$

Region C ($3/8 < b < 1/2$) :

There are now two different expressions for λ :

$$\lambda_{C-I} = \ln |4b-2| \quad \text{and} \quad \lambda_{C-II} = [\ln |8b-4|]/2.$$

Region D ($1/2 < b < 5/8$):

In this case the sequence of iterated points is LRR :

$$\lambda_D = \{ \ln [2(4b-2)^2] \}/3.$$

Region E ($5/8 < b \lesssim 0.666$) :

The analysis of the window with period 11 shows that the iterated points hit four times the left side of the TB-map and seven times the right side. Taking this approximation for getting λ_E, in this range, we find

$$\lambda_e \cong \{ \ln [2^4(4b-2)^7]/11 \}.$$

This quantity vanishes when $b \cong 0.667$.

Region F ($0.666 \lesssim b < 1$) :

We take the approximation that the iterated points do the same number of hits on the left side and on the right side (sequence LR):

$$\lambda_F \cong [\ln |8b-4|]/2.$$

In figure 6a and 6b we compare the results of our approximation with the numerical results for λ.

We have no elements for saying that this map has some kind of generic behavior; however, we have observed that similar transitions between regular and chaotic behavior occur for several discontinuous maps constructed with straight lines.

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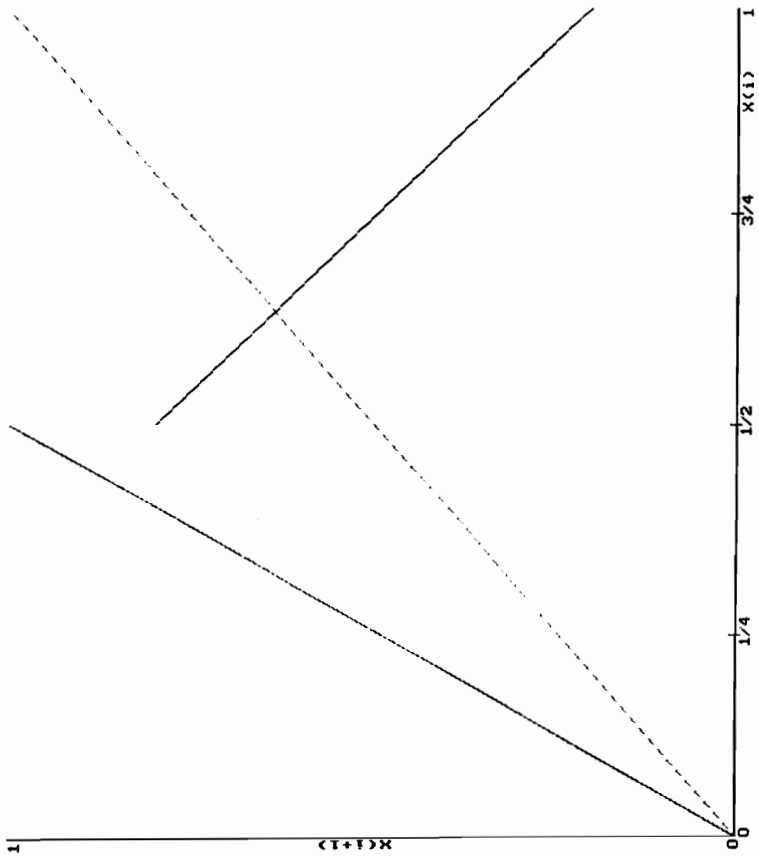
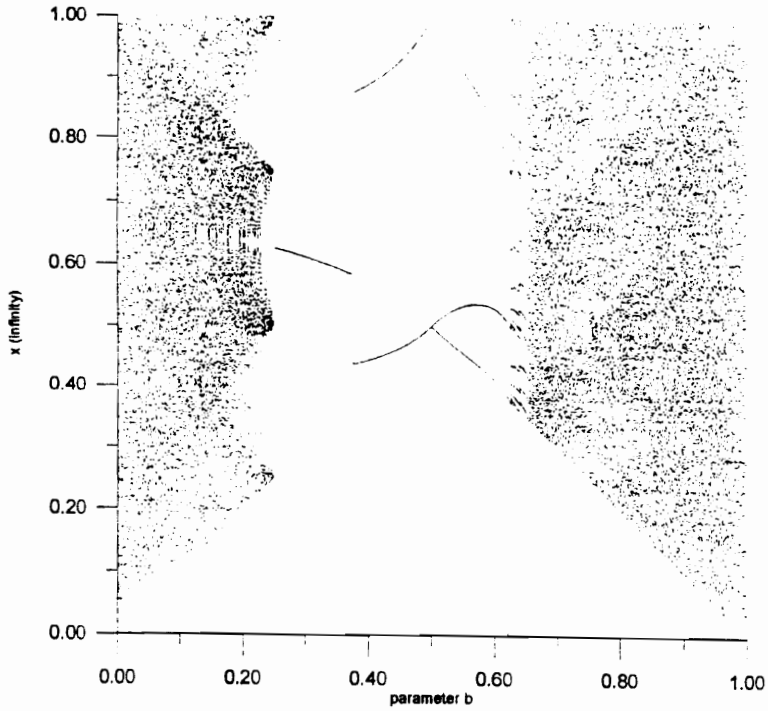


Fig 1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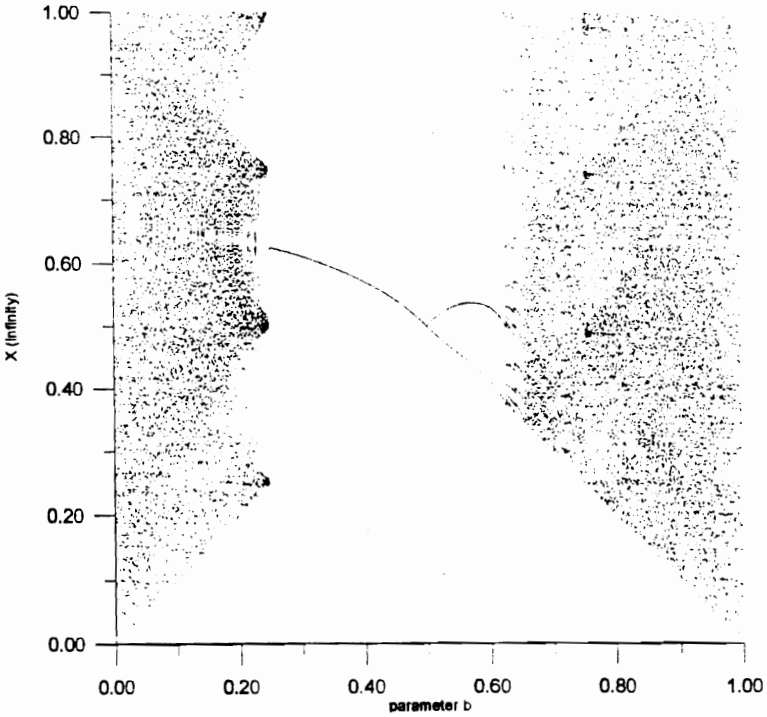
parameter b

Fig 2a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parameter b

Fig 2b

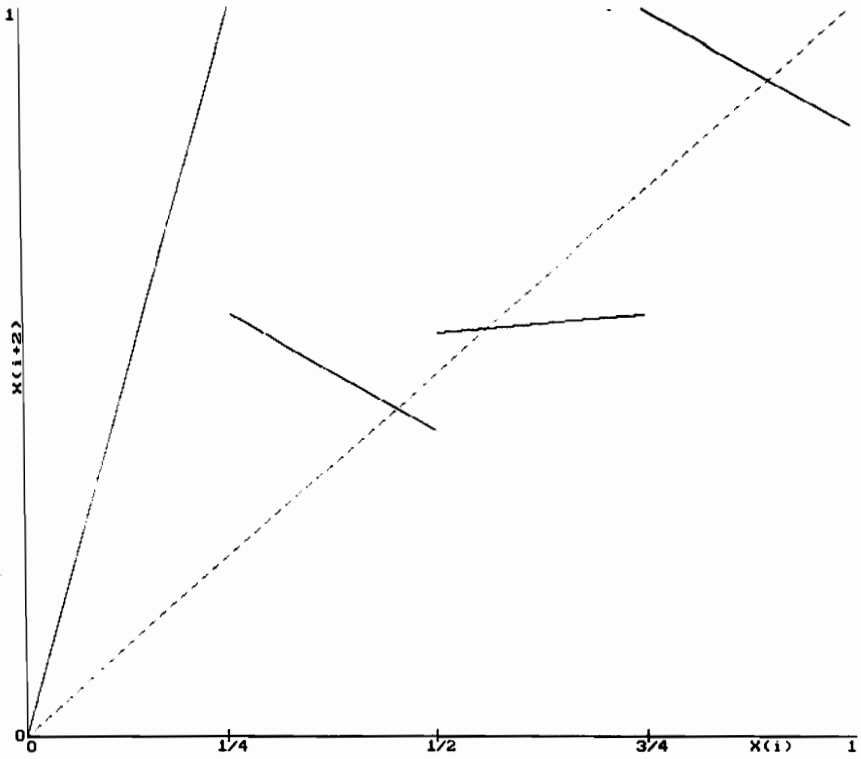
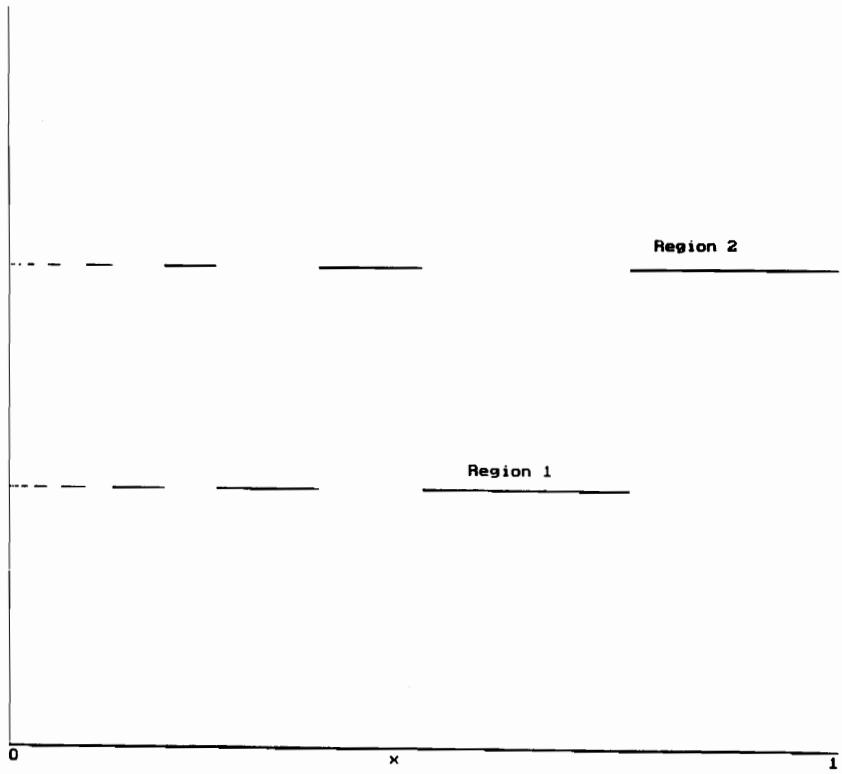


Fig 3



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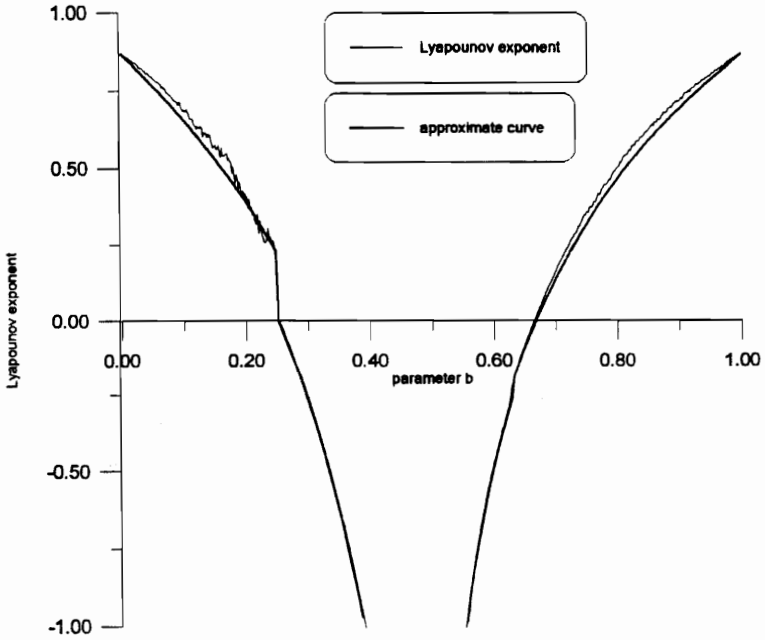
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-0.50
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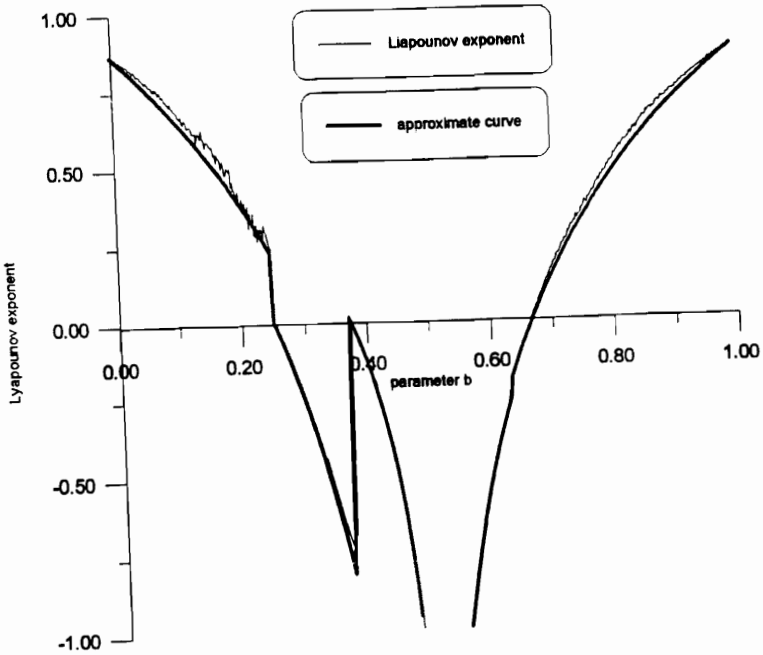
0.40
0.60
0.80
1.00
parameter b
Lyapounov exponent

Fig. 6a


Liapounov exponent
approximate curve
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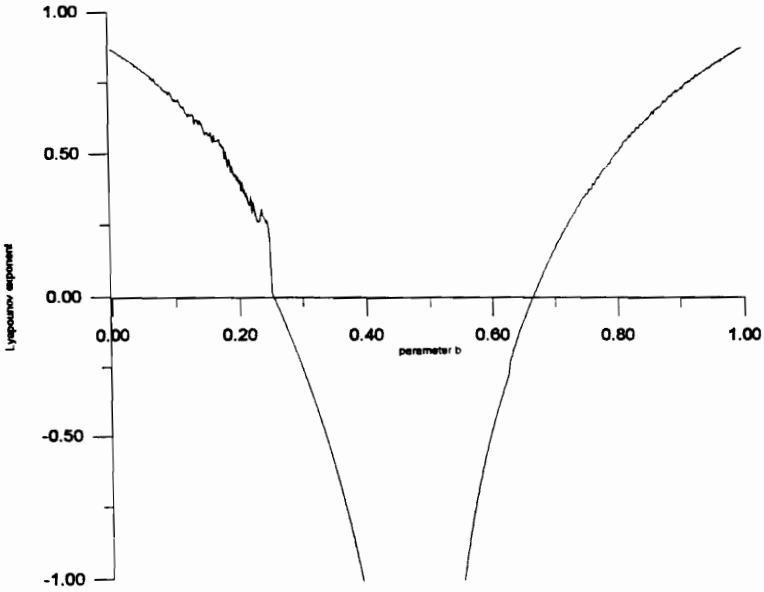
0.80
1.00
parameter b
Lyapounov exponent

Fig. 6b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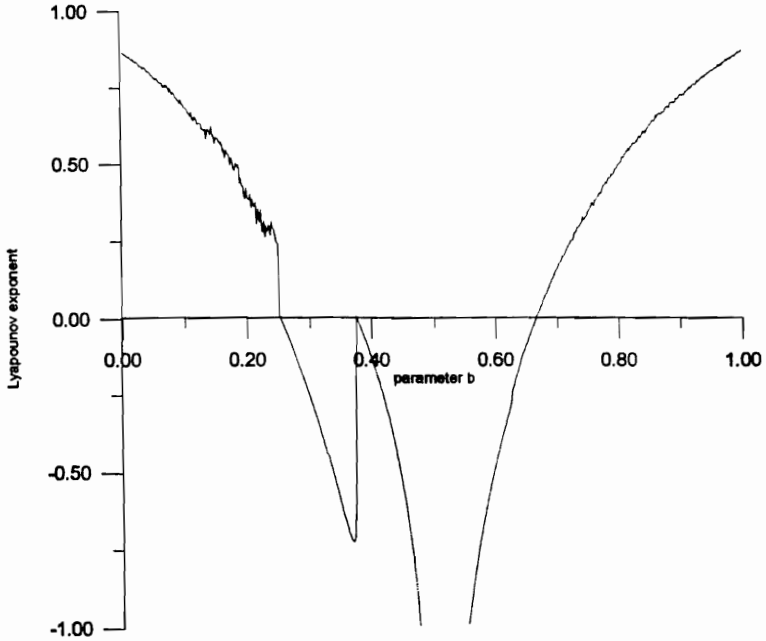
parameter b

Fig. 5a.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1.00
parameter b

Fig. 5b