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VACUUM POLARIZATION: A PROPER TIME PATH INTEGRAL APPROACH

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Abstract

We obtain the exact result for the vacuum polarization in scalar electrodynamics, to the first order α, in the presence of an external homogeneous electromagnetic field using a new method. We reduce a problem of QED to one of quantum mechanics which is treated by using the path integral approach.

I - Introdu

In Quantum Electrodynamics (QED) the matrix elements for physical processes are calculated pertubatively in terms of the electron and photon propagators. When we are interested in QED with external electromagnetic fields it is possible to calculate the modified propagators exactly for certain configurations of these fields.

For the cases of homogeneous and plane wave fields the electron propagator was calculated by Schwinger [1] and a slight modification of his results gives the spin zero propagator as well. If the external field is a homogeneous magnetic field these propagators have a momentum representation, which was used by Tsai and Erber [2] to calculate the lowest order vacuum polarization exactly. A similar procedure for the spin zero contribution is possible. For the case of homogeneous electromagnetic field Baier et al [3] used an operator technique to calculate the vacuum polarization tensor in scalar and spinor QED. All of the calculations cited are based on the use of the proper time method for the solution of the differential equation for charged particle propagator.

A different approach to scalar QED was given by Feynman [4] who wrote the Klein-Gordon equation with external electromagnetic field as a Schrödinger equation in a "five dimensional space of space-time coordinates and a "time" parameter which can be related to the proper time in Schwinger formulation. The physical states are a definite mass projection of the five dimensional states.

Then, one can introduce the charged particle propagator which may be given as a proper time path integral representation. For external homogeneous field one can calculate the propagator very simply [5].

Although not relevant for the calculations described above but relevant for this paper we mention the technique introduced by Schwinger [6] for the calculation of the mass operator of a spinless charged particle in an external homogeneous electromagnetic field. Replacing the integration of the photon momentum by an expectation value in "photon coordinate" states he related the mass operator to the matrix elements of proper time evolution operator. This led to an elegant calculation of the mass operator. On the other hand if the coordinate space matrix elements of the mass operator are considered we need the calculation of a propagator which may be given by a proper time path integral representation using the idea proposed by Feynman [4]. Such a calculation is possible [7].

In the present paper we consider the calculation, in order α, of the vacuum polarization tensor in external homogeneous field for the scalar QED. Using the technique of Schwinger [6], mentioned above, we replace the electron loop momentum integration by an expectation value which allows us to relate the vacuum polarization operator to an amplitude which obeys the differential equation for a "nonrelativistic propagator in a five dimensional space". Using a proper time path integral representation the amplitude can be calculated reducing the calculation of the vacuum polarization reduces to the evaluation of certain functional derivatives. The final result coincides with that of Baier et al [3].

This paper is organized as follows: In section II we formulate the problem and rewrite the vacuum polarization in the form of an expectation value. In section III we evaluate a related transformation function by the path integral technique. In section IV we calculate the vacuum polarization by making the functional derivatives of the amplitude evaluated in section III. Section V contains conclusions.

II - Formulation of the problem

When one is analyzing higher-order process in QED, one of the most important objects is the modification, to order α, of the photon propagator. In the scalar case this contribution, when the particle is interacting with an external electromagnetic field, is given by the vacuum polarization operator

$$\Pi^{(a)}_{\mu\nu}(k) = -ie^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{\pi^2-m^2} (2\pi-k)_\mu \frac{1}{(\pi-k)^2-m^2} (2\pi-k)_\nu, \qquad (1)$$

where

$$\pi_\mu = p_\mu - e\mathcal{A}^e_\mu(x). \qquad (2)$$

$\mathcal{A}^e_\mu(x)$ is the external vector potential which for a homogeneous electromagnetic field can be written in the form

$$\mathcal{A}^e_\mu(x) = 1/2\ F_{\mu\nu}\ (x-x'')^\nu, \tag{3}$$

where $F_{\mu\nu}$ is the field-strength tensor and x" is a point of reference.

When a particle is interacting with an external electromagnetic field it is possible to demonstrate that the bubble graphs as that of the figure 1 will contribute to guarantee the gauge invariance condition

$$k^\mu\ \Pi_{\mu\nu}(k) = 0. \tag{4}$$

This contribution is

$$\Pi^{(b)}_{\mu\nu} = 2ie^2\ g_{\mu\nu} \int \frac{d^4p}{(2\pi)^4}\ \frac{1}{\pi^2-m^2} \tag{5}$$

So, the photon propagator is modified by

$$\mathcal{D}^{(0)}_{\rho\tau} \quad \text{—} \quad \mathcal{D}^{(1)}_{\rho\tau} = \mathcal{D}^{(0)}_{\rho\mu}(k)\ \Pi^{\mu\nu}(k)\ \mathcal{D}^{(0)}_{\nu\tau}(k),$$

where $\mathcal{D}^{(0)}_{\rho\tau}(k)$ is the photon propagator in the free-field case and $\Pi_{\mu\nu}(k)$ is the sum of equations (1) and (5).

In equation (1) the integral over the momenta can be substituted by the expectation value of the integrand between position eigenstates

$$\int \frac{d^4p}{(2\pi)^4}\ f(p) = \langle x'=0|f(p)|x''=0\rangle, \tag{6}$$

since

$$\langle x|p\rangle = (2\pi)^{-4}\ e^{-ip.x}. \tag{7}$$

fig.1

Besides, we can express the propagators of the particle in a formal integral representations

$$\frac{1}{\pi^2-m^2} = -i \int_0^\infty ds_1\ \exp\left[is_1(\pi^2-m^2)\right] \tag{8.a}$$

$$\frac{1}{(\pi-k)^2-m^2} = -i \int_0^\infty ds_2\ \exp\left[is_2((\pi-k)^2-m^2)\right]. \tag{8.b}$$

Moreover, using the commutation relation

$$\left[\pi_\mu, \pi_\nu\right] = i\ e\ F_{\mu\nu} \tag{9}$$

and the fact that $F_{\mu\nu}$ is constant, we can prove that

$$\exp(i\pi^2 s_1)\ \pi_\mu\ \exp(-i\pi^2 s_1) = \left[\exp(2eFs_1)\ \pi\right]_\mu. \tag{10}$$

Then, substituting equations (10), (8) and (6) in equation (1) we get,

$$\Pi^{(a)}_{\mu\nu}(k) = ie^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \, \exp[-im^2(s_1+s_2)] \langle x'=0 | [2 \exp(2eFs_1)\pi - k]_\mu$$

$$\exp[i\pi^2 s_1] \exp[i(\pi-k)^2 s_2] \, (2\pi-k)_\nu | x''=0 \rangle. \qquad (11)$$

Since π^2 does not commute with $(\pi-k)^2$ we attempt to put the expression of the vacuum polarization in a convenient form. First we define the quantity

$$M(a) = \exp[-i\, a\, H_1\, s_1] \exp[-i\, a\, H_2\, s_2] \qquad (12)$$

where

$$H_1 = -\pi^2 \qquad (13.a)$$

$$H_2 = -(\pi-k)^2. \qquad (13.b)$$

Thus we have

$$i \frac{\partial M}{\partial a} = [s_1 H_1 + s_2 \exp(-i\, a\, H_1\, s_1)\, H_2 \exp(i\, a\, H_1\, s_1)]\, M(a). \qquad (14)$$

On the other hand it can be shown that

$$\exp[-i\, a\, H_1\, s_1]\, H_2 \exp[i\, a\, H_1\, s_1] =$$

$$= H_2 + \sum_{n=1}^{\infty} (-i\, a\, s_1)^n C_n / n!, \qquad (15)$$

where

$$C_1 = \left[H_1, H_2\right] = -\left[\pi^2, 2\pi.k\right] = -2\, k\, (-2\, i\, e\, F)\, \pi =$$

$$= 2\, k\, (2\, i\, e\, F)\, \pi; \qquad (16)$$

$$C_n = 2\, k\, (2i\, e\, F)^n\, \pi. \qquad (17)$$

Substituting equation (17) in (15) we get

$$\exp[-i\, a\, H_1\, s_1]\, H_2 \exp[i\, a\, H_1\, s_1] =$$

$$= H_2 + 2 \sum_{n=1}^{\infty} (-i\, a\, s_1)^n\, k\, (2i\, e\, F)^n\, \pi / n!. \qquad (18)$$

Going back to equation (14) we obtain

$$i \frac{\partial M}{\partial a} = \left\{ s_1 H_1 + s_2 H_2 + 2\, s_2\, k \left[\exp(2\, e\, a\, F\, s_1) - 1\right] \pi \right\} M(a) \qquad (19)$$

which after integrating gives

$$i \ln M(a) = (s_1 H_1 + s_2 H_2) a +$$

$$+ 2\, s_2\, k \left[\frac{\exp(2\, e\, a\, F\, s_1) - 1}{2\, e\, a\, F\, s_1} - a \right] \pi. \qquad (20)$$

Putting a=1 and substituting equations (13) in (20) we get

$$i \ln M(1) = -(s_1+s_2)\pi^2 - s_2 k^2 + 2 k s_2 \left[\frac{\exp(2 e F s_1) - 1}{2 e F s_1}\right] \pi \quad (21)$$

Thus

$$M(1) = \exp[i s_1 \pi^2] \exp[i s_2 (\pi-k)^2] =$$

$$= \exp\left[- i \int_0^s ds' \; H(s')\right] \exp\left[i \frac{(1+v)}{2} k^2 s\right], \quad (22)$$

where

$$H(s') = - \pi^2 - 2\pi . J(s'), \quad (23)$$

with

$$J(s') = - \frac{(1+v)}{2} \exp[-(1-v) e F s'] k. \quad (24)$$

Next using the transformation of variables

$$s_1 = s \frac{(1-v)}{2} \quad ; \quad s_2 = s \frac{(1+v)}{2}, \quad (25)$$

equation (11) can be rewritten as

$$\Pi^{(a)}_{\mu\nu}(k) = i e^2 \int_0^\infty \frac{s}{2} ds \int_{-1}^{+1} dv \; e^{-ims^2} \exp\left[i \frac{(1+v)}{2} k^2\right].$$

$$.\langle x'=0| \left[2 \exp[(1-v)eFs]\pi-k\right]_\mu \exp\left[-i \int H(s')ds'\right] (2\pi-k)_\nu |x''=0\rangle \quad (26)$$

where we used the commutation relations

$$\left[k_\mu, \pi_\nu\right] = \left[k_\mu, k_\nu\right] = 0. \quad (27)$$

III - A proper time path integral approach

In this section we calculate the matrix element

$$K(x',x'') = \langle x'| \exp\left[-i \int_0^s H(s') \, ds'\right] |x''\rangle = \langle x';s|x'';0\rangle, \quad (28)$$

which will be used later to calculate the vacuum polarization tensor.

It can be seen that the transformation function K(x',x") satisfies

$$(i \partial_s - H)K(x',x'') = i \, \delta(s) \, \delta(x'-x'') \qquad \text{for } s \geq 0, \quad (29)$$

where

$$\lim_{s \to 0^+} K(x',x'') = \delta(x'-x''). \quad (30)$$

Equation (29) is an inhomogeneous Schrödinger eqution in "five

dimensions" with s playing the role of "time". In this form the transformation function can be expressed in the path integral representation

$$K(x',x'') = \int \mathcal{D}^4x \, \exp\left[-i\int_0^s ds' \; L(s')\right], \tag{31}$$

where

$$L(s') = \left[\frac{\dot{x} - 2J}{2}\right]^2 + e\, \mathcal{A}^e.\dot{x} \tag{32}$$

is the Lagrangian with the classical sources J(s') given by (24) where k is a c-number now.

Looking at the Lagrangian we can see that it is a quadratic one. So, the path integral in (31) can be written as

$$K(x',x'') = N \exp\left[-i\, W_{cl}\right], \tag{33}$$

where N is a convenient normalization constant and W_{cl} is the classical action which is found through the solution of the Euler-Lagrange equation of motion

$$x - 2\, e\, F\, \dot{x} = 2\, \dot{J}, \tag{34}$$

subject to the boundary conditions

$$x(s) = x' \quad , \quad x(0) = x''. \tag{35}$$

Here we are using the matrix notation

$$\begin{aligned} x^{\mu} &= (x)_{\mu} & F^{\mu}{}_{\nu} &= (F)_{\mu\nu} \\ g_{\mu\nu} &= (g)_{\mu\nu} & g^{\mu\nu} &= (g^{-1})_{\mu\nu} \end{aligned} \tag{36}$$

which will also be used later on if convenient.

Integrating equation (34) we get

$$\dot{x}(s') + 2\, e\, F\, (x(s') - x') = 2\, J(s') + (\dot{x}(s) - 2\, J(s)). \tag{37}$$

Integrating once more we have

$$e^{2eFs'}(x(s') - x') + \Delta x = 2\int_0^{s'} ds''\; e^{2eFs''}\, J(s'') + \frac{e^{2eFs'} - 1}{2\, e\, F}\,(\dot{x}(s) - 2\, J(s)). \tag{38}$$

Making s'=s in the above equation we have

$$\frac{\dot{x}(s) - 2\, J(s)}{2} = \frac{e\, F}{e^{2eFs} - 1}\left[\Delta x - 2\int_0^s ds''\; e^{2eFs''}\, J(s'')\right], \tag{39}$$

Substitution of (38) and (39) in (37) leads us to

$$e^{2eFs'}\left[\frac{\dot{x}(s') - 2\, J(s')}{2}\right] = \frac{\dot{x}(s) - 2\, J(s)}{2} +$$

$$+ e F \left[\Delta x - 2 \int_0^{s'} ds'' \, e^{2eFs''} J(s'') \right]. \tag{40}$$

Thus, using the equation of motion (34) and the first integrals in equations (39) and (40) the classical action is given by

$$W_{cl} = 1/2 \, \Delta x^t \, g \, \frac{e F}{A} \, \Delta x - 2 \, \Delta x^t \, g \, \frac{e F}{A} \left\{ \int_0^s ds' \, e^{2eFs'} J(s') \right\} -$$

$$- 2 \left\{ \int_0^s ds' \, e^{2eFs'} J(s') \right\}^t g \, e F \, \Delta x +$$

$$+ 2 \left\{ \int_0^s ds' \, e^{2eFs'} J(s') \right\}^t g \, \frac{e F}{A} \left\{ \int_0^s ds' \, e^{2eFs'} J(s') \right\} +$$

$$+ 2 \left\{ \int_0^s ds' \, e^{2eFs'} J(s') \right\}^t g \, e F \left\{ \int_0^{s'} ds'' \, e^{2eFs''} J(s'') \right\}, \tag{41}$$

where $A = 1 - e^{-2eFs}$ and the symbol t designates the transpose.

It is easy to see that

$$W_{cl} \Big|_{x'=x''=0} = 2 \left\{ \int_0^s ds' \, e^{2eFs'} J(s') \right\}^t g \, \frac{e F}{A} \left\{ \int_0^s ds' \, e^{2eFs'} J(s') \right\} +$$

$$+ 2 \left\{ \int_0^s ds' \, e^{2eFs'} J(s') \right\}^t g \, e F \left\{ \int_0^{s'} ds'' \, e^{2eFs''} J(s'') \right\} =$$

$$= - \left\{ \int_0^s ds' \, e^{2eFs'} J(s') \right\}^t g \, \frac{[e^{(1+v)eFs} - 1]}{A} \, k -$$

$$- \left[\int_0^s ds' \left[e^{2eFs'} J(s') \right]^t g \, [e^{(1+v)eFs'} - 1] \, k \right. =$$

$$= k^t \, g \, \frac{[e^{-(1+v)eFs} - 1]}{A} \, \frac{(1+v)}{2} \int_0^s ds' \, e^{(1+v)eFs'} \, k +$$

$$+ \frac{(1+v)}{2} \, k^t \, g \int_0^s ds' \, e^{(1+v)eFs'} [e^{-(1+v)eFs'} - 1] \, k =$$

$$= \frac{(1+v)s}{2} \, k^t \, g \, k + k^t \, g \, \frac{[\cosh(v \, e \, F \, s) - \cosh(e \, F \, s)]}{2 \, e \, F \sinh(e \, F \, s)} \, k. \tag{42}$$

Now, we must determine the normalization constant N. For this we can take the Lagrangian (32) without the source term J(s') since it does not contribute to N.

$$L^0 = \frac{\dot{x}^2}{4} - \frac{1}{2} (x(s') - x'') \, e \, F \, \dot{x}. \tag{43}$$

Making the transformation of variables

$$x(s') - x'' = \Lambda(s') \, y(s'), \tag{44}$$

and imposing that the kinetic energy term stays invariant

$$\dot{x}^2 \text{ ——— } \dot{y}^2$$

and that there are no crossed terms, we obtain

$$\Lambda^t \; g \; \Lambda = g \tag{45.a}$$

$$\Lambda(s') = e^{-eFs}\Lambda(0). \tag{45.b}$$

So, we obtain the following transformed Lagrangian

$$L^0 = \frac{1}{4}\,\dot{y}^t \; g \; \dot{y} - \frac{1}{4}\, y^t \; g \; \Omega^2 \; y, \tag{46}$$

where the matrix

$$\Omega^2 = (\Lambda(0))^{-1} \; (e\;F)^2 \; \Lambda(0) \tag{47}$$

is diagonal.

Thus we are left with a harmonic oscillator Lagrangian with mass 1/2 and frequency matrix Ω. In this form the normalization constant is given by

$$N = \frac{i}{(4\pi)^4 \; s^2} \det \left[\frac{e\;F\;s}{\sinh(e\;F\;s)}\right]^{1/2}, \tag{48}$$

where we have taken into account the four-dimensional nature of the oscillator and the metric.

IV - Calculation of the Vacuum Polarization

The expectation value which appears in equation (26)

$$\langle x'=0| \left[2\,\exp[(1-v)eFs]\;\pi - k\right]_\mu \exp\left[-i\int_0^s H\,ds'\right] (2\pi-k)_\nu |x''=0\rangle, \tag{46}$$

can be decomposed in the form

$$\langle \left[2\,\exp[(1-v)\;e\;F\;s]\pi\right]_\mu e^{-i\;H\;s}\; 2\pi_\nu\rangle =$$

$$= \left[\exp[(1-v)\;e\;F\;s\right]_\mu^\sigma \langle e^{-iHs}\; 2\pi_\sigma(s)\; 2\pi_\nu(0)\rangle = \left[\exp[(1-v)eFs]\right]_\mu^\sigma .$$

$$\cdot \left\{ - \frac{\delta^2}{\delta J^\sigma(s)\;\delta J^\nu(0)} \langle x'|T\,\exp\left[-i\int_0^s ds'\;H(s')\right]|x''\rangle \right\}_{x'=x''=0} ; \tag{50.a}$$

$$- \langle \left[2\,\exp[(1-v)\;e\;F\;s]\;\pi\right]_\mu e^{-i\;H\;s}\; k_\nu\rangle = \left[\exp[(1-v)\;e\;F\;s]\right]_\mu^\sigma .$$

$$\cdot \left\{ i\,\frac{\delta}{\delta J^\sigma(s)} \langle x'|T\,\exp\left[-i\int_0^\infty ds'\;H(s')\right]|x''\rangle \right\}_{x'=x''=0} k_\nu ; \tag{50.b}$$

$$- \langle k_\mu\; e^{-i\;H\;s}\; 2\pi_\nu\rangle = k_\mu \left\{ i\,\frac{\delta}{\delta J^\nu(0)}\; K(x',x'') \right\}_{x'=x''=0} \tag{50.c}$$

and

$$<k_\mu\, e^{-i\,H\,s}\, k_\nu> = k_\mu\, k_\nu\, K(x',x'')\Big|_{x'=x''=0}. \qquad (50.d)$$

where we have used a compact notation for the evolution operator and the eigenstates.

Substituting equation (33) in equations (50) we have

$$<\Big[2\,\exp[(1-v)\,e\,F\,s]\,\pi\Big]_\mu\, e^{-i\,H\,s}\, 2\,\pi_\nu> =$$

$$=\Big[\exp[(1-v)\,e\,F\,s\Big]_\mu^\sigma \left\{-\frac{\delta^2}{\delta J^\sigma(s)\,\delta J^\nu(0)}\, N\,\exp\Big[-i\,W_{cl}\Big]\right\}_{x'=x''=0} =$$

$$=\Big[\exp[(1-v)\,e\,F\,s\Big]_\mu^\sigma \left\{\left[i\,\frac{\delta^2 W_{cl}}{J^\sigma(s)\,\delta J^\nu(0)} + \frac{\delta W_{cl}}{\delta J^\sigma(s)}\,\frac{\delta W_{cl}}{\delta J^\nu(0)}\right].\right.$$

$$\left..N\,\exp\Big[-i\,W_{cl}\Big]\right\}_{x'=x''=0}; \qquad (51.a)$$

$$-<\Big[2\,\exp[(1-v)\,e\,F\,s]\,\pi\Big]_\mu\, e^{-i\,H\,s}\, k_\nu> =$$

$$= \Big[\exp[(1-v)\,e\,F\,s]\Big]_\mu^\sigma \left\{i\,\frac{\delta}{\delta J^\sigma(s)}\, N\,\exp\Big[-i\,W_{cl}\Big]\right\}_{x'=x''=0} k_\nu =$$

$$=\Big[\exp[(1-v)\,e\,F\,s]\Big]_\mu^\sigma \left\{\frac{\delta W_{cl}}{\delta J^\sigma(s)}\, N\,\exp\Big[-i\,W_{cl}\Big]\right\}_{x'=x''=0} k_\nu; \qquad (51.b)$$

$$-<k_\mu\, e^{-i\,H\,s}\, 2\,\pi_\nu> = k_\mu \left\{i\,\frac{\delta}{\delta J^\nu(0)}\, N\,\exp\Big[-i\,W_{cl}\Big]\right\}_{x'=x''=0} =$$

$$= k_\mu \left\{\frac{\delta W_{cl}}{\delta J^\nu(0)}\, N\,\exp\Big[-i\,W_{cl}\Big]\right\}_{x'=x''=0} \qquad (51.c)$$

and

$$<k_\mu\, e^{-i\,H\,s}\, k_\nu> = k_\mu\, k_\nu \left\{N\,\exp\Big[-i\,W_{cl}\Big]\right\}_{x'=x''=0}. \qquad (51.d)$$

The functional derivatives can be evaluated as follows:

$$\frac{\delta W_{cl}}{\delta J^\sigma(s)}\Big|_{x'=x''=0} = \frac{\delta}{\delta J^\sigma(s)}\left\{2\,\left[\int_0^s ds'\; e^{2eFs'} J(s')\right]^t g\,\frac{e\,F}{A}\,.\right.$$

$$.\left[\int_0^s ds'\; e^{2eFs'} J(s')\right] + 2\,\left[\int_0^s ds'\; e^{2eFs'} J(s')\right]^t g\,e\,F.$$

$$\left..\left[\int_0^{s'} ds''\; e^{2eFs''} J(s'')\right]\right\} = 2\,\left\{\left[\int_0^s ds'\; e^{2eFs'} J(s')\right]^t g\,\frac{e\,F}{A}\,.\right.$$

$$.\left[\int_0^s ds'\; e^{2eFs'}\,\delta(s'-s)\right]\Bigg\}_\sigma - 2\,\left\{\left[\int_0^s ds'\; e^{2eFs'} J(s')\right]^t g\,\frac{e\,F}{A}\,.\right.$$

$$.\left[\int_0^s ds'\; e^{2eFs'}\,\delta(s'-s)\right]\Bigg\}_\sigma - 2\,\left[\int_0^s ds' \int^{s'} ds''\; \left(e^{2eFs''} J(s'')\right)^t g\,e\,F.\right.$$

$$\left..e^{2eFs'}\,\delta(s'-s)\right]\Bigg\}_\sigma = 2\,\left\{\left[\int_0^s ds'\; e^{2eFs'} J(s')\right]^t g\,\frac{e\,F}{A}\, e^{2eFs} -\right.$$

$$- 2 \left[\int_0^{s} ds'\ e^{2eFs'} J(s')\right]^t g\ \frac{e\ F}{A}\ e^{2eFs} - 2 \left[\int_0^{s} ds''\ \left(e^{2eFs''} J(s'')\right)^t .\right.$$

$$\left.\left. .g\ e\ F . e^{2eFs'}\right]\right\}_\sigma = \left\{2\ k^t\ g\ \frac{[1 - e^{-(1+v)eFs}]}{A}\right\}_\sigma , \qquad (52)$$

where we used equation (24) for $J(s')$ and $A=1-e^{2eFs}$.

We can, in a straightforward way, obtain

$$\left.\frac{\delta W_{cl}}{\delta J^{\nu}(0)}\right|_{x'=x''=0} = - \left\{2\ k^t\ g\ \frac{[1 - e^{-(1+v)eFs}]}{A}\right\}_\nu \qquad (53)$$

Taking the functional derivative relative to $J^{\nu}(0)$ of the third equality of equation (52), we have

$$\left.\frac{\delta^2 W_{cl}}{\delta J^{\sigma}(s)\delta J^{\nu}(0)}\right|_{x'=x''=0} = - \left[\frac{4\ g\ e\ F}{A}\right]_{\sigma\nu} \qquad (54)$$

Substituting equations (52-54) in (51) we get

$$<\left[2\ [e^{(1-v)eFs}]\ \pi\right]_\mu e^{-i\ H\ s}\ 2\ \pi_\nu> = \left\{-i\ \left[\frac{4\ g\ e\ F\ e^{(1-v)eFs}}{A}\right]_{\mu\nu} + \right.$$

$$\left. + 4\left[k^t\ g\ e^{-(1-v)eFs}\ \frac{[1 - e^{-(1+v)eFs}]}{A}\right]_\mu \left[k^t\ g\ \frac{[1 - e^{-(1+v)eFs}]}{A}\right]_\nu \right\}.$$

$$\left. .N \exp\left[-i\ W_{cl}\right]\right|_{x'=x''=0} \qquad (55.a)$$

$$-<\left[2\ [e^{(1-v)eFs}]\ \pi\right]_\mu e^{-i\ H\ s}\ k_\nu> =$$

$$= - 2 \left[k^t\ g\ e^{-(1-v)eFs}\ \frac{[1 - e^{-(1+v)eFs}]}{A}\right]_\mu k_\nu\ N \exp\left[-i\ W_{cl}\right]\Big|_{x'=x''=0} \qquad (55.b)$$

$$- <k_\mu\ e^{-i\ H\ s}\ 2\ \pi_\nu> = - 2\ k_\mu \left[k^t\ g\ \frac{[1 - e^{-(1+v)eFs}]}{A}\right].$$

$$\left. .N \exp\left[-i\ W_{cl}\right]\right|_{x'=x''=0} \qquad (55.c)$$

$$<k_\mu\ e^{-i\ H\ s}\ k_\nu> = k_\mu\ k_\nu\ N \exp\left[-i\ W_{cl}\right]\Big|_{x'=x''=0} \qquad (55.d)$$

Substituting equations (42) and (48) in equations (55) and these in (26) we obtain finally

$$\Pi^{(a)}_{\mu\nu} = \frac{-\ e^2}{(4\pi)^2} \int_0^{s} \frac{ds}{s} \det\left[\frac{e\ F\ s}{\sinh(eFs)}\right]^{1/2} e^{-im^2 s} \int_{-1}^{+1} \frac{dv}{2} .$$

$$.\left\{- \frac{2i}{s} \left[\frac{2\ e\ F\ e^{(1-v)eFs}}{A}\right]_{\mu\nu} + 4 \left[k^t\ g\ e^{-(1-v)eFs}\ \frac{(1-e^{-(1+v)eFs})}{A}\right]_\mu \right.$$

$$\left[k^t\ g\ \frac{(1-e^{-(1+v)eFs})}{A}\right]_\nu - 2 \left[k^t\ g\ e^{-(1-v)eFs}\ \frac{(1-e^{-(1+v)eFs})}{A}\right]_\mu k_\nu -$$

$$- 2\, k_\mu \left[k^t\, g\, \frac{(1-e^{-(1+v)eFs})}{A} \right]_\nu + k_\mu k_\nu \Bigg\}.$$

$$\exp\left\{ - i\, k^t\, g \left[\frac{\cosh(veFs) - \cosh(eFs)}{2\, e\, F\, \sinh(eFs)} \right] k \right\}, \qquad (56)$$

which, after some simple algebraic manipulations, can be rewritten in a more compact form

$$\Pi^{(a)}_{\mu\nu} = \frac{-e^2}{(4\pi)^2} \int_0^s \frac{ds}{s} \det\left[\frac{e\, F\, s}{\sinh(eFs)}\right]^{1/2} e^{-im^2 s} \int_{-1}^{+1} \frac{dv}{2}\, .$$

$$\Bigg\{ \frac{2i}{s}\, \partial \mathbb{R}_{\mu\nu}/\partial s + \frac{2i}{s}\, \partial \mathbb{L}_{\mu\nu}/\partial s + (\mathbb{R}k)_\mu (\mathbb{R}k)_\nu - (\mathbb{L}k)_\mu (\mathbb{L}k)_\nu +$$

$$+ (\mathbb{L}k)_\mu (\mathbb{R}k)_\nu - (\mathbb{R}k)_\mu (\mathbb{L}k)_\nu \Bigg\} \exp\left[-i\, k\, \mathbb{M}\, k\right], \qquad (57)$$

where

$$\mathbb{R} = \frac{\sinh(veFs)}{\sinh(eFs)}\, , \qquad \mathbb{L} = \frac{\cosh(eFs) - \cosh(veFs)}{\sinh(eFs)}, \qquad (58)$$

$$\mathbb{M} = \frac{\cosh(veFs) - \cosh(eFs)}{2\, e\, F\, \sinh(eFs)}.$$

Integrating by parts, in variable v, the first two terms of the expression (57) we obtain

$$\Pi^{(a)}_{\mu\nu} = \frac{-e^2}{2\,(4\pi)^2} \int_0^s \frac{ds}{s} \det\left[\frac{e\, F\, s}{\sinh(eFs)}\right]^{1/2} e^{-im^2 s}.$$

$$\cdot \left\{ \frac{2i}{s}\, \mathbb{R}_{\mu\nu} \exp\left[-i\, k\mathbb{M}k\right]\Bigg|_{v=-1}^{v=+1} + \frac{2i}{s}\, \mathbb{L}_{\mu\nu} \exp\left[-i\, k\mathbb{M}k\right]\Bigg|_{v=-1}^{v=+1} + \right.$$

$$+ \int_{-1}^{+1} dv \left[(\mathbb{R}k)_\mu (\mathbb{R}k)_\nu - (\mathbb{L}k)_\mu (\mathbb{L}k)_\nu - \mathbb{R}_{\mu\nu} (k\mathbb{R}k) \right] \exp\left[-i\, k\, \mathbb{M}\, k\right] +$$

$$+ \int_{-1}^{+1} dv \left[(\mathbb{L}k)_\mu (\mathbb{R}k)_\nu - (\mathbb{R}k)_\mu (\mathbb{L}k)_\nu - \mathbb{L}_{\mu\nu} (k\mathbb{R}k) \right] \exp\left[-i\, k\, \mathbb{M}\, k\right], \qquad (59)$$

where we are using the equality

$$\partial \mathbb{M}/\partial v = s/2\, \mathbb{R}.$$

Looking at expressions (58) we can see that

$$\mathbb{R}_{\mu\nu} \exp\left[-i\, k\mathbb{M}k\right]\Bigg|_{v=-1}^{v=+1} = 2\, g_{\mu\nu},$$

$$\mathbb{L}_{\mu\nu} \exp\left[-i\, k\mathbb{M}k\right]\Bigg|_{v=-1}^{v=+1} = 0. \qquad (60)$$

Further the last term of equation (59) is zero since the integrand is an odd function of v. So,

$$\Pi^{(a)}_{\mu\nu} = \frac{-e^2}{2\,(4\pi)^2} \int_0^s \frac{ds}{s} \det\left[\frac{e\, F\, s}{\sinh(eFs)}\right]^{1/2} e^{-im^2 s} \cdot \left\{ \frac{4i}{s}\, g_{\mu\nu} + \right.$$

$$+ \int_{-1}^{+1} dv \left[(\mathbb{R}k)_\mu (\mathbb{R}k)_\nu - (\mathbb{L}k)_\mu (\mathbb{L}k)_\nu - \mathbb{R}_{\mu\nu} (k\mathbb{R}k) \right] \exp\left[-i\, k\, \mathbb{M}\, k \right] \Bigg\} \quad (61)$$

The contribution of equation (5) for the polarization operator is obtained by making v = -1 (or k 0) in expression (34). So,

$$\Pi^{(b)}_{\mu\nu} = \frac{2i\, e^2}{(4\pi)^2}\, g_{\mu\nu} \int_0^\infty \frac{ds}{s^2} \det\left[\frac{e\, F\, s}{\sinh(eFs)}\right]^{1/2} e^{-im^2 s}, \quad (62)$$

which cancels out the first term of equation (61). In this form we get the follow expression for the vacuum polarization operator

$$\Pi_{\mu\nu} = \frac{-\, e^2}{2\,(4\pi)^2} \int_0^s \frac{ds}{s} \det\left[\frac{e\, F\, s}{\sinh(eFs)}\right]^{1/2} e^{-im^2 s}.$$

$$\cdot \int_{-1}^{+1} dv \left[(\mathbb{R}k)_\mu (\mathbb{R}k)_\nu - (\mathbb{L}k)_\mu (\mathbb{L}k)_\nu - \mathbb{R}_{\mu\nu} (k\mathbb{R}k) \right] \exp\left[-i\, k\, \mathbb{M}\, k \right]. \quad (63)$$

The renormalization of $\Pi_{\mu\nu}(k)$ can be done by the standard methods as in the paper by Baier et al [3].

V - Conclusions

In this paper we have calculated the order α contribution to the vacuum polarization in scalar electrodynamics in the presence of an external homogeneous electromagnetic field using proper time path integration and functional differentiation. The final results agree with previous calculation as they should.

For spinor electrodynamics the method cannot be used since we do not have a natural definition of path integral in the sense used here. Perhaps the formulation given by Barut and Duru [8] can be used to obtain analogous results in the spin 1/2 case.

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