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THE PROPER TIME PATH INTEGRAL APPROACH TO THE MASS OPERATOR

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Abstract

We obtain the exact result for the mass operator in the coordinate representation, to the first order in α, for a spin-zero particle in the presence of an external homogeneous electromagnetic field. We transform this quantum electrodynamics problem to one in quantum mechanics, which is solved using the proper time path integral formalism.

I - Introduction

The problem of charged particles interacting with external electromagnetic fields is interesting not only from the academic point of view but also from one of applications For the special configurations of a homogeneous external field the problem is fairly simple to treat at the level of classical mechanics and of nonrelativistic quantum mechanics. The treatment of the problem by quantum electrodynamics is fairly complicated since radiative corrections have to be taken into account.

As is well known the building blocks of QED are the Green functions and the one-particle irreducible graphs such as, mass operator and vacuum polarization. Many techniques have appeared in the literature seeking to calculate these quantities exactly at the level of the QED with external fields. In this context we must to cite the techniques of Schwinger [1], Feynman [2], Baier et al [3], Dittrich/Reuter [4] and Batalin et al [5]. The common point of these methods is the use of the proper-time method [6].

In the Schwinger technique the proper-time parameter plays the role of "time", in the equations of motion, of a fictitious quantum system which are then solved exactly. Later Schwinger generalized his technique to calculate exactly the mass operator, to the first order in α, for a particle interacting with an external homogeneous electromagnetic field [7].

On the other hand in the Feynman technique he guessed a form for the path integral representation of the propagator of a particle interacting with an external field, such that it satisfies an inhomogeneous Schrödinger equation in "five dimensions" and is associated to the zero-order Green function for QED by a Schwinger transformation. Essentially in both formulations a problem of QED is transformed to a quantum mechanics one.

We intend in this work to broaden the Feynman technique by applying it to the calculation of the mass operator, to the first order in α, for a particle in the presence of an external homogeneous electromagnetic field. Although the exact expression of the mass operator is known through other approaches, the present technique to calculate the relevant Feynman graphs has not been used before; besides it permits us to obtain a coordinate representation for the quantity in question.

The paper is organized as follows: in section II we give a formulation of the problem seeking to write the matrix element of the mass operator between position eigenstates in a form that permits us to use the formalism of path integrals; in section III we explain the formalism and find the exact expression for the mass operator in coordinate representation; we conclude the paper discussing the applicability of the method and its difficulties. There is also an appendix where we calculate the mass operator by using the Schwinger technique in the coordinate representation and we reach the same result as that obtained by applying the path integral technique.

II - Formulation of the problem

Consider a spinless particle of mass m and charge e in an external electromagnetic field $\mathcal{A}^e_\mu$. The propagator $G^0(x',x'')$ satisfies the equation

$$(\pi^2-m^2)G^0 = \delta^4(x'-x''), \tag{1}$$

where

$$\pi_\mu = i\partial_\mu - e\mathcal{A}^e_\mu. \tag{2}$$

Using a mixed representation one may write

$$G^0(x',x'') = \langle x'|(\pi^2-m^2)^{-1}|x''\rangle, \tag{3}$$

where the ket $|x'\rangle$ is the eigenvector of the operator x with eigenvalue x'.

Next if we consider the corrections due to the radiation field, one may use the Furry picture and write the order α correction to the propagation function in terms of the mass operator M where

$$M(x',x'') = \langle x'|ie^2 \int \frac{d^4k}{(2\pi)^4} (2\pi-k)^\mu \frac{1}{k^2} \frac{1}{(\pi-k)^2-m^2} (2\pi-k)_\mu |x''\rangle + \text{contact terms} \tag{4}$$

and one may write the modified Green function as

$$\langle x'|(\pi^2-m^2-M)^{-1}|x''\rangle = G(x',x''). \tag{5}$$

The contact terms arise due to the fact that in Furry picture the vacuum expectation value of the current is nonzero. The normalization conditions for $G(x',x'')$ require that the operator M and its first derivative with respect to π^2 vanish in the null field situation for $\pi^2=m^2$.

Next using the Schwinger representation

$$\frac{1}{D+i\varepsilon} = -i\int_0^\infty e^{is(D+i\varepsilon)}\, ds, \tag{6}$$

one can write

$$-\frac{1}{k^2[(\pi-k)^2-m^2]} = \int_0^\infty ds_1 \int_0^\infty ds_2 \exp i[-s_1 m^2 + s_1(\pi-k)^2 + s_2 k^2]. \tag{7}$$

Making the transformation of variables

$$s_1 = su \ , \ s = s_1 + s_2 \tag{8}$$

we get

$$-\frac{1}{\pi^2[(\pi-k)^2-m^2]} = \int_0^\infty ds \ s \int_0^1 du \ e^{-im^2 us} \ e^{-isH}, \tag{9}$$

where

$$H = -u(\pi-k)^2 - (1-u)k^2 = -(k-u\pi)^2 - u(1-u)\pi^2. \tag{10}$$

Also

$$\int \frac{d^4k}{(2\pi)^4} f(k) = \langle y'=0|f(k)|y''=0\rangle, \tag{11}$$

where $|y'\rangle$ is an eigenstate of the operator y conjugate to the momentum k with

$$\left[y_\mu, k_\nu\right] = -i \ g_{\mu\nu}. \tag{12}$$

Thus the expression (4) becomes

$$M(x',x'') = -ie^2 \int_0^\infty ds \ s \int_0^1 du \ e^{-im^2 su}$$

$$\langle x',y'=0|(2\pi-k)^\mu \ e^{-isH} \ (2\pi-k)_\mu|x'',y''=0\rangle. \tag{13}$$

III - Calculation of the mass operator

Let

$$|x',y';s\rangle = e^{isH} \ |x',y'\rangle \tag{14.a}$$

and

$$|x'',y'';0\rangle = |x'',y''\rangle \tag{14.b}$$

then the transformation function

$$\langle x',y';s|x'',y'';0\rangle = \langle x',y'|e^{-isH}|x'',y''\rangle = \langle e^{-isH}\rangle \tag{15}$$

satisfies

$$(i\partial_s - H) \ \langle e^{-isH}\rangle = 0 \qquad \text{for } s=0 \tag{16}$$

$$\lim_{s\to 0} \langle e^{-isH}\rangle = \delta(x'-x'')\,\delta(y'-y''). \tag{17}$$

Since that we need $\langle e^{-isH}\rangle$ for $s\geq 0$ we may impose the condition

$$\langle e^{-isH}\rangle = 0 \qquad \text{for } s<0 \tag{18}$$

and replace equations (16) and (17) by

$$(i\partial_s - H)\,\langle e^{-isH}\rangle = i\,\delta(s)\,\delta(x'-x'')\,\delta(y'-y''). \tag{19}$$

Since equation (19) has the form of a Schrödinger equation in "five dimensions", with s playing the role of "time" we may use the usual procedure and write the transformation function as a path integral.

$$\langle e^{-isH}\rangle = \int \mathcal{D}^4p\,\mathcal{D}^4x\,\mathcal{D}^4k\,\mathcal{D}^4y\,\exp\left[-i\int_0^s ds'\,(p.\dot{x} + k.\dot{y} + H)\right], \tag{20}$$

where $a.b = a_\mu b^\mu$ and the overdot denotes derivative with respect to s'.

Equation (20) is correct is spite of the sign of H and is consistent with the usual conventions since

$$p.\dot{x} = p_0\,\dot{x}_0 - \vec{p}.\dot{\vec{x}}.$$

It may be noted however that here the "Hamiltonian" H is a Lorentz scalar. The integration over the momenta p and k can be done easily to get $\langle e^{-isH}\rangle$ in the form

$$\langle e^{-isH}\rangle = \int \mathcal{D}^4x\,\mathcal{D}^4y\,\exp\left[-i\int_0^s L\,ds'\right], \tag{21}$$

where

$$L = \frac{\dot{x}^2}{4u} + \frac{(\dot{x}+\dot{y})^2}{4(1-u)} + e\,\dot{x}.\mathcal{A}^e. \tag{22}$$

In equation (13) we need to calculate

$$(2\pi-k)^\mu\,e^{-isH}\,(2\pi-k)_\mu = e^{-isH}\,(2\pi(s)-k(s))^\mu\,(2\pi(0)-k(0))_\mu, \tag{23}$$

where we introduced the operators

$$\mathcal{O}(s) = e^{-isH}\,\mathcal{O}(s)\,e^{isH}. \tag{24}$$

It is easy to see that one may write

$$\langle x',y'|(2\pi-k)^\mu\,e^{-isH}\,(2\pi-k)_\mu|x'',y''\rangle =$$

$$= -\frac{\delta^2}{\delta J_\mu(s)\,\delta J^\mu(0)}\,\langle x',y'|\exp(-i\int_0^s ds' H^J)|x'',y''\rangle\Big|_{J=0}, \tag{25}$$

where $H^J = H - J.(2\pi-k)$ and $J(s')$ is an external classical current. Also we can rewrite $\langle\exp(-i\int_0^s ds'\,H^J\rangle$ in the form

$$\langle x',y'|\exp(-i\int_0^s ds'\,H^J)|x'',y''\rangle = \langle x',y';s|x'',y'';0\rangle^J =$$

$$= \int \mathcal{D}^4x\, \mathcal{D}^4y \exp(-iW^J), \tag{26}$$

where W^J is the action given by

$$W^J = \int_0^s ds' \left[\frac{\dot{x}^2}{4u} + \dot{x}.\left[\mathcal{A}^e - \frac{J}{u}\right] + \frac{J^2}{u} + \frac{1}{4(1-u)}(\dot{x}+\dot{y}-J)^2 \right]. \tag{27}$$

For the case of a constant electromagnetic field $F_{\mu\nu}$ we choose the potential $\mathcal{A}^e_\mu$ to be

$$\mathcal{A}^e_\mu = 1/2\, F_{\mu\nu}\, (x-x'')^\nu. \tag{28}$$

The integrations are trivial since we are dealing with a quadratic Lagrangian. So the result can be written as

$$\langle x',y';s|x'',y'';0\rangle = N \exp(-iW^J_{cl}). \tag{29}$$

where W^J_{cl} is to be evaluated for the classical trajectory and N is a suitable normalization constant. To calculate W^J_{cl} we need to solve the classical equations of motion

$$d(\dot{x} + \dot{y})/ds' = \dot{J} \tag{30.a}$$

and

$$\ddot{x} + 2eu\, F\dot{x} = 2\, \dot{J}, \tag{30.b}$$

subject to the boundary conditions

$$x(s) = x' \qquad x(0) = x''$$
$$y(s) = y' \qquad y(0) = y'' \tag{31}$$

In equations (30) and henceforth (when convenient) we use a matrix notation

$$x^\mu = (x)_\mu \qquad F^\mu{}_\nu = (F)_{\mu\nu}$$

$$g_{\mu\nu} = (g)_{\mu\nu} \qquad g^{\mu\nu} = (g^{-1})_{\mu\nu}$$

where on the left hand side we have tensorial quantities and on the right hand side we have matrix elements. Henceforth the parenthesis indicating matrices will be omitted. It may be noted that the matrix F has the property

$$F^t = -\, g\, F\, g^{-1}, \tag{32}$$

where F^t is the transpose of F.

One can solve equation (30.a), subject to equation (31) in the form

$$\dot{x} + \dot{y} - J(s) = \frac{1}{s}\left[\Delta x + \Delta y - \int_0^s J(s')\, ds'\right], \tag{33}$$

where $\Delta x = x'-x''$ and $\Delta y = y'-y''$.

Equation (30.b) gives

$$\dot{x}(s') + 2eu\, F(x(s') - x') = 2\, J(s') + (\dot{x}(s) - 2J(s)). \tag{34}$$

Integrating we get

$$e^{2eus'F}(x(s')-x') + \Delta x = 2\int_0^s e^{2euFs'}\, J(s')\, ds' +$$

$$+ \frac{e^{2euFs'}-1}{2eu\,F}\,(\dot{x}(s) - 2\,J(s)). \tag{35}$$

Making s'=s we have

$$\Delta x = 2\int_0^s e^{2euFs'}\, J(s')\, ds' + \frac{e^{2euFs}-1}{2eu\,F}\,(\dot{x}(s) - 2\,J(s)). \tag{36}$$

Hence

$$\frac{(\dot{x}(s) - 2\,J(s))}{2u} = \frac{e\,F}{e^{2euFs}-1}\left[\Delta x - 2\int_0^s e^{2euFs'}J(s')\, ds'\right]. \tag{37}$$

Going back to equation (34) we get, using (35),

$$e^{2euFs'}\,\frac{\dot{x}(s') - 2\,J(s')}{2u} = \frac{\dot{x}(s) - 2\,J(s)}{2u} +$$

$$+ e\,F\left[\Delta x + 2\int_0^{s'} e^{2euFs''}J(s'')\, ds''\right]. \tag{38}$$

Now, using equations (27), (33), (37) and (38) we get

$$W^J_{cl} = \frac{1}{2}\,\Delta x^t g\,\frac{e\,F}{A}\,\Delta x - 2\,\Delta x^t g\,\frac{e\,F}{A}\left\{\int_0^s e^{2euFs'}J(s')\, ds\right\} -$$

$$- 2\left\{\int_0^{s'} e^{2euFs'}J(s')\, ds'\right\}^t g\,e\,F\,\Delta x +$$

$$+ 2\left\{\int_0^s e^{2euFs'}J(s')\, ds'\right\}^t g\,\frac{e\,F}{A}\left\{\int_0^s e^{2euFs'}J(s')\, ds'\right\} +$$

$$+ 2\left\{\int_0^s e^{2euFs'}J(s')\, ds'\right\}^t g\,e\,F.\left\{\int_0^{s'} e^{2euFs''}\,J(s'')\, ds''\right\} + \frac{(\Delta x + \Delta y)^2}{4(1-u)s} +$$

$$+ \frac{1}{4(1-u)s}\left\{-2\,(\Delta x + \Delta y)^t\, g\left[\int_0^s J(s')\, ds'\right] + \left[\int_0^s J(s')\, ds'\right]^2\right\}, \tag{39}$$

where $A = 1 - e^{-2euFs}$.

In order to calculate the expectation value which appears in equation (13) we must, according to equations (25) and (29), to take the limit y'=y"=0 in the classical action. In this form we obtain

$$\langle x', y'=0|(2\pi-k)^{\mu}\, e^{-isH}\,(2\pi-k)_{\mu}|x'', y''=0\rangle =$$

$$= \frac{-\,\delta^2\, N\,\exp\left[-iW^J_{cl}\right]}{\delta J_{\mu}(s)\,\delta J^{\mu}(0)}\Bigg|_{\substack{J=0\\ y'=y''=0}} = N\left\{i\,\frac{\delta^2 W^J_{cl}}{\delta J_{\mu}(s)\,\delta J^{\mu}(0)} + \frac{\delta W^J_{cl}}{\delta J_{\mu}(s)}\,\frac{\delta W^J_{cl}}{\delta J^{\mu}(0)}\right\}.$$

$$.\exp\left[-iW^J_{cl}\right]\Bigg|_{\substack{J=0\\ y'=y''=0}} \tag{40}$$

Looking at equation (39) we can see that only the second, third and seventh terms will be contribute for the first order functional derivatives

which appear in (40). On the other hand, for the second order derivative, we need only of the fourth, fifth and eighth terms. So we obtain

$$\left.\frac{\delta W^J_{cl}}{\delta J_\mu(s)}\right|_{\substack{J=0\\ y'=y''=0}} = -\left[\Delta x^t\ g\ \frac{2\ e\ F}{A} + \frac{1}{2(1-u)s}\ \Delta x\ g\right]^\mu, \tag{41.a}$$

$$\left.\frac{\delta W^J_{cl}}{\delta J^\mu(0)}\right|_{\substack{J=0\\ y'=y''=0}} = -\left[\frac{2\ e\ F}{A}\ \Delta x + \frac{1}{2(1-u)s}\ \Delta x\right]_\mu \tag{41.b}$$

and

$$\left.\frac{\delta^2 W^J_{cl}}{\delta J_\mu(s)\ \delta J^\mu(0)}\right|_{\substack{J=0\\ y'=y''=0}} = \mathrm{Tr}\left[\frac{1}{2(1-u)s} + 2\ e\ F\ \coth(euFs)\right], \tag{41.c}$$

Substituting equations (41) in equation (40) and taking J=0 and y'=y"=0 in the classical action (39) we get

$$\langle x', y'=0|(2\pi-k)_\mu\ e^{-iHs}\ (2\pi-k)^\mu|x'', y''=0\rangle =$$

$$= N\left\{\Delta x^t\ g\left[\frac{2\ e\ F}{A} + \frac{1}{2(1-u)s}\right]^2 \Delta x + i\ \frac{2}{(1-u)s} + 2i\ \mathrm{Tr}[e\ F\ \coth(euFs)]\right\}$$

$$\exp\left\{-\frac{i}{4}\left[\Delta x^t\ g\ e\ F\ \coth(euFs)\ \Delta x + \frac{\Delta x^2}{(1-u)s}\right]\right\}. \tag{42}$$

Now, it is necessary to determine the normalization constant. Since N is independent of the classical source J, it can be determined as follow: In

equation (22) we make the transformation of variables.

$$\xi(s') = \Lambda(s')\ (x(s')-x'')$$
$$\eta(s') = x(s') + y(s') \tag{41}$$

We restrict $\Lambda(s')$ by the conditions

i) Under the transformation considered the kinectic terms are form invariant.

$$\dot{x}^2 \text{ —— } \dot{\xi}^2 \tag{44.a}$$

ii) The terms

$$\frac{\dot{x}^2}{u} + e\ \dot{x}.A^e, \tag{44.b}$$

after the transformation, do not result in cross terms in ξ and $\dot{\xi}$.

It is easy to see that the above conditions imply that

$$\Lambda(s') = e^{-euFs}\ \Lambda(0) \tag{45.a}$$

$$\Lambda(0)^t\ g\ \Lambda(0) = g. \tag{45.b}$$

The quadratic terms in (45.b) have the form

$$\frac{e^2\ u}{4}\ \xi^t\ g\ \Lambda^{-1}(0)\ F^2\ \Lambda(0)\ \xi. \tag{46}$$

We impose that the above expression is of the form

$$- \frac{1}{4u} \xi^t g \, \Omega^2 \, \xi, \tag{47}$$

where Ω^2 is diagonal. It can be shown that the matrix $A(0)$ exists. Thus the terms which result of (44) after the transformation can be interpreted as a Lagrangian of a harmonic oscillator of mass $\frac{1}{2u}$, and frequency matrix Ω. The remaining terms in the original Lagrangian result in a free-particle Lagrangian with mass $\frac{1}{2(1-u)}$. These two results imply that

$$N = - \frac{1}{(4\pi)^4 \, s^4 \, u^2 \, (1-u)^2} \, \mathrm{Det}\left[\frac{e \, u \, F \, s}{\sinh \, (euFs)} \right]^{1/2}. \tag{48}$$

Substituting equations (48) and (42) in equation (13) the mass operator $M(x',x'')$ becomes

$$M(x',x'') = - \frac{ie^2}{(4\pi)^4} \int_0^\infty s \, ds \int_0^1 du \, e^{-im^2 s} \, \frac{1}{(1-u)^2 u^2 s^4} \, \mathrm{Det}\left[\frac{e \, u \, F \, s}{\sinh \, (euFs)} \right]^{1/2}$$

$$\left\{ \Delta x^t \, g \left[\frac{2 \, e \, F}{A} + \frac{1}{2(1-u)s} \right]^2 \Delta x + \frac{2i}{(1-u)s} + 2i \, \mathrm{Tr}\left[e \, F \, \coth(euFs) \right] \right\}$$

$$\exp\left\{ - \frac{i}{4} \left[\Delta x^t \, g \, e \, F \, \coth(euFs) \, \Delta x + \frac{\Delta x^2}{(1-u)s} \right] \right\}. \tag{49}$$

The contact terms mentioned previously can be found using the results by Tsai [8].

First we write them as

$$\text{c.t.} = - m_0^{\,2} - \zeta_0^{\,2} \, (p^2 - m^2), \tag{50}$$

being

$$m_0^{\,2} = - \frac{e^2}{(4\pi)^2} \int \frac{ds \, du}{s} \, e^{-ism^2u^2} \, [-(2-u)^2 m^2 - 2i/s] \tag{51}$$

and

$$\zeta_0^{\,2} = - \frac{e^2}{(4\pi)^2} \int \frac{ds \, du}{s} \, e^{-ism^2u^2}$$

$$[- i \, s \, u \, (1-u) \, (-(2-u)^2 m^2 - 2i/s) + (2-u)^2]. \tag{52}$$

In order to write the contact terms in coordinate representation we need to take the expectation value of (50) between position eigenstates. In this way we get

$$\text{c.t.} = - m_0^{\,2} \, \delta(x'-x'') - \zeta_0^{\,2} \left[\frac{\partial^2}{\partial x'^2} - m^2 \right] \delta(x'-x''). \tag{53}$$

IV - Conclusions

We derived the mass operator, to the first order in α and in the coordinate representation, for a spinless particle in the presence of an external homogeneous electromagnetic field. We have transformed a problem in QED to one in quantum mechanics which then has been solved using the path

integral formulation that resemble the usual functional techniques of the Quantum Field Theories when one wants to derive the Feynman rules.

Here we are treating with scalar QED, but if we go to the spinorial QED we find some difficulties since there is not a well established method to work with particles of spin 1/2 in the path integral representation. This difficulty does not arise in the Schwinger technique, this can be verified in the reference [9]. In spite of this one could try to use the prescription given by Barut and Duru [10] in order to treat particles of spin 1/2 in the path integral representation.

If the external field has a configuration that gives origin to a nonquadratic Lagrangian, then equation (29) is not true, and the problem is difficult to solve. This problem also appears in the Schwinger technique because the equation of motion become very laborious to integrate, but the "operator approach" by Baier et al overcome this difficulty, as for example on the case of a plane-wave electromagnetic field [11].

The present technique can also be applied to derive the exact expression, in order α approximation, for the vacuum polarization operator in the presence of an external homogeneous electromagnetic field. This calculation will be presented elsewhere

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Appendix:

Here we are going to use the formulation developed by Schwinger to obtain the Green function for particles interacting with external electromagnetic fields, to reobtain the mass operator in coordinate representation.

To begin with it, let us write the equations of motion for the operators whose evolution is governed by equation (24).

$$\pi^{\mu} = -i\,[\pi^{\mu}, H] = -\,2\,e\,u\,F^{\mu}{}_{\nu}\,(\pi - k)^{\nu}, \tag{A.1}$$

$$k^{\mu} = 0, \tag{A.2}$$

$$x^{\mu} = 2\,u\,(\pi - k)^{\mu}, \tag{A.3}$$

$$y^{\mu} = 2\,(k - u\pi)^{\mu}. \tag{A.4}$$

Integrating equation (A.1) we obtain

$$\pi^{\mu}(s) = \left[e^{-2euFs}\,\pi(0)\right]^{\mu} + \left[\left(1 - e^{-2euFs}\right)k(0)\right]^{\mu}. \tag{A.5}$$

Substituting (A.3) in (A.1) and integrating we have

$$\pi^{\mu}(s) = \pi^{\mu}(0) - [e\,F\,(x(s) - x(0))]^{\mu}. \tag{A.6}$$

Besides

$$(y(s) - y(0))^{\mu} + (x(s) - x(0))^{\mu} = 2\ (1-u)\ s\ k^{\mu}, \tag{A.7}$$

and

$$\pi^{\mu}(0) - k^{\mu}(0) = \left[\frac{e\ F}{A}\ (x(s) - x(0)\right]^{\mu}. \tag{A.8}$$

Defining

$$M(x',x";y',y") = -\ i\ e^{2} \int_{0}^{\infty} s\ ds \int_{0}^{1} du\ \exp(-im^{2}su).$$

$$<x',y'|(2\pi - k)^{\mu}\ \exp(-iHs)\ (2\pi - k)_{\mu}|x",y">, \tag{A.9}$$

and using equations (23) and (A.5)-(A.8) we get

$$M(x',x";y',y") = -i\ e^{2} \int_{0}^{\infty} s\ ds \int_{0}^{1} du\ \exp(-im^{2}su).$$

$$.<x',y'|\exp(-iHs)\left[\frac{2\ e\ F\ e^{-2euFs}}{A}\ \Delta x + \frac{(\Delta x + \Delta y)}{2\ s\ (1-u)}\right]_{\mu}.$$

$$.\left[\frac{2\ e\ F}{A}\ \Delta x + \frac{(\Delta x + \Delta y)}{2\ s\ (1-u)}\right]^{\mu}|x",y">, \tag{A.10}$$

where $\Delta x = x(s) - x(0)$ and $\Delta y = y(s) - y(0)$.

In order to make the product of quadrivectors which appears in above equation is necessary to make the time-ordering the operators. For this we use the follow commutation relations

$$\left[x^{\mu}(0), x^{\nu}(s)\right] = -i\ \left[\frac{A}{e\ F}\right]^{\nu\mu}, \tag{A.11}$$

$$\left[y^{\mu}(0), y^{\nu}(s)\right] = -i\ \left[\frac{D}{e\ F}\right]^{\nu\mu}, \tag{A.12}$$

$$\left[y^{\mu}(0), x^{\nu}(s)\right] = i\ \left[\frac{A}{e\ F}\right]^{\nu\mu}, \tag{A.13}$$

$$\left[x^{\mu}(0), y^{\nu}(s)\right] = i\ \left[\frac{A}{e\ F}\right]^{\nu\mu}, \tag{A.14}$$

where $D = A + 2\ (1-u)\ e\ F\ s$.

In this form, after to realize the products and calculate the expectation values, we obtain

$$M(x',x";y',y") = -i\ e^{2} \int_{0}^{\infty} s\ ds \int_{0}^{1} du\ \exp(-im^{2}su)\ \left\{\Delta x\ \left[\frac{2\ e\ F}{A} + \frac{1}{2\ s\ (1-u)}\right] \Delta x + \right.$$

$$\left. + \frac{1}{(1-u)\ s}\ \Delta x\ \left[e\ F\ \left[\frac{1}{A} - \frac{1}{A}\right] + \frac{1}{2\ s\ (1-u)}\right] \Delta y + \frac{\Delta y^{2}}{4\ s^{2}\ (1-u)^{2}}\right\}.$$

$$.<x',y'|\exp(-iHs)|x",y">. \tag{A.15}$$

We know that the amplitude $<e^{-iHs}>$ obeys the equation

$$i\ \frac{\partial}{\partial s}\ <\exp(-iHs)> = <\exp(-iHs)\ H>. \tag{A.16}$$

Then, using once more (A.7) and (A.8) we reach

$$i\ \frac{\partial}{\partial s}\ \ln <\exp(-iHs)> = u\ \Delta x\ \left[\frac{(e\ F)}{A\ A}\right]^{2} \Delta x + i\ \mathrm{Tr}\ \left[\frac{e\ F\ u}{A}\right] -$$

$$- \frac{2\,i}{s} - \frac{(\Delta x + \Delta y)^2}{4\,s^2\,(1-u)} \,. \qquad (A.17)$$

In this form

$$\langle \exp(-iHs) \rangle = C\, s^{-2}\, \mathrm{Det}\left[\frac{e\,F}{\sinh(eFus)}\right]^{1/2} .$$

$$\exp - \frac{i}{4}\left\{\Delta x \left[e\,F\,\coth(euFs)\right] \Delta x + \frac{(\Delta x + \Delta y)^2}{(1-u)\,s}\right\}. \qquad (A.18)$$

Here we have introduced a factor eF in the determinant to reestablish the correct limit when F 0. In this case

$$\langle e^{-iHs} \rangle = \frac{C}{s^4\,u^2} \exp\left\{\frac{-i\,\Delta x^2}{4\,(1-u)\,u\,s}\right\} \exp\left\{\frac{-i\,\Delta y^2}{4\,(1-u)\,s}\right\}. \qquad (A.19)$$

Using equation (17) we get

$$C = (4\pi)^{-4}\,(1-u)^4 . \qquad (A.20)$$

Now, if we want to reproduce the matrix element defined in equation (4) we must to make $\Delta y = 0$ in (A.18). So,

$$M(x',x'') = \frac{-i\,e^2}{(4\pi)^4} \int_0^\infty s\,ds \int_0^1 du\, \exp\,(-im^2 us) \quad \frac{1}{(1-u)^4\,u^2\,s^4}$$

$$\mathrm{Det}\left[\frac{e\,F\,u\,s}{\sinh(eFus)}\right] \left\{\Delta x \left[\frac{2\,e\,F}{A} + \frac{1}{2\,(1-u)\,s}\right]^2 \Delta x + \frac{2\,i}{(1-u)\,s} + \right.$$

$$\left. + 2i\,\mathrm{Tr}\left[e\,F\,\coth(e\,F\,u\,s)\right]\right\}.$$

$$\cdot\ \exp - \frac{i}{4}\left\{\Delta x \left[e\,F\,\coth(eFus)\right]\Delta x + \frac{\Delta x^2}{(1-u)\,s}\right\}, \qquad (A.21)$$

which is the same result obtained via path integral technique.

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