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EVALUATION OF SCHWINGER-DE WITT KERNELS FOR SPIN 1/2 FIELDS IN ANISOTROPIC SPACE-TIMES THROUGH THE SO(2,1) LIE ALGEBRA

S.J. Rabello[†] and A.N. Vaidya

Instituto de Fisica

Universidade Federal do Rio de Janeiro

P.O. Box 68528

21945 Rio de Janeiro

Brasil

Abstract

Using the noncompact SO(2,1)~SU(1,1) Lie algebra we obtain the Schwinger-DeWitt kernels of spin 1/2 fields propagating in the classical background gravitational fields of the Bianchi type-I models .The method makes use of the Schwinger-DeWitt proper time representation and a Baker-Campbell-Hausdorff formula.

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[†]Present Address: Physics Laboratory

University of Kent at Canterbury

CT2 7NR, U.K.

I. Introduction

In a previous paper [1] we evaluated the Schwinger-DeWitt kernels for massive spin $\frac{1}{2}$ fields propagating in some spatially flat Robertson-Walker universes. The main feature of these models is that they are conformally static, in another words in the $m \to 0$ limit the solutions can be conformally mapped to the flat space ones. General homogeneus anisotropic models do not share this property. The anisotropy should give rise to particle creation which, as has been suggested by Zel'dovich [2], might react back on its source to reduce the effect - i.e. to isotropize the models. This could account for the present high degree of isotropy of the universe.

For most of these models we only know perturbative results [3], but for some particular cases there are exact solutions , mainly for the scalar field. In the following we obtain some exact results for Dirac particles in the background of Bianchi type I models .

The procedure explained below is a variant of the Schwinger-DeWitt method , that makes use of the Schwinger representation [4] and the Lie algebra of the SO(2,1)~SU(1,1) group [5]. This method has been applied to the relativistic Coulomb problem by Mil'shtein and Strakhovenko [6] , to numerous problems of non relativistic quantum mechanics by Boschi-Filho and Vaidya [7] and recently by us to the scalar and spinor fields propagating in cosmological backgrounds [8][1]. In the next section we set up the basic machinery necessary for the study of fermionic fields in curved space-time.

II. BASIC EQUATIONS

The generally covariant propagator for a "free" massive spin $\frac{1}{2}$ field satisfies the equation [3] (h=c=1)

$$\left[i\gamma^{\mu}(x)\left(\partial_{\mu} - \Gamma_{\mu}(x)\right) - m\right]S(x,x') = \frac{1}{\sqrt{-g}}\,\delta^{4}(x-x') \qquad (1)$$

where $\gamma^{\mu}(x)$ are the curved space Dirac matrices that verify the anticommutation rule

$$\{\gamma^{\mu}(x),\gamma^{\nu}(x)\}=2g^{\mu\nu}(x)\ , \qquad (2)$$

where $g^{\mu\nu}$ is the spacetime metric. The quantity Γ_{μ} that appears in (1) is the spinor affine connection given by [3]

$$\Gamma_{\mu}(x) = -\frac{1}{8}\,[\gamma^{\nu}(x),\gamma_{\nu}(x)_{;\mu}] \qquad (3)$$

where the semicolon denotes the vector covariant derivative.

To find an expression for the $\gamma^{\mu}(x)$ in terms of its flat space counterparts γ^{α} (hereafter the indices from the beginning of the Greek alphabet transform by the local Lorentz group) let us write the metric $g^{\mu\nu}$ using the Vierbein $V^{\mu}_{\alpha}(x)$ [3] and the Minkowski metric $\eta^{\alpha\beta}$

$$g^{\mu\nu}(x)=V^{\mu}_{\alpha}(x)V^{\nu}_{\beta}(x)\eta^{\alpha\beta}. \qquad (4)$$

Using this representation for $g^{\mu\nu}$ we have for the curved space Dirac matrices:

$$\gamma^{\mu}(x)=V^{\mu}_{\alpha}(x)\gamma^{\alpha}\ . \qquad (5)$$

We shall use the following representation for the γ^{α} [9]:

$$\gamma^{0}=\gamma_{0}=\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad , \quad \gamma^{i}=-\gamma_{i}=\begin{pmatrix} 0 & \sigma_{i} \\ \sigma_{i} & 0 \end{pmatrix} \qquad (6)$$

where σ_{i} are the Pauli matrices.

III. THE MODELS

Let us now imagine that our fields are propagating on the classical background of linearly expanding Bianchi Type-I universes [10]. The metrics of these cosmological models are of the form

$$g_{\mu\nu}(t) = \mathrm{diag}(1, -t^{2p_1}, -t^{2p_2}, -t^2) \qquad (7)$$

$$0 \le t < \infty$$

where p_1 and p_2 are constant parameters equal to 1 or 0. The case $p_1=p_2=1$ is the linearly expanding , spatially flat, Robertson-Walker universe that has been studied in our previous paper [1]. Both this model as well the anisotropic model with $p_1=1$, $p_2=0$ have a curvature singularity at $t=0$. The case $p_1=p_2=0$ corresponds to the degenerate Kasner universe, that is flat and singularity free.

The above metrics can be described by the Vierbein $V^{\alpha}_{\mu}(t)$:

$$V^0_0 = 1 \qquad V^1_1 = t^{p_1} \qquad V^2_2 = t^{p_2} \qquad V^3_3 = t \qquad (8)$$

The $\gamma^{\mu}(t)$ and $\Gamma_{\mu}(t)$ therefore read

$$\gamma^0(t) = \gamma^0 \ , \quad \gamma^1(t) = t^{-p_1}\gamma^1 \ , \quad \gamma^2(t) = t^{-p_2}\gamma^2 \ , \quad \gamma^3(t) = t^{-1}\gamma^3 \qquad (9a)$$

$$\Gamma_0 = 0 \ , \quad \Gamma_1 = \frac{1}{2} p_1 t^{p_1 - 1} \gamma_0\gamma_1 \ , \quad \Gamma_2 = \frac{1}{2} p_2 t^{p_2 - 1} \gamma_0\gamma_2 \ , \quad \Gamma_3 = \frac{1}{2} \gamma_0\gamma_3 \qquad (9b)$$

Due to the translational invariance in the spatial sector , we can write $S(x,x')$ as

$$S(x,x') = \frac{1}{(2\pi)^3} \int d^3k \, e^{-i\vec{k}.(\vec{x}-\vec{x}')} \, S_k(t,t') \ . \qquad (10)$$

Inserting (10) in (1) we have that $S_k(t,t')$ verifies

$$\left[i\left[\gamma^0\partial_0 - it^{-p_1}\gamma^1 k_1 - it^{-p_2}\gamma^2 k_2 - it^{-1}\gamma^3 k_3 + \frac{\gamma^0}{2t}(p_1 + p_2 + 1) \right] - m \right] S_k(t,t') = t^{-(p_1+p_2+1)} \, \delta(t-t') \qquad (11)$$

In order to deal with a second order equation, let us write $S_k(t,t')$ as

$$S_k(t,t')=\left[i\left(\gamma^0\partial_0 - it^{-p_1}\gamma^1 k_1 - it^{-p_2}\gamma^2 k_2 - it^{-1}\gamma^3 k_3 + \frac{\gamma^0}{2t}(p_1+p_2+1)\right)+m\right]G_k(t,t') \tag{12}$$

Substituting (12) in (11) we have

$$\left\{\partial_0^2 + (p_1+p_2+1)\frac{1}{t}\partial_0 + \frac{1}{t^2}\left[-\frac{1}{2}(p_1+p_2+1)+\frac{1}{4}(p_1+p_2+1)^2+\mathbf{k}^2\right]\right.$$

$$\left. + \frac{i}{t^2}\gamma^0\gamma^a \mathbf{k}_a + (1-p_1)k_1^2 + (1-p_2)k_3^2 + m^2\right\} G_k(t,t') = -t^{-(1+p_1+p_2)}\delta(t-t') \tag{13}$$

where $\mathbf{k}$ is the vector

$$\mathbf{k}_1 = p_1 k_1 \quad , \quad \mathbf{k}_2 = p_2 k_2 \quad , \quad \mathbf{k}_3 = k_3 \tag{14}$$

to get rid of the non-diagonal term of (13)

$$\frac{i}{t^2}\gamma^0\gamma^a \mathbf{k}_a \tag{15}$$

we apply a kind of Foldy-Wouthuysen transformation [9] to (13). For the operator that maps (15) to a diagonal form we take the *Ansatz*

$$e^{\gamma^a \mathbf{k}_a \theta} = \cos|\mathbf{k}|\theta + \frac{\gamma^a \mathbf{k}_a}{|\mathbf{k}|}\sin|\mathbf{k}|\theta \tag{16}$$

With θ a parameter to be determined by

$$e^{\gamma^a \mathbf{k}_a \theta}\left[\frac{i}{t^2}\gamma^0\gamma^a \mathbf{k}_a\right]e^{-\gamma^a \mathbf{k}_a \theta} = \frac{i}{t^2}\gamma^0\gamma^a \mathbf{k}_a \cos 2|\mathbf{k}|\theta + \frac{i}{t^2}\gamma^0|\mathbf{k}|\sin 2|\mathbf{k}|\theta \tag{17}$$

To make the above expression diagonal we choose :

$$\theta = \frac{\pi}{4|\mathbf{k}|} \tag{18}$$

With this choice, the transformed Green function

$$\tilde{G}_k(t,t') = e^{\gamma^a \mathbf{k}_a \theta} G_k(t,t')\, e^{-\gamma^a \mathbf{k}_a \theta} \tag{19}$$

can be writen as

$$\tilde{G}_k(t,t') = \begin{bmatrix} G_k^+(t,t')\mathbb{I} & 0 \\ 0 & G_k^-(t,t')\mathbb{I} \end{bmatrix} \tag{20}$$

and $G_k^{\pm}(t,t')$ verifies

$$\left\{ \partial_0^2 + (p_1 + p_2 + 1)\frac{1}{t}\,\partial_0 + \frac{1}{t^2}\left[-\frac{1}{2}(p_1 + p_2 + 1) + \frac{1}{4}(p_1 + p_2 + 1)^2 + k^2 \pm i|k| \right] \right.$$

$$\left. + (1-p_1)k_1^2 + (1-p_2)k_3^2 + m^2 \right\} G_k^{\pm}(t,t') = -t^{-(1+p_1+p_2)}\,\delta(t-t') \tag{21}$$

Collecting the t dependent terms of the above differential equation in the operators $T_1^{\pm}$, equation (21) can be recast to a more compact form

$$\left[T_1^{\pm} + (1-p_1)k_1^2 + (1-p_2)k_3^2 + m^2 \right] G_k^{\pm}(t,t') = -t^{-(1+p_1+p_2)}\,\delta(t-t') \tag{22}$$

IV. EVALUATION OF THE KERNELS

We are now ready to evaluate $G_k^{\pm}(t,t')$, to this end we use the Schwinger-DeWitt proper time representation for $G_k^{\pm}(t,t')$ [3]

$$G_k^{\pm}(x,x') = -i\int_0^{\infty} ds\; e^{-i[T_1^{\pm} + (1-p_1)k_1^2 + (1-p_2)k_3^2 + m^2]s}\;\delta(t-t')\; t^{-(1+p_1+p_2)} \tag{23}$$

Boundary conditions must be imposed in the above expression, by choosing a convenient path of integration in the complex s plane. This problem is related to the definition of the vacuum state in curved space-time [3], which is not unique. For spin $\frac{1}{2}$ fields there are numerous attempts in the literature to define the vacuum state, for some results see [11][12]. In this paper our intention is to evaluate the kernel of (23) leaving the question of the s integration untouched.

Our task now is to find the action of the exponential in (23) on the delta function. This can be done using the algebraic method [1][7].

First we note that along with the operators

$$T_2 = -\frac{1}{2}\left[t\partial_t + \frac{1}{2}(p_1 + p_2 + 2)\right] \tag{24}$$

$$T_3 = -\frac{1}{8}t^2 \tag{25}$$

the operators $T_1^{\pm}$ verify the commutation relations of the SO(2,1)~SU(1,1) Lie algebra generators [5].

$$[T_1^{\pm}, T_2] = -iT_1^{\pm} \quad , \quad [T_2, T_3] = -iT_3 \quad , \quad [T_1^{\pm}, T_3] = -iT_2 \tag{26}$$

To find the action of the exponential of this linear operator on the delta function, we make use of the following BCH formula, that can be obtained by use of a faithful representation of SO(2,1)~SU(1,1) constructed with the Pauli matrices σ_i [7]:

$$T_1^{\pm} = \frac{\sigma_1 - i\sigma_2}{2\sqrt{2}} \qquad T_2 = \frac{-i\sigma_3}{2} \qquad T_3 = \frac{\sigma_1 + i\sigma_2}{2\sqrt{2}} \tag{27}$$

To obtain the BCH formula we use the above representation in the following equation

$$\exp(-icT_1^{\pm})\exp(pT_3) = \exp(-ia_1T_3)\exp(-ib_1T_2)\exp(-ic_1T_1^{\pm}) \tag{28}$$

Expanding both sides of the above formula and identifying the coefficients we have:

$$a_1 = ip(1-ipc/2)^{-1} \qquad b_1 = 2\mathrm{Ln}(1-ipc/2) \qquad c_1 = c(1-ipc/2)^{-1} \tag{29}$$

We need also the action of $\exp[-ibT_2]$ on arbitrary functions of t, which can be obtained by a simple Taylor expansion:

$$\exp[-ibT_2]f(t) = e^{-\frac{1}{4}(p_1+p_2+2)b} f(te^{-b/2}) \tag{30}$$

For $\delta(t-t')$ we take the following representation :

$$\delta(t-t') = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma\, e^{\frac{\sigma}{2}(t^2-t'^2)}\, t^{\rho} t'^{1-\rho} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma\, e^{-\frac{\sigma}{2}t'^2}\, e^{-4\sigma T_3}\, t^{\rho} t'^{1-\rho} \tag{31}$$

where ρ is a parameter chosen to make $T_1^{\pm}\, t^{\rho} = 0$.

$$\rho = \frac{1}{2} \mp i|k| - \frac{1}{2}(p_1 + p_2) \tag{32}$$

To find the action of $\exp[-icT_1^{\pm}]$ on $\delta(t-t')$ we use the above representation and formulae (28) and (30):

$$\exp(-icT_1^{\pm})\,\exp(-4\sigma T_3)t^{\rho} = \exp\left\{\frac{-\sigma t^2}{2(1+2i\sigma c)}\right\}(1+2i\sigma c)^{-\rho-\frac{1}{2}(p_1+p_2+2)}\,t^{\rho} \qquad (33)$$

with this result we have

$$e^{-icT_1^{\pm}}\,\delta(t-t') = \frac{t'^{1-\rho}\,t^{\rho}}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma\,\exp\left\{\frac{-\sigma t^2}{2(1+2i\sigma c)} - \frac{\sigma}{2}t'^2\right\}(1+2i\sigma c)^{-\rho-\frac{1}{2}(p_1+p_2+2)} \qquad (34)$$

Expanding the exponential in t and integrating in σ we get

$$e^{-icT_1^{\pm}}\,\delta(t-t') = \frac{(-i)^{\nu_{\pm}+1}}{2c}\,t\,\left(\frac{t'}{t}\right)^{\frac{1}{2}(p_1+p_2+2)}\exp\left\{-\frac{i}{4c}(t^2+t'^2)\right\}J_{\nu_{\pm}}\left(-\frac{tt'}{2c}\right)$$

$$\nu_{\pm} = \frac{1}{2} \mp i|k| \qquad (35)$$

where $J_{\nu}(z)$ is the type I Bessel function [13].

(23) and (35) we obtain the $G_k^{\pm}$'s as :

$$G_k^{\pm}(x,x') = -\frac{(tt')^{\frac{1}{2}(p_1+p_2)}}{2}\int_0^{\infty} ds\,\frac{e^{-i[(1-p_1)k_1^2+(1-p_2)k_3^2+m^2]s}}{s} \times \exp\left\{-\frac{i}{4s}(t^2+t'^2)\right\}\,I_{\nu_{\pm}}\left[i\frac{tt'}{2s}\right]$$

$$\nu_{\pm} = \frac{1}{2} \mp i|k| \qquad (36)$$

where $I_{\nu}(z)$ is the modified type I Bessel function obtained by analytic continuation of the ordinary Bessel function:

$$I_{\nu}(z) = (-i)^{\nu}\,J_{\nu}(iz) \qquad (37)$$

Now ,using formulae (19) , (12) and (10) we regain $S(x,x')$ in a form useful for the calculation of effective actions, expectation values , etc.

The above result resembles the one found by Charach [10] in his paper on scalar fields propagating in the Bianchi Type-I models background, where he utilized the Schwinger-DeWitt representation and path integral techniques to obtain the scalar field kernels.

V. CONCLUSIONS

Using a BCH formula and the SO(2,1)~SU(1,1) Lie algebra , avoiding the need of solving differential equations, we evaluated the exact kernels of the Schwinger-De Witt representation for Dirac particles in the background of the Bianchi Type-I models. It may be noted that the addition theorem for the Bessel functions [9] may be used to separate the s-integration in (36) which can be done once a path of integration in the complex s-plane has been specified. This allows also the deduction of normalized wave functions. This procedure has been used in non-relativistic problems , where the $i\varepsilon$ prescription works [5].

In our case it is not clear what boundary conditions should be used. In any case since the subsequent steps do not really involve algebraic techniques, we have not considered the question of s-integration.

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