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14

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Abstract

Using the non-compact SO(2,1) Lie algebra we obtained the kernels of the conformally coupled, massive scalar field, in some classical background gravitational fields. The method makes use of the Schwinger proper time representation and Baker-Campbell-Hausdorff formulae.

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I. Introductio

In the last years there has been a growing interest in the study of quantum systems described by Hamiltonians linear in the generators of some non-compact Lie algebra, to which one could apply algebraic techniques to obtain the energy spectrum and eigenfunctions. For these systems one could choose a representation together with a complete set of states and a convenient pseudo-rotation, or "tilting", is performed on the Hamiltonian through a non-compact generator of the group, so that the resulting Hamiltonian is diagonal on the chosen states. This procedure gives the spectrum and the eigenstates of the system [1].

An alternative approach is to write the Schwinger [2] representation for the resolvent of the system and use a Baker-Campbell-Hausdorff (BCH) formula to disentangle the exponential that involves the linear combination of the generators of the Lie Algebra. This method has been applied to the relativistic Coulomb problem , that has a SO(2,1) spectrum generating group [1] ,by Mil'shtein and Strakhovenko [3] and to numerous problems of non relativistic quantum mechanics by Boschi-Filho and Vaidya [4][5]. In the case of relativistic systems, Boschi-Filho and Vaidya [6] utilized the Perelomov SO(2,1) coherent states to study the Klein-Gordon Coulomb problem.

In this paper we use the SO(2,1) spectrum generating Lie algebra and BCH formulae to evaluate the kernels of the 3+1D conformally coupled scalar field, propagating in some classical background gravitational fields. The Schwinger representation in this case is [7] ($\hbar=c=1$):

$$G(x,x') = -i\int_0^\infty ds\, \exp[-i(\Delta_{LB}+\tfrac{1}{6}R+m^2)s]\, \frac{1}{\sqrt{-g}}\,\delta^4(x-x')\ . \qquad (1)$$

Where $g=\det g_{\mu\nu}$, R is the Ricci scalar curvature and Δ_{LB} is the Laplace-Beltrami operator,

$$\Delta_{LB} = \frac{1}{\sqrt{-g}}\,\partial_\mu[\sqrt{-g}\, g^{\mu\nu}\partial_\nu] \qquad (2)$$

The coupling to R is included here both to study the conformally invariant equation (m=0), and because analogous terms appear in the equations used to generate higher spin Green functions [7].

For some particular forms of $g_{\mu\nu}$ one can find a convenient separation of variables for the operator $(\Delta_{LB}+\frac{1}{6}R+m^2)$ and identify the generators of the SO(2,1) group. In the next section we explain the algebraic method by calculating $G(x,x')$ for some cosmological backgrounds.

II. Some Examples

a) Radiation dominated universe.

As our first example we consider the case described by the line element

$$ds^2 = dt^2 - t(dx^2+dy^2+dz^2) \qquad (3)$$

$$0 \le t < \infty$$

With this choice we have a Robertson-Walker radiation dominated spacetime, with spatially flat sections [8] and zero scalar curvature. To solve this problem more easily we write (3) using the conformal time $\eta=\sqrt{2t}$ [3]:

$$ds^2 = \frac{\eta^2}{4}(d\eta^2-dx^2-dy^2-dz^2) \qquad (4)$$

$$0 \le \eta < \infty$$

Due to the translational invariance in the spatial sector of (4) we decompose $G(x,x')$ as

$$G(x,x') = \frac{1}{(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, G_k(\eta,\eta') \qquad (5)$$

and $G_k(\eta,\eta')$ satisfies

$$\left[\partial_\eta^{\,2} + \frac{2}{\eta}\partial_\eta + k^2 + \frac{m^2\eta^2}{4}\right]G_k(\eta,\eta') = -\frac{4\delta(\eta-\eta')}{\eta'^{\,2}} \tag{6}$$

We see that the above equation resembles the radial wave equation of the harmonic oscilator of non relativistic quantum mechanics, with zero angular momentum.

To proceed with our calculation we note that the above expression can be written as

$$[T_1 - 2m^2T_3]G_k(\eta,\eta') = -\frac{4\delta(\eta-\eta')}{\eta'^{\,2}} \tag{7}$$

where

$$T_1 = \partial_\eta^{\,2} + \frac{2}{\eta}\partial_\eta \tag{8}$$

$$T_3 = -\frac{1}{8}\eta^2 \tag{9}$$

The above operators , along with

$$T_2 = -\frac{i}{2}(\eta\partial_\eta + \frac{3}{2}) \tag{10}$$

satisfy the commutation relations of the SO(2,1) Lie algebra generators [1]

$$[T_1,T_2]=-iT_1 \quad , \quad [T_2,T_3]=-iT_3 \quad , \quad [T_1,T_3]=-iT_2 \tag{11}$$

In terms of these operators the Schwinger representation (1) takes the form

$$G_k(\eta,\eta') = -i\int_0^\infty ds\; e^{-is(T_1-2m^2T_3+k^2)}\;\frac{\delta(\eta-\eta')}{\eta'^{\,2}} \tag{12}$$

Our task now is to find the action of the exponential of this linear combination of operators on the delta function. To do this we use the following two BCH formulae , that can be obtained by use of a faithful representation of SO(2,1) constructed with the Pauli matrices σ_i [4]:

$$T_1 = \frac{\sigma_1 - i\sigma_2}{2\sqrt{2}} \qquad T_2 = \frac{-i\sigma_3}{2} \qquad T_3 = \frac{\sigma_1 + i\sigma_2}{2\sqrt{2}} \tag{13}$$

To obtain the BCH formulae we use the above representation in the following equations

$$\exp[-i(g_1T_1 + g_3T_3)s] = \exp(-iaT_3)\exp(-ibT_2)\exp(-icT_1) \tag{14}$$

$$\exp(-icT_1)\exp(pT_3) = \exp(-ia_1T_3)\exp(-ib_1T_2)\exp(-ic_1T_1) \tag{15}$$

Expanding both sides of the above formulae and identifying the **coefficients** we have:

$$a=2\frac{\xi}{g_1}\,tg(\xi s) \qquad b=2Ln[\cos(\xi s)] \qquad c=\frac{g_1}{\xi}\,tg(\xi s)$$

$$\xi=(g_1 g_3/2)^{1/2} \tag{16}$$

$$a_1=ip(1-ipc/2)^{-1} \quad b_1=2Ln(1-ipc/2) \quad c_1=c(1-ipc/2)^{-1}$$

We need also the action of $\exp[-ibT_2]$ on arbitrary functions of η, which can be obtained by a simple Taylor expansion:

$$\exp[-ibT_2]f(\eta) = e^{-\frac{3}{4}b}\,f(\eta e^{-b}) \tag{17}$$

Let us write the Laplace transform of $\delta(\eta-\eta')$:

$$\delta(\eta-\eta') = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma\;\eta' e^{\frac{\sigma}{2}(\eta^2-\eta'^2)} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma\;\eta' e^{-\frac{\sigma}{2}\eta'^2}\,e^{-4\sigma T_3} \tag{18}$$

To find the action of $\exp[-icT_1]$ on $\delta(\eta-\eta')$ we use the above representation and the result obtained from formulae (15) and (17) :

$$\exp(-icT_1)\,\exp(-4\sigma T_3)\mathbb{1} = \exp\left\{-\frac{\sigma\eta^2}{(1+i2\sigma c)}\right\}(1+i2\sigma c)^{-3/2} \tag{19}$$

with this result we have

$$e^{-icT_1}\,\delta(\eta-\eta') = \frac{\eta'}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma\;\frac{\exp\left[\frac{\sigma\eta^2}{2(1+i2\sigma c)} - \frac{\sigma\eta'^2}{2}\right]}{(1+i2\sigma c)^{3/2}} \tag{20}$$

Expanding the exponential in η and integrating in σ we get

$$e^{-icT_1}\,\frac{\delta(\eta-\eta')}{\eta'^2} = \frac{-\sqrt{i}}{4c}\,(\eta\eta')^{-1/2}\,e^{-\frac{i}{4c}(\eta^2+\eta'^2)}\,J_{1/2}\left[-\frac{\eta\eta'}{2c}\right] \tag{21}$$

where $J_\nu(z)$ is the type I Bessel function [9].

Using equs. (14) and (17) we obtain the Green function as

$$G_k(\eta,\eta') = \frac{2im}{\sqrt{\eta\eta'}} \int_0^\infty ds\, e^{-isk^2} \frac{e^{-i\frac{m}{4}(\eta^2+\eta'^2)\coth(ms)}}{\sinh(ms)} I_{1/2}\left[\frac{im\eta\eta'}{2\sinh(ms)}\right] \tag{22}$$

where $I_\nu(z)$ is the modified type-I Bessel function obtained by analytic continuation of the ordinary Bessel function:

$$I_\nu(z) = (-i)^\nu J_\nu(iz) \tag{23}$$

The above integral representation do agree with the one found by Peak and Inomata [10] for the radial harmonic oscillator with l=0 . A similar calculation was done by Duru and Ünal through path integrals [11], but for an extended manifold with $-\infty<\eta<+\infty$, that is why their result is somewhat different of ours.

b) Bianchi type-I universes.

Let us imagine now that our fields are propagating on the classical background of linearly expanding Bianchi Type-I universes [12]. The metrics of these cosmological models are of the form

$$ds^2 = dt^2 - t^2dx^2 - t^{2p_1}dy^2 - t^{2p_2}dz^2 \tag{24}$$

$$0 \le t < \infty$$

where p_1 and p_2 are constant parameters equal to 1 or 0. The case $p_1=p_2=1$ is the linearly expanding , spatially flat, Robertson-Walker universe. Both this model as well the anisotropic model with $p_1=1$, $p_2=0$ have a curvature singularity at t=0. The case $p_1=p_2=0$ corresponds to the degenerate Kasner universe, that is flat and singularity free.

As in the previous model we have translational invariance in the spatial sector, this means that we can decompose G(x,x') as in (5). The function $G_k(t,t')$ therefore satisfies:

$$\left[\partial_t^{\,2} + (1+p_1+p_2)\frac{1}{t}\partial_t + \frac{1}{t^2}k_x^{\,2} + \frac{1}{t^{2p_1}}k_y^{\,2} + \frac{1}{t^{2p_2}}k_z^{\,2} + m^2 + \frac{\gamma}{t^2}\right]G_k(t,t')$$

$$= -t^{-(1+p_1+p_2)}\delta(t-t') \tag{25}$$

where γ is a constant determined by p_1 and p_2 :

$$\gamma = \frac{(p_1 p_2 + p_1 + p_2)}{3} \tag{26}$$

we can identify in (25) the generator T_1

$$\partial_t^{\,2} + (1+p_1+p_2)\frac{1}{t}\partial_t + \frac{1}{t^2}(k_x^{\,2} + p_1 k_y^2 + p_2 k_z^2 + \gamma) \tag{27}$$

For this generator we have

$$T_2 = -\frac{i}{2}\, t\partial_t - \frac{i}{4}(2+p_1+p_2) \quad , \qquad T_3 = -\frac{1}{8}\, t^2 \tag{28}$$

In this case we proceed as in the previous example, but with a modified representation for the delta function:

$$\delta(t-t') = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma \quad e^{\sigma(t^2-t'^2)}\; t^{\rho}\; t'^{\,1-\rho} \tag{29}$$

We have chosen this representation in order to have T_3 in the exponential, and ρ is a parameter chosen to make $T_1 t^{\rho}=0$.

By the same steps of the previous example we obtain:

$$G_k(t,t') = -\frac{i}{2}(tt')^{-(p_1+p_2)/2}\int_0^{\infty} ds\; e^{-is\mu^2}\; \frac{e^{\frac{-i}{4s}(t^2+t'^2)}}{s}\; I_\nu\!\left(\frac{itt'}{2s}\right)$$

where $\nu = \frac{1}{2}[(p_1+p_2)^2 - 4(k_x^2 + p_1 k_y^2 + p_2 k_z^2 + \gamma)]^{1/2}$

(30)

and $\qquad \mu^2 = m^2 + (1-p_1)k_y^2 + (1-p_2)k_z^2$

The above results agree with previous calculations utilizing path integrals [12] and the Schwinger-DeWitt proper time method [13].

The connection of the above Green functions with the Feynman propagator must be discussed for each particular form of $g_{\mu\nu}$, a convenient path of integration must be chosen in the complex s plane for each case. For a discussion of this kind see Buchbinder et al [7], where some of the models treated in this paper are discussed.

III. Conclusion

In this paper we obtained the Green functions of the conformally coupled scalar field , propagating in the classical background of some model universes. The method utilized here, is an adaptation of Lie algebraic techniques applied before to the relativistic Coulomb problem and to many other problems of ordinary quantum mechanics.

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