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THE SPECTRUM OF AN ANISOTROPIC RING-SHAPED OSCILLATOR VIA WIGNER-HEISENBERG ALGEBRAIC TECHNIQUE

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The Wigner-Heisenberg technique is applied to the potential $ar^2 + b(r\sin\vartheta)^{-2} + s(r\cos\vartheta)^{-2} + dr^2 \sin^2\vartheta + er^2\cos^2\vartheta$. The spectral resolution is exactly obtained. The supersymmetric quantum mechanics algebra is realized in terms of ladder operators of the Wigner Hamiltonian system for this problem.

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Recently, using an algebraic technique related to the SO(2,1) group Boschi Filho and Vaidya (BFV) showed that the potential

$$V(r) = ar^2 + b(r\sin\vartheta)^{-2} + s(\cos\vartheta)^{-2} + dr^2 \sin^2\vartheta + er^2\cos^2\vartheta \qquad (1)$$

possesses SO(2,1) ⊗ SO(2,1) dynamical symmetry [1], and calculated the Green function, energy spectrum and the wave functions. This potential has as particular cases the ring-shaped oscillator, discussed by Quesne [2], and the double ring-shaped oscillator for which the spectral resolution was obtained by Carpio-Bernido and Bernido using the path integral technique [3] and the SO(2,1) algebraic approach [4].

We will show in this letter that Jayaraman and Rodrigues (JR) operator method [5] , which is based on general Wigner-Heisenberg (WH) oscillator algebra [6-10], can be applied to provide an easy spectral resolution of the potential (1). It is convenient to use the circular cylindrical coordinates ρ , ϕ and z so that the time-independent Schrödinger equation associated to the above potential becomes

$$H\,\psi(z,\rho,\varphi) = E\,\psi(z,\rho,\varphi) \quad , \qquad (2a)$$

$$H = H_1 + H_2 \quad , \qquad (2b)$$

$$\psi(z,\rho,\varphi) = e^{im\varphi}\,\chi(z)\,R(\rho) \quad , \qquad (3)$$

$$H_1 = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial z^2} + \frac{1}{2}M\omega_1^2 z^2 + \frac{\hbar^2}{2Mz^2}(\mu^2 - 1/4) \quad , \qquad (4)$$

$$H_2 = -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial}{\rho\,\partial\rho}\right) + \frac{M\,\omega_2^{\,2}}{2}\rho^2 + \frac{\hbar^2}{2M\rho^2}\nu^2 \tag{5}$$

In equation (4) and (5) we have defined

$$\nu = \left(m^2 + \frac{2Mb}{\hbar^2}\right)^{1/2} \tag{6.a}$$

and

$$\mu = \frac{1}{2}\left(1 + \frac{8Mc}{\hbar^2}\right)^{1/2} . \tag{6.b}$$

Following the operator technique developed by Jayaraman and Rodrigues [5], we begin with the super-realized mutually adjoint operators:

$$A_1^{\pm} = A_1^{\pm}\left(\mu + \frac{1}{2}\right) \equiv \frac{1}{\sqrt{2\,M\,\hbar\,\omega_1}}\,\Sigma_1 \times$$

$$\times\left\{\pm\hbar\frac{\partial}{\partial z} \mp \frac{\hbar}{2z}(1 + 2\mu)\Sigma_3 - M\,\omega_1 z\right\} . \tag{7.a}$$

and

$$A_2^{\pm} = A_2^{\pm}\left(\nu + \frac{1}{2}\right) \equiv \frac{1}{\sqrt{2\,M\,\hbar\,\omega_2}}\,\Sigma_1 \times$$

$$\times\left\{\pm\hbar\left[\frac{\partial}{\partial\rho} + \frac{1}{2\rho}\right] \mp \frac{\hbar}{2\rho}(1 + 2\nu)\Sigma_3 - M\,\omega_2\,\rho\right\} , \tag{7.b}$$

which will be interpreted as annihilation and creation operators. From the algebra of Pauli matrices, Σ_i (i=1,2,3), and canonical commutation relations between coordinates and momenta, we obtain the following Wigner Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left[A_1^{+}, A_1^{-}\right]_+ \omega_1\,\hbar + \frac{1}{2}\left[A_2^{+}, A_2^{-}\right]_+ \omega_2\,\hbar \tag{8.a}$$

$$= \begin{bmatrix} \mathcal{H}_{1-}(\mu - 1/2) + \mathcal{H}_{2-}(\nu - 1/2) & 0 \\ 0 & \mathcal{H}_{1+}(\mu - 1/2) + \mathcal{H}_{2+}(\nu - 1/2) \end{bmatrix} . \tag{8.b}$$

where the Hamiltonians of the bosonic and fermionic sectors are respectively given by

$$\mathcal{H}_{1-}(\mu - 1/2) = H_1 , \tag{9.a}$$

$$\mathcal{H}_{2-}(\nu - 1/2) = H_2 , \tag{9.b}$$

$$\mathcal{H}_{1+}(\mu - 1/2) = \mathcal{H}_{1-}(\mu + 1/2) \quad , \tag{10.a}$$

and

$$\mathcal{H}_{2+}(\nu - 1/2) = \mathcal{H}_{2-}(\nu + 1/2) \quad . \tag{10.b}$$

The WH algebra ladder relations for this super-realization are given by

$$\left[\mathcal{H} \, , \, A^{\pm} \right]_{-} = \pm\omega_1 \hbar \, A_1^{\pm} \pm \omega_2 \hbar \, A_2^{\pm} \, , \tag{11}$$

where

$$A^{\pm} = A_1^{\pm} + A_2^{\pm} \quad . \tag{12}$$

It easy to see that the general quantum commutation relation for A^+ and A^- can be written in the form

$$\left[A^- \, , \, A^+ \right]_{-} = 2\{1 + (\nu + \mu)\Sigma_3\} \tag{13}$$

From the annihilation condition

$$A^- \, \Psi^{(0)} \, (z,\rho) = 0 \quad . \tag{14.a}$$

$$\Psi^{(0)} \, (z,\rho) = \begin{bmatrix} \psi^{(0)}(z,\rho) \\ 0 \end{bmatrix} \quad , \tag{14.b}$$

we obtain the energy and eigenfunction of the ground state, viz,

$$\Psi^{(0)} \, (z,\rho) = \begin{pmatrix} \chi^{(0)}(z) \, R^{(0)}(\rho) \\ 0 \end{pmatrix} = z^{\mu+1/2} \, \rho^{\nu} \quad \times$$

$$\times \exp\left\{ - \frac{M}{2\hbar} \left[\omega_1 z^2 + \omega_2 \rho^2 \right] \right\} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{15}$$

and

$$\varepsilon^{(0)} = \varepsilon_1^{(0)} + \varepsilon_2^{(0)} = (1 + \mu) \, \hbar \, \omega_1 + (1 + \nu) \, \hbar \, \omega_2 \quad . \tag{16}$$

This value of the ground-state energy is readily obtained with the following form for the Wigner Hamiltonian:

$$\mathcal{H} = \left\{ A_1^+ \, A_1^- + \frac{1}{2} \left[1 + (1 + 2 \mu) \, \Sigma_3 \right] \right\} \hbar \, \omega_1 +$$

$$+ \left\{ A_2^+ \, A_2^- + \frac{1}{2} \left[1 + (1 + 2 \nu) \, \Sigma_3 \right] \right\} \hbar \, \omega_2 \quad . \tag{17}$$

Once $\Psi^{(0)}(z,\rho)$ is determined, the whole set of eigenfunctions can be built up by means of the two operators A_i^+, $i=1,2$.

Now, we consider the complete resolution of the even quantum

eigenvalue problem associated with the Wigner Hamiltonian, which has as bosonic sector which corresponds exataly to the anisotropic ring-shaped oscillator Hamiltonian. The eigenvalue equation

$$\mathcal{H}\ \Psi^{(2m_1,2m_2)}(z,\rho) = \varepsilon^{(2m_1,2m_2)}\ \Psi^{(2m_1,2m_2)}(z,\rho) \quad . \tag{18}$$

can be solved to give

$$\Psi^{(2m_1,2m_2)}(z,\rho) \propto (A_1^{+})^{(2m_1)}\ (A_2^{+})^{(2m_2)} \begin{bmatrix} \psi^{(0)}(z,\rho) \\ 0 \end{bmatrix} \tag{19}$$

and

$$\varepsilon^{(2m_1,2m_2)} = \varepsilon^{(0)} + 2\ m_1\hbar\ \omega_1 + 2\ m_2\ \hbar\ \omega_2 \quad , \ m_{1,2} = 0,1,2,\ldots \tag{20}$$

Substituting in (18) the creation operators given by (7.a,b), and the ground-state wave function deduced before , we get

$$\Psi^{(2m_1,2m_2)}(z,\rho) = \begin{bmatrix} \psi^{(m_1,m_2)}(z,\rho) \\ 0 \end{bmatrix} \quad , \tag{21}$$

where

$$\psi^{(m_1,m_2)}(z,\rho) = \chi^{(m_1)}(z)\ R^{(m_2)}(\rho) \propto z^{(\mu + 1/2)} \times$$

$$\times \exp\left[- \frac{M\omega_1}{2\hbar} z^2 \right] L^{\mu}_{m_1}\left[\frac{M\omega_1}{\hbar} z^2 \right] \rho^{\nu} \times$$

$$\times \exp\left[- \frac{M\omega_2}{2\hbar} \rho^2 \right] L^{\nu}_{m_2}\left[\frac{M\omega_2}{\hbar} \rho^2 \right] \quad , \tag{22}$$

which are exactly the z and ρ dependent eigenfunction of the anisotropic ring-shaped oscillator. From (3), (16) and (20) we obtain the complete energy spectrum of this system, viz,

$$E^{(m_1,m_2)} = \varepsilon_{-}^{(m_1,m_2)} = \{(1 + \mu + 2m_1)\omega_1 + (1 + \nu + 2m_2)\omega_2\}\hbar \ . \tag{23}$$

Calling ψ_{BFV} the normalized eigenfunction obtained in ref.[1] we have the relation

$$\psi_{BFV} \propto \psi \ .$$

We have applied the WH algebraic technique developed by JR [5] for simpler spectral resolution of the anisotropic ring-shaped oscillator. We have obtained the complete energy spectrum and

unormalized eigenfunctions, which are identical to those obtained by BFV [1], up to multiplicative factor which is the normalization constant. Since the Witten construction of supersymmetric quantum mechanics (SUSYQM) formalism [11] one has seen several applications of it as an operator technique for the spectral resolution of non-relativistic quantum systems [12], and as well as to a relativistic eigenvalue problem [13]. The factorization of the Wigner Hamiltonian (17) provides us a formulation of supersymmetry (SUSY) for the anisotropic ring-shaped oscillator. Here we will only present the realization of SUSYQM algebra by corresponding generators in terms of ladder operators of the Wigner system

$$Q_{i-} = \frac{1}{2}\left[1 - \sum_3\right] A_i^- \qquad (24.a)$$

and

$$Q_{i+} = \frac{1}{2}\left[1 + \sum_3\right] A_i^+ \quad (i=1,2) \quad . \qquad (24.b)$$

From (12), (13), (17) and (24.ab) it is trivial to check that the super-charges satisfy the following relations,

$$H_{SUSY} = \sum_{i=1}^{2} \left[Q_{i-}, \; Q_{i+} \right]_+ \quad , \qquad (25)$$

$$\left[Q_{i\pm} \; , \; H_{SUSY} \right]_- = Q_{i-}^{\;2} = Q_{i+}^{\;2} = 0 \; , \; (i=1,2). \qquad (26)$$

which constitute the SUSYQM algebra.

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