IEKP-KA/93-14

A Convenient Parametrization for the General Matrix Element of Leptonic Tau Decays

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8 July 1993

Abstract

The matrix element for the leptonic decay of polarized τ leptons is calculated for all possible four-fermion couplings and non vanishing ν_τ mass. A Lorentz invariant expression is presented, which leads to the well known Michel formula with minimal simplifications. An implementation of the Michel parameters in the τ decay library TAUOLA is provided. In addition, for the $V - A$ interaction the version of TAUOLA 1.5 is compared to the version 2.4 with respect to the difference of the ansatz for the radiative corrections in leptonic τ decays.

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The most general matrix element of leptonic τ decay for the four-fermion-point interaction is given by [1]

$$\mathcal{M} = 4\frac{G_0}{\sqrt{2}} \sum_{\substack{\gamma=S,V,T \\ \epsilon,\mu=R,L}} g^{\gamma}_{\epsilon\mu} < \bar{l}_{\epsilon}|\Gamma^{\gamma}|(\nu_l) >< (\bar{\nu}_{\tau})|\Gamma_{\gamma}|\tau_{\mu} > \quad . \tag{1}$$

The Γ^{γ} matrices define the properties under Lorentz transformations of the two currents, where $\gamma = S, V, T$ means scalar, vector and tensor couplings. The right- or left-handedness (R or L) of the τ lepton and lepton l (electron or muon) are fixed by the indices μ and ϵ. From the twelve complex coupling constants $g^{\gamma}_{\epsilon\mu}$ only ten are relevant. The T_{LL} and the T_{RR} couplings are, due to linear dependency, excluded. The total strength of the interaction is specified by the constant G_0. So the $g^{\gamma}_{\epsilon\mu}$ give only the relative strengths of the different interactions and are normalized by

$$\begin{aligned} A \equiv\ & 4\left\{|g^S_{LL}|^2 + |g^S_{LR}|^2 + |g^S_{RL}|^2 + |g^S_{RR}|^2\right\} + 16\left\{|g^V_{LL}|^2 + |g^V_{LR}|^2 + |g^V_{RL}|^2 + |g^V_{RR}|^2\right\} \\ & +48\left\{|g^T_{LR}|^2 + |g^T_{RL}|^2\right\} := 16 \quad . \end{aligned} \tag{2}$$

This gives 19 independent real parameters, which have to be measured to determine the specific kind of the interaction. Since the neutrinos are not detectable and the polarization of the outgoing electron or muon is not easily measurable in present day experiments, only the four Michel parameters ρ, η, ξ and δ remain.

In the Standard Model, where the leptonic τ decay takes place exclusively via the W boson, the phase of the coupling constant can be chosen arbitrarily. No other interaction is allowed and consequently no interference between other interactions can occur. The absolute value of the coupling constant is, due to the normalization, fixed to 1. So the Standard Model is represented in formula (1) by $g^V_{LL} = 1$ and all other couplings being zero.

Some extensions of the Standard Model predict charged Higgs bosons. This would lead to interferences between amplitudes mediated by Higgs bosons and the W boson and could be seen in the differential decay rate of the leptonic τ decay. This effect would change the low energy part of the muon spectrum. For the electrons this effect would be roughly $5 \cdot 10^{-3}$ times smaller, because this effect scales with the mass of the emitted lepton. Not only the isotropic part of the spectrum would be affected by the existence of these bosons but the anisotropic part of the spectrum, too, which depends on the energy of the emitted lepton and the angle between the direction of flight of this particle and the polarization vector of the τ lepton.

Squaring formula (1) for a given polarization of the τ lepton and a finite ν_{τ} mass gives:

$$
\begin{aligned}
&\left|\mathcal{M}\left(\tau^{\mp} \to e^{\mp}\bar{\nu}\nu\right)\right|^2 = 32G_0^2\cdot \\
&\Big[\left\{4\Re(g^S_{LR}g^{T*}_{LR}) + 4\Re(g^S_{RL}g^{T*}_{RL}) + 4|g^V_{LR}|^2 + 4|g^V_{RL}|^2 + 8|g^T_{LR}|^2 + 8|g^T_{RL}|^2\right\}(qp)(hk) \\
&+ \left\{|g^S_{LL}|^2 + |g^S_{LR}|^2 + |g^S_{RL}|^2 + |g^S_{RR}|^2 - 4|g^T_{LR}|^2 - 4|g^T_{RL}|^2\right\}(qh)(pk) \\
&+ \left\{4|g^V_{LL}|^2 + 4|g^V_{RR}|^2 - 4\Re(g^S_{LR}g^{T*}_{LR}) - 4\Re(g^S_{RL}g^{T*}_{RL}) + 8|g^T_{LR}|^2 + 8|g^T_{RL}|^2\right\}(qk)(ph) \\
&+ \left\{2\Re(g^S_{LL}g^{V*}_{RR}) + 2\Re(g^S_{LR}g^{V*}_{RL}) + 2\Re(g^S_{RL}g^{V*}_{LR}) + 2\Re(g^S_{RR}g^{V*}_{LL}) + 12\Re(g^V_{LR}g^{T*}_{RL}) + 12\Re(g^V_{RL}g^{T*}_{LR})\right\} \\
&\quad m_\tau m_l(hk) \\
&+ \left\{2\Re(g^S_{LL}g^{S*}_{LR}) + 2\Re(g^S_{RR}g^{S*}_{RL}) - 4\Re(g^V_{LL}g^{V*}_{LR}) - 4\Re(g^V_{RR}g^{V*}_{RL})\right\} m_{\nu_\tau}m_\tau(pk) \\
&+ \left\{2\Re(g^S_{LL}g^{V*}_{RL}) + 2\Re(g^S_{RR}g^{V*}_{LR}) + 2\Re(g^V_{LL}g^{S*}_{RL}) + 2\Re(g^V_{RR}g^{S*}_{LR}) - 12\Re(g^V_{LL}g^{T*}_{RL}) - 12\Re(g^V_{RR}g^{T*}_{LR})\right\} \\
&\quad m_{\nu_\tau}m_l(qk) \\
&\pm \left\{4\Re(g^S_{LR}g^{T*}_{LR}) - 4\Re(g^S_{RL}g^{T*}_{RL}) + 4|g^V_{LR}|^2 - 4|g^V_{RL}|^2 + 8|g^T_{LR}|^2 - 8|g^T_{RL}|^2\right\} m_\tau(hk)(\omega p) \\
&\pm \left\{|g^S_{RR}|^2 + |g^S_{LR}|^2 - |g^S_{RL}|^2 - |g^S_{LL}|^2 + 4|g^T_{RL}|^2 - 4|g^T_{LR}|^2\right\} m_\tau(pk)(\omega h) \\
&\pm \left\{4|g^V_{RR}|^2 - 4|g^V_{LL}|^2 + 4\Re(g^S_{RL}g^{T*}_{RL}) - 4\Re(g^S_{LR}g^{T*}_{LR}) + 8|g^T_{LR}|^2 - 8|g^T_{RL}|^2\right\} m_\tau(ph)(\omega k) \\
&\pm \left\{2\Re(g^S_{LL}g^{V*}_{RR}) - 2\Re(g^S_{LR}g^{V*}_{RL}) + 2\Re(g^S_{RL}g^{V*}_{LR}) - 2\Re(g^S_{RR}g^{V*}_{LL}) + 4\Re(g^V_{RL}g^{T*}_{LR}) - 4\Re(g^V_{LR}g^{T*}_{RL})\right\} \\
&\quad m_l(qh)(k\omega) \\
&\mp \left\{2\Re(g^S_{LL}g^{V*}_{RR}) - 2\Re(g^S_{LR}g^{V*}_{RL}) + 2\Re(g^S_{RL}g^{V*}_{LR}) - 2\Re(g^S_{RR}g^{V*}_{LL}) + 4\Re(g^V_{RL}g^{T*}_{LR}) - 4\Re(g^V_{LR}g^{T*}_{RL})\right\} \\
&\quad m_l(qk)(h\omega) \\
&\pm \left\{4\Re(g^S_{RR}g^{T*}_{RL}) - 4\Re(g^S_{LL}g^{T*}_{LR}) + 4\Re(g^V_{LL}g^{V*}_{LR}) - 4\Re(g^V_{RR}g^{V*}_{RL})\right\} m_{\nu_\tau}(qp)(k\omega) \\
&\mp \left\{4\Re(g^S_{RR}g^{T*}_{RL}) - 4\Re(g^S_{LL}g^{T*}_{LR}) + 4\Re(g^V_{LL}g^{V*}_{LR}) - 4\Re(g^V_{RR}g^{V*}_{RL})\right\} m_{\nu_\tau}(qk)(p\omega) \\
&\pm \left\{2\Re(g^S_{RR}g^{V*}_{LR}) - 2\Re(g^S_{LL}g^{V*}_{RL}) + 2\Re(g^V_{RR}g^{S*}_{LR}) - 2\Re(g^V_{LL}g^{S*}_{RL}) + 12\Re(g^V_{LL}g^{T*}_{RL}) - 12\Re(g^V_{RR}g^{T*}_{LR})\right\} \\
&\quad m_{\nu_\tau}m_\tau m_l(k\omega) \\
&+ \left\{4\Im(g^S_{LR}g^{T*}_{LR}) + 4\Im(g^S_{RL}g^{T*}_{RL})\right\} m_\tau\epsilon^{\alpha\beta\gamma\delta}p_\alpha k_\beta h_\gamma\omega_\delta \\
&+ \left\{2\Im(g^S_{LL}g^{V*}_{RR}) + 2\Im(g^S_{LR}g^{V*}_{RL}) + 2\Im(g^S_{RL}g^{V*}_{LR}) + 2\Im(g^S_{RR}g^{V*}_{LL}) + 4\Im(g^V_{LR}g^{T*}_{RL}) + 4\Im(g^V_{RL}g^{T*}_{LR})\right\} \\
&\quad m_l\epsilon^{\alpha\beta\gamma\delta}k_\alpha h_\beta q_\gamma\omega_\delta \\
&+ \left\{4\Im(g^S_{LL}g^{T*}_{LR}) + 4\Im(g^S_{RR}g^{T*}_{RL}) - 4\Im(g^V_{LL}g^{V*}_{LR}) - 4\Im(g^V_{RR}g^{V*}_{RL})\right\} m_{\nu_\tau}\epsilon^{\alpha\beta\gamma\delta}p_\alpha k_\beta q_\gamma\omega_\delta\Big]
\end{aligned}
\tag{3}
$$

with

q^μ :	momentum of the τ lepton
h^μ :	momentum of the τ neutrino ν_τ
p^μ :	momentum of the electron or muon
k^μ :	momentum of the electron- or muon neutrino ν_l
ω^μ :	polarization vector of the τ lepton

For the charge conjugated decay one has to change the sign of the terms, which are antisymmetric under the exchange of the L and R indices. These terms have two possible signs in formula (3), where the upper sign always corresponds to the τ^- decay and the lower sign to the τ^+ decay. This convention is valid for all following formulae. The polarization vector ω^μ of the τ lepton gives the expected direction of the τ spin and fulfils the conditions $(\omega q) = 0$ and $(\omega\omega) = -1$.

Six of the 17 terms in the sum of equation (3) vanish by setting m_{ν_τ} equal to zero, which is done in all following formulae. The integration over the momenta of the two neutrinos yields [2]:

$$\int \frac{d^3h}{2E_{\nu_\tau}} \frac{d^3k}{2E_{\nu_l}} h^\alpha k^\beta \, \delta^4(Q-h-k) = \frac{\pi}{24}\left(g^{\alpha\beta}Q^2 + 2Q^\alpha Q^\beta\right) \quad \text{with} \quad Q = q - p \tag{4}$$

Due to the symmetry of this tensor all terms in equation (3) with the ϵ tensor drop off and terms, where the momenta of the neutrinos are exchanged between the dot products, become equal. So the integration reduces the remaining terms to five and the partial width of a τ lepton with momentum q^μ and polarization vector ω^μ reads:

$$\begin{aligned} d\Gamma\left(\tau^\mp \to l^\mp \bar{\nu}\, \nu\right) &= \frac{4G_0^2}{3m_\tau(2\pi)^4} \frac{d^3p}{2E_l} \left\{ \left(3 - \frac{10\rho}{3}\right) Q^2(qp) + \frac{4\rho}{3}(Qp)(Qq) \right. \\ &\quad \left. + 3\eta m_\tau m_l Q^2 \pm (\omega p) m_\tau \xi \left(\left(1 - 2\delta\right) Q^2 + \frac{4\delta}{3}(Qp) \right) \right\} \end{aligned} \tag{5}$$

Going into the τ lepton restframe and neglecting terms, which are quadratic in m_l, one can verify, that equation (5) is the Michel formula [3] . The parameters ρ, η, ξ and δ are the Michel parameters, which are defined by the coupling constants $g^\gamma_{\epsilon\mu}$ in the common way

$$\rho = \frac{3}{4} - \frac{3}{4}\left\{ |g^V_{LR}|^2 + |g^V_{RL}|^2 + 2|g^T_{LR}|^2 + 2|g^T_{RL}|^2 + \Re(g^S_{LR} g^{T*}_{LR} + g^S_{RL} g^{T*}_{RL}) \right\} \tag{6}$$

$$\eta = \frac{1}{2}\Re(6 g^V_{RL} g^{T*}_{LR} + 6 g^V_{LR} g^{T*}_{RL} + g^S_{RR} g^{V*}_{LL} + g^S_{RL} g^{V*}_{LR} + g^S_{LR} g^{V*}_{RL} + g^S_{LL} g^{V*}_{RR}) \tag{7}$$

$$
\begin{aligned}
\xi &= 4\Re(g^S_{LR}g^{T*}_{LR}) - 4\Re(g^S_{RL}g^{T*}_{RL}) + |g^V_{LL}|^2 + 3|g^V_{LR}|^2 - 3|g^V_{RL}|^2 - |g^V_{RR}|^2 \\
&\quad +5|g^T_{LR}|^2 - 5|g^T_{RL}|^2 + \frac{1}{4}|g^S_{LL}|^2 - \frac{1}{4}|g^S_{LR}|^2 + \frac{1}{4}|g^S_{RL}|^2 - \frac{1}{4}|g^S_{RR}|^2 \qquad (8) \\
\xi\delta &= \frac{3}{16}|g^S_{LL}|^2 - \frac{3}{16}|g^S_{LR}|^2 + \frac{3}{16}|g^S_{RL}|^2 - \frac{3}{16}|g^S_{RR}|^2 - \frac{3}{4}|g^T_{LR}|^2 + \frac{3}{4}|g^T_{RL}|^2 \\
&\quad +\frac{3}{4}|g^V_{LL}|^2 - \frac{3}{4}|g^V_{RR}|^2 + \frac{3}{4}\Re(g^S_{LR}g^{T*}_{LR}) - \frac{3}{4}\Re(g^S_{RL}g^{T*}_{RL}) \quad . \qquad (9)
\end{aligned}
$$

These definitions of the Michel parameters together with the normalization condition (2) limit the allowed regions of these parameters, and the boundaries of one parameter depend on the values of the other parameters.

One sees from formula (3), that the matrix element for the decay of a polarized τ lepton can be written in the form:

$$
|\mathcal{M}|^2 = |\overline{\mathcal{M}}|^2 (1 + \omega_\mu H^\mu) \qquad (10)
$$

Here $|\overline{\mathcal{M}}|^2$ denotes the spin averaged matrix element and H^μ is the so called polarimetric vector. This vector carries the full spin information of the decay and its magnitude is a measure for the efficiency of the decay as a polarimeter. For an implementation of formula (3) or (5) in the τ decay library TAUOLA [4, 5, 6] one has to program the spin averaged matrix element and the polarimetric vector following the convention of formula (10). In TAUOLA 1.5 [4] these changes have to be done in the subroutine DPHSEL for the electron and DPHSMU for the muon case. In the versions 2.1 [5] and 2.4 [6] one has to modify the function THB. Formula (3) has the advantage that the whole phase space is covered correctly, but is not very comfortable to handle, because the user of the program has to initialize 19 coupling constants. If one is not interested in the momenta of the neutrinos one can use the more convenient formula (5), where the Michel parameters enter directly. The modified subroutines DPHSEL and DPHSMU for version 1.5 and the function THB for versions 2.1/2.4 may be obtained from the author. In these routines formula (5) is not explicitly coded but

$$
\begin{aligned}
\left|\mathcal{M}\left(\tau^\mp \to l^\mp \bar{\nu}\nu\right)\right|^2 &= 128G_0^2\Bigg[\left(1 - \frac{4\rho}{3}\right)(qp)(kh) + \frac{2\rho}{3}\Big((qh)(pk) + (qk)(ph)\Big) \\
&\quad + \eta m_\tau m_l (kh) \pm m_\tau \frac{\xi}{3} \cdot \\
&\quad \left[\left(1 - \frac{4\delta}{3}\right)(kh)(p\omega) - 2\delta\Big((pk)(h\omega) + (ph)(k\omega)\Big)\right]\Bigg] \quad . \qquad (11)
\end{aligned}
$$

One gets this formula out of formula (3) for $m_{\nu_\tau} = 0$ and keeping in mind that—after the integration over the momenta of the neutrinos—the terms, which are antisymmetric under the exchange of the neutrinos, drop off and the terms, which are symmetric, become equal. This allows to introduce the Michel parameters in a convenient way at the prize of

generating fake neutrinos, which are distributed differently from the real neutrinos in this decay. Using formula (4), it is easily shown that formula (11) yields formula (5). So—from a technical point of view—formula (11) is equivalent to formula (5).

TAUOLA 1.5 simulates the decay of a τ lepton on the Born level and calls internally the program package PHOTOS [7], which generates a radiative photon independently from the Born matrix element of the decay. Evidently this factorization ansatz is an approximation to the real physics. For example the interference between the radiation of the photon from the τ lepton and from the electron cannot be taken into account. But on the other hand it provides radiative corrections for all combinations of couplings on a certain level of accuracy. A more precise treatment of the radiative corrections means loss of generality, since one has to introduce then a specific model of the interaction. Especially one has to know the bosons, which intermediate the interaction. This is done in the new versions 2.1 [5] and 2.4 [6] of TAUOLA for the $V-A$ interaction. In this versions the exact matrix element for a $V-A$ coupling at the τ vertex up to the first order in α is programmed. If the user chooses any other combination of vector to axialvector coupling at the τ vertex as the standard $V-A$ interaction, then the program uses in the Born part the new matrix element, but for the first order in α the matrix element of the standard $V-A$ interaction. The modified function THB works in the same way. This means, if one allows only small deviations of the Michel parameters from their standard $V-A$ values, then the radiative corrections in the new versions are treated more precise than in the old version. The function THB stops with an error message, if the virtual and soft corrections (calculated for $V-A$) exceeds ten per cent of the Born matrix element. This can be regarded as the limit how far one can go away from the $V-A$ interaction in the Born part of the matrix element. In the modified function THB this limit is kept. Therefore the new versions do not allow to choose all possible combinations of the Michel parameters, if the radiative corrections are switched on.

If one does not want to give up the generality of the interaction, one has to live with the approximation of the factorization ansatz. To illustrate the problems of the factorization ansatz, I have compared the TAUOLA version 1.5 with the version 2.4 for the standard $V-A$ interaction. This comparision is shown in figure (1) and (2). Plotted is $dN_{\text{version 1.5}}/dN_{\text{version 2.4}}$. In both versions the same number of τ events were generated. So the deviation of the shown distributions from 1.0 is a measure for the difference of the versions. For the production of the τ events the Monte-Carlo program KORAL-B [8] was used, where a center of mass energy of 10.0 GeV was chosen. Figure (1) corresponds to the τ restframe and figure (2) to the laboratory system. Only events with a radiative photon and an energy of this photon greater than 20 MeV in the τ restframe were filled. Figure (1a) shows, that the energy spectra of the photon in the old and new version are in good agreement. The energy spectra of the electron in the τ restframe (figure 1b) differ systematically at the low energy part of the spectrum. For this part of the spectrum the interference becomes important and the factorization ansatz of version 1.5 must fail. The $\cos(\theta)$ distributions (figure 1c), where θ is the angle between electron and radiated photon, are very different in both versions. This deviation in the $\cos(\theta)$ spectra enlarge the difference between the electron spectra in the laboratory system. Figure (2) shows, that for electron energies smaller than 600 MeV the crosses with the errors are systematically smaller than the expected value of 1.0 and greater for energies above 1.3 GeV. For all these considerations one has to keep in mind that only this part of the energy spectrum

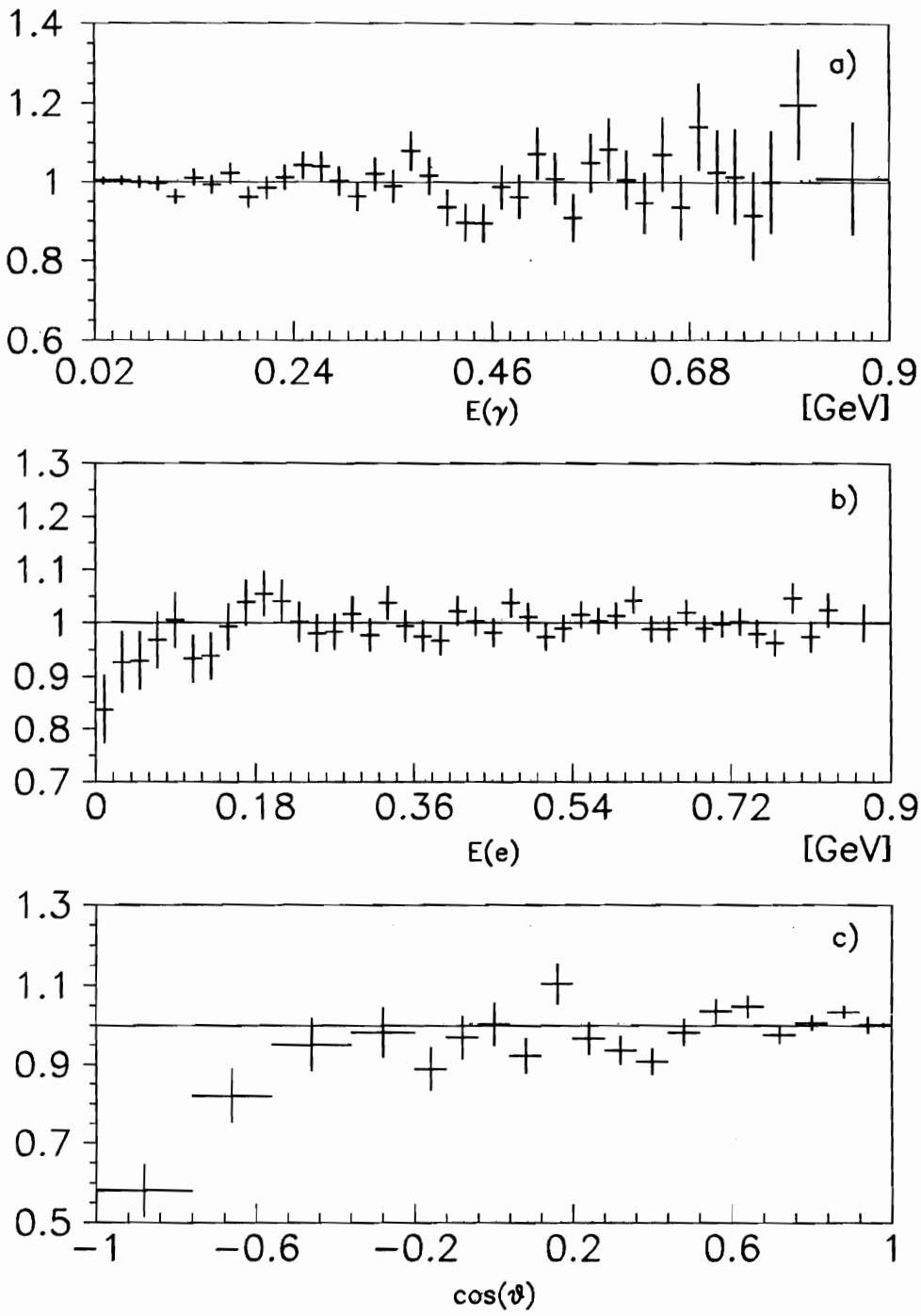

Figure 1: Comparison of the radiative corrections in leptonic τ decay between the old and new version of TAUOLA. Plotted is $dN_{\text{version 1.5}}/dN_{\text{version 2.4}}$. Figure (a) shows the ratio of the energy spectra of the radiated photon, figure (b) the ratio of the energy spectra of the electron and figure (c) the ratio of the $\cos(\theta)$ distributions, where θ is the angle between electron and photon, in the τ restframe.

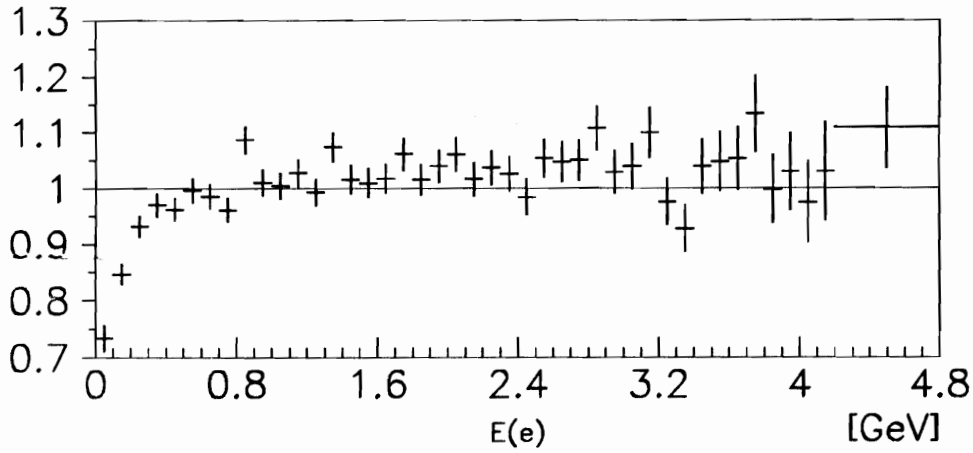

Figure 2: Comparison of the radiative corrections in leptonic τ decay between the old and new version of TAUOLA. Plotted is $dN_{\text{version 1.5}}/dN_{\text{version 2.4}}$ for the electron energy in the laboratory system.

of the electron is plotted, in which a difference can occur. For events without photon the energy spectra of the electron is equal and this will reduce the difference in total.

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