26
FERMILAB
APR 17 1995
LIBRARY

ICN-UNAM-94-08
DIAS-STP-94-42
HD-THEP-94-50

ICN-UNAM-94-08
Fermilab Library
0 1160 0041532 7

The Specific Heat of a Ferromagnetic Film

F. Freire

Inst. für Theor. Phys., Universität Heidelberg

Philosophenweg 16, 69120 Heidelberg, Germany

Denjoe O'Connor

DIAS, 10 Burlington Road, Dublin 4, Ireland

C. R. Stephens

DIAS, 10 Burlington Road, Dublin 4, Ireland

and

Instituto de Ciencias Nucleares, UNAM,

Circuito Exterior, A. Postal 70-543, México D.F. 04510. †

Abstract: We analyze the specific heat for the $O(N)$ vector model on a d-dimensional film geometry of thickness L using "environmentally friendly" renormalization. We consider periodic, Dirichlet and antiperiodic boundary conditions, deriving expressions for the specific heat and an effective specific heat exponent, α_{eff}. In the case of $d = 3$, for $N = 1$, by matching to the exact exponent of the two dimensional Ising model we capture the crossover for $\xi_L \to \infty$ between power law behaviour in the limit $\frac{L}{\xi_L} \to \infty$ and logarithmic behaviour in the limit $\frac{L}{\xi_L} \to 0$ for fixed L, where ξ_L is the correlation length in the transverse dimensions.

M.S.
I.D. No
NAME
RETURN TO FERMILAB LIBRARY

† Permanent address

ICN-UNAM-94-08

DIAS-STP-94-42

HD-THEP-94-50

The Specific Heat of a Ferromagnetic Film

F. Freire

Inst. für Theor. Phys., Universität Heidelberg

Philosophenweg 16, 69120 Heidelberg, Germany

Denjoe O'Connor

DIAS, 10 Burlington Road, Dublin 4, Ireland

C. R. Stephens

DIAS, 10 Burlington Road, Dublin 4, Ireland

and

Instituto de Ciencias Nucleares, UNAM,

Circuito Exterior, A. Postal 70-543, México D.F. 04510. †

Abstract: We analyze the specific heat for the $O(N)$ vector model on a d-dimensional film geometry of thickness L using "environmentally friendly" renormalization. We consider periodic, Dirichlet and antiperiodic boundary conditions, deriving expressions for the specific heat and an effective specific heat exponent, α_{eff}. In the case of $d = 3$, for $N = 1$, by matching to the exact exponent of the two dimensional Ising model we capture the crossover for $\xi_L \to \infty$ between power law behaviour in the limit $\frac{L}{\xi_L} \to \infty$ and logarithmic behaviour in the limit $\frac{L}{\xi_L} \to 0$ for fixed L, where ξ_L is the correlation length in the transverse dimensions.

† Permanent address

§ 1. Introduction

Thermodynamic quantities generally depend on many details of the system, and are therefore functions of a large number of variables, however, in the critical regime this dependence drops to a smaller number. The resulting functions are referred to as scaling functions. Scaling functions generically describe a crossover, wherein the effective degrees of freedom of a system can change dramatically as a function of scale. Calculating such scaling functions in critical phenomena is generally accepted to be much more difficult than calculating critical exponents. From a renormalization group (RG) point of view one can think of this as being due to the fact that to calculate a critical exponent one only needs a local RG linearized around the fixed point of interest whereas, generally speaking, to calculate a scaling function one needs a global, non-linear RG that is capable of encompassing more than one fixed point. One of the chief difficulties in the latter is developing a "uniform" approximation scheme that can describe the crossover between two fixed points perturbatively. Conventional small parameters such as ε and $1/N$ might be adequate for certain crossovers but not others.

Crossovers are induced by some asymmetry parameter which often can be fruitfully thought of as an "environmental" variable, such as temperature, system size, magnetic field etc. The formalism of "environmentally friendly" renormalization [1,2] offers a quite general approach to the solution of crossover problems and the calculation of scaling functions. Given that the key idea behind the notion of a crossover is the qualitatively changing nature of the effective degrees of freedom as a function of "scale" and "environment" it implements a renormalization which is capable of tracking the evolving effective degrees of freedom in a perturbatively controllable manner. However, it is based on reparametrization invariance, as in the original field theoretic RG, rather than Wilson/Kadanoff coarse graining.

The basic idea is that the relation of the bare couplings to the renormalized ones, which can be used to describe, parametrically, a physical system, can be thought of as a coordinate transformation in the space of these couplings. In thinking of the renormalized couplings as new "coordinates" the conventional field theoretic RG simply expresses the invariance of physical quantities under changes of coordinate system. This coordinate invariance is an *exact* invariance of field theory. When calculating a physical quantity perturbatively, in spite of the fact that physics doesn't depend on coordinates, the particular choice of coordinates can be quite crucial in obtaining a reliable approximation scheme. One can understand this clearly in the context of the crossover studied in this paper — dimensional crossover induced by finite size effects.

Reverting for the moment to a coarse graining RG, if we thought of possible coarse grainings in a d dimensional ferromagnetic film of size L, one would find that block spins of size $\xi \ll L$ were d dimensional, whilst those of size $\xi \gg L$ were $d-1$ dimensional. Thus this coarse graining procedure reflects a crucial property of the "environment" of the system — that it is finite in one dimension. Block spinning is therefore an environmentally friendly form of renormalization. In the context of reparametrization, an environmentally friendly renormalization is one that yields a set of parameters that give a perturbatively reliable description of the crossover. In the finite size context a necessary condition for the reparametrization to be environmentally friendly is that it be L dependent.

In previous papers environmentally friendly renormalization has been used to describe various physical quantities for particular crossovers of interest [3—5]. In this paper we consider dimensional crossover of the specific heat as it is one of the more readily accessible quantities from an experimental point of view. In the context of films and experimental tests of finite size scaling this was the first experimentally measured quantity [6].

From a theoretical point of view the case of a totally finite geometry has been successfully investigated numerically [7] and analytically for both periodic boundary conditions

[8] and Dirichlet boundary conditions [9]. In the case of film geometries some progress has been made [10] but no theoretical work has been able to access the complete crossover other than for two dimensional films [11].

The format of the paper is as follows: in section 2 we analyze the connection between the specific heat and the vertex functions of a Landau-Ginzburg-Wilson effective Hamiltonian. By choosing as mass parameter $t_B = \Lambda^2 \frac{(T-T_c)}{T}$ we include all the non-analytic dependence of the specific heat in the vertex function $\Gamma_B^{(0,2)}$. We then present a renormalization of the theory as a reparametrization through normalization conditions on certain renormalized vertex functions using a fiducial correlation length as our RG scale. Section 3 is devoted to perturbative calculations. In particular we calculate the specific heat and a specific heat effective exponent to one loop. In section 4, by matching to the known asymptotic exponents for a three dimensional Ising film, we access the crossover between power law behaviour at the three dimensional end and logarithmic behaviour at the two dimensional end. The results in sections 3 and 4 are illustrated in the figures. The paper ends with our conclusions.

§ 2. Renormalization of the Specific Heat

We consider an $O(N)$ symmetric order parameter described by the "microscopic" Landau-Ginzburg-Wilson Hamiltonian

$$H[\varphi_B] = \int_0^L \int d^d x \left(\frac{1}{2}(\nabla\varphi_B)^2 + \frac{1}{2}m_B^2\varphi_B^2 + \frac{1}{2}t_B(x)\varphi_B^2 + \frac{\lambda_B}{4!}\varphi_B^4 - H_B(x)\varphi_B \right) \quad (2.1)$$

which describes a d dimensional film geometry of thickness L. The variable t_B when taken to be homogeneous has analytic dependence on temperature T, and we choose its origin to be the critical temperature of the film. Hence m_B^2 is determined by the difference between the L dependent critical temperature and the mean field critical temperature, i.e. the

temperature at which the potential in (2.1) acquires a non-zero minimum. λ_B is assumed to be temperature independent. The subscript B refers to bare parameters as distinct from renormalized parameters which will be introduced below. We will restrict attention to the case when the film also exhibits a phase transition and consider $3 \leq d \leq 4$ for $N = 1$, and $3 < d \leq 4$ for $N > 1$. We will present results for periodic, antiperiodic and Dirichlet boundary conditions. Note that in the case of periodic boundary conditions our results for $N = 1$ could equally well be reinterpreted to describe the quantum to classical crossover of an Ising model in a transverse magnetic field Γ, where now $t_B = \Gamma - \Gamma_c(T)$ and $L = \hbar/T$ [12]. However, in this paper we restrict our considerations to the film geometry.

The partition function for the the model (2.1) is given by the path integral

$$Z = \int [d\varphi_B] e^{-H[\varphi_B] - \frac{V}{T} F^b}. \tag{2.2}$$

The free energy density is $F = -\frac{T}{V} \ln Z = F^b - \frac{T}{V} \ln Z_{LGW}$, where V is the volume and F^b is the background free energy density obtained after coarse graining from the underlying microscopic degrees of freedom to those of the effective field theory description in terms of the LGW Hamiltonian (2.1), Z_{LGW} being the partition function of this Hamiltonian. F^b is assumed to be an analytic function of the thermodynamic variables. The internal energy density is

$$U = F - T\frac{\partial F}{\partial T}$$

and the specific heat, by definition $\partial U/\partial T$, is given by

$$C = -T^2 \frac{\partial^2 F}{\partial T^2}. \tag{2.3}$$

The assumption in working with this LGW Hamiltonian is that the only one of its parameters to retain a dependence on temperature is the mass parameter t_B. Thus the internal energy density

$$U = U^b - \frac{T^2}{2V} \int d^d x \frac{\partial t_B(x)}{\partial T} G^{(0,1)} \tag{2.4}$$

and the specific heat

$$C = C^b - \frac{1}{2V}\int d^dx(\frac{\partial}{\partial T}T^2\frac{\partial t_B(x)}{\partial T})G^{(0,1)}(x) + \frac{T^2}{4V}\int d^dx\int d^dy\frac{\partial t_B(x)}{\partial T}G^{(0,2)}(x,y)\frac{\partial t_B(y)}{\partial T} \tag{2.5}$$

where

$$G^{(0,1)}(x) = \langle\varphi^2(x)\rangle \quad \text{and} \quad G^{(0,2)}(x,y) = \langle\varphi^2(x)\varphi^2(y)\rangle - \langle\varphi^2(x)\rangle\langle\varphi^2(y)\rangle. \tag{2.6}$$

Concentrating on $T > T_c$, where $< \varphi >= 0$, and denoting

$$\Gamma[t_B] = -\ln Z_{LGW} \tag{2.7}$$

we have that

$$\Gamma^{(0,1)}(x) = \frac{1}{2}G^{(0,1)}(x) \qquad \text{and} \qquad \Gamma^{(0,2)}(x,y) = -\frac{1}{4}G^{(0,2)}(x,y). \tag{2.8}$$

So for homogeneous t_B we find

$$U = U^b - T^2\frac{\partial t_B}{\partial T}\Gamma_B^{(0,1)} \tag{2.9}$$

$$C = C^b - (\frac{\partial}{\partial T}T^2\frac{\partial t_B}{\partial T})\Gamma_B^{(0,1)} - T^2(\frac{\partial t_B}{\partial T})^2\Gamma_B^{(0,2)}, \tag{2.10}$$

where $\Gamma_B^{(0,1)}$ and $\Gamma_B^{(0,2)}$ are to be evaluated at zero external momentum.

If we wish to incorporate all of the non-analytic dependence of the internal energy and the specific heat into $\Gamma^{(0,1)}$ and $\Gamma^{(0,2)}$ respectively, then a natural choice of the dependence of t_B is

$$t_B = \Lambda^2\,\frac{(T - T_c)}{T} \tag{2.11}$$

where Λ is a microscopic mass scale. In the vicinity of the critical temperature the results with this variable will be the same as those obtained with the linear measure $\Lambda^2\,\frac{(T-T_c)}{T_c}$. With the choice (2.11) the internal energy density becomes

$$U = U^b - \Lambda^2 T_c\Gamma_B^{(0,1)} \tag{2.12}$$

and the specific heat is given by

$$C = C^b - \frac{\Lambda^2}{T^2}\Gamma_B^{(0,2)}. \tag{2.13}$$

For an $O(N)$ model $G_B^{(0,2)}$ is manifestly positive and either diverges or goes to zero at the critical temperature according to the value of N. Thus we anticipate that $\Gamma_B^{(0,2)}$ should diverge to $-\infty$ or vanish at the critical temperature. Our problem is therefore to calculate $\Gamma_B^{(0,2)}$.

The correlation length in the transverse dimensions, $\xi_L = m^{-1}$, we define via the second moment of the two point function, $G^{(2)}$. On Fourier transforming ξ_L is obtained from

$$m^2 = \left.\frac{\Gamma_B^{(2)}(p, t_B(m), \lambda_B, L)}{\partial_{p^2}\Gamma_B^{(2)}(p, t_B(m), \lambda_B, L)}\right|_{p^2=0} \tag{2.14}$$

where p is the transverse momentum and $t_B(m)$ is that bare mass parameter which produces the inverse correlation length m. The origin for the variable $t_B(m)$ is specified by requiring that

$$\Gamma_B^{(2)}(0, 0, \lambda_B, L) = 0 \tag{2.15}$$

which insures that t_B is proportional to $T - T_c(L)$ as the critical temperature is approached. Changing the mass parameter t_B, by changing the temperature in (2.11), allows us to tune the correlation length. Note that the physical correlation length of the film geometry (2.14) depends on L and will be infinite at the film critical temperature $T_c(L)$.

We will define renormalized parameters by

$$t(m, \kappa) = Z_{\varphi^2}^{-1}(\kappa) t_B(m) \qquad \text{and} \qquad \lambda(\kappa) = Z_\lambda(\kappa)\lambda_B \tag{2.16}$$

and renormalized vertex functions by

$$\Gamma^{(N,M)}(m, \kappa) = Z_\varphi^{\frac{N}{2}}(\kappa) Z_{\varphi^2}^M(\kappa)\Gamma_B^{(N,M)}(m) + \delta_{N0}\delta_{Mn}A^{(n)}(\kappa) \qquad n = 0, 1, 2 \tag{2.17}$$

which is just a reparametrization of the original theory, where κ is an arbitrary renormalization scale.

Contrary to the renormalization of other vertex functions, like $\Gamma_B^{(2)}$ and $\Gamma_B^{(4)}$, the vertex functions $\Gamma_B^{(0,n)}$ $(n = 0, 1, 2)$ have to be renormalized additively via

$$\Gamma^{(0,n)} = Z_{\varphi^2}^n \Gamma_B^{(0,n)} + A^{(n)} \tag{2.18}$$

Γ and $\Gamma^{(0,1)}$ determine the Gibbs free energy density and the energy density of the LGW Hamiltonian respectively * .

Before discussing the renormalization of $\Gamma^{(0,n)}$ for $n = 0, 1, 2$ we will specify the Z's associated with the reparametrization (2.16) and (2.17). Here we will restrict ourselves to $T > T_c(L)$. The case of $T < T_c(L)$ will be considered in conjunction with crossover amplitude ratios elsewhere. For $T > T_c(L)$ the conditions which specify our Z's are

$$Z_\varphi^{-1} = \partial_{p^2} \Gamma_B^{(2)}(p, t_B(\kappa), \lambda_B, L)\Big|_{p^2=0} \tag{2.19}$$

$$Z_{\varphi^2}^{-1} = \frac{\Gamma_B^{(2,1)}(p, t_B(\kappa), \lambda_B, L)}{\partial_{p^2} \Gamma_B^{(2)}(p, t_B(\kappa), \lambda_B, L)}\Bigg|_{p^2=0} \tag{2.20}$$

$$Z_\lambda = \frac{\Gamma_B^{(4)}(0, t_B(\kappa), \lambda_B, L)}{\lambda_B} \tag{2.21}$$

where the relation between $t_B(\kappa)$ and κ is specified by

$$\kappa^2 = \frac{\Gamma_B^{(2)}(p, t_B(\kappa), \lambda_B, L)}{\partial_{p^2} \Gamma_B^{(2)}(p, t_B(\kappa), \lambda_B, L)}\Bigg|_{p^2=0} \tag{2.22}$$

and the origin of t_B is fixed by (2.15). Note that the Z's are obtained from the vertex functions of the system specified at an arbitrary, fiducial, transverse correlation length κ^{-1}, as opposed to the correlation length of interest, m^{-1}. Furthermore the conditions are all

* For homogeneous t_B, $H_B = 0$ and $T > T_c$ we use the convention $\Gamma = \frac{1}{V} \ln Z_{LGW}$

L dependent. As has been emphasized on previous occasions [1,2] such "environmentally friendly" conditions are essential in order to obtain a perturbatively controllable description of the finite size crossover.

We define the Wilson functions as the logarithmic derivatives

$$\gamma_\varphi = \frac{1}{Z_\varphi}\kappa\frac{dZ_\varphi}{d\kappa} \tag{2.23}$$

$$\gamma_{\varphi^2} = -\frac{1}{Z_{\varphi^2}}\kappa\frac{dZ_{\varphi^2}}{d\kappa} \tag{2.24}$$

$$\gamma_\lambda = \frac{1}{Z_\lambda}\kappa\frac{dZ_\lambda}{d\kappa}. \tag{2.25}$$

The Wilson functions γ_{φ^2}, γ_φ and γ_λ are explicitly L dependent and interpolate between those of a d and $d-1$ dimensional $O(N)$ model in the limits $\kappa L \to \infty$, $\kappa \to 0$ and $\kappa L \to 0$, $\kappa \to 0$ respectively.

The invariance of the bare vertex functions, $\Gamma_B^{(N,L)}$, under the one parameter group of reparametrizations indexed by the arbitrary renormalization scale κ (they don't know which reference correlation length κ^{-1} will be picked to define the reparametrization) yields the RG equation

$$\kappa\frac{d\Gamma^{(N,L)}}{d\kappa} + (L\gamma_{\varphi^2} - \frac{N}{2}\gamma_\varphi)\Gamma^{(N,L)} = \delta_{N0}\delta_{Ln}B^{(n)} \tag{2.26}$$

where $n = 0, 1, 2$. The equation is inhomogeneous for the three vertex functions Γ, $\Gamma^{(0,1)}$ and $\Gamma^{(0,2)}$, where the "source" term

$$B^{(n)} = \kappa\frac{dA^{(n)}}{d\kappa} + n\gamma_{\varphi^2}A^{(n)} \tag{2.27}$$

is finite order by order in the loop expansion.

The relationship between temperature and κ can be obtained by using

$$\Gamma^{(2)}(t) = \int_0^t \Gamma^{(2,1)}(t')dt' \tag{2.28}$$

and conditions (2.19-2.22) with the definitions of the Wilson functions (2.23-2.25) to find

$$t(m,\kappa) = \kappa^2 \int_0^m \frac{dx}{x}(2-\gamma_\varphi)e^{\int_\kappa^x (2-\gamma_{\varphi^2})\frac{dy}{y}}. \tag{2.29}$$

We see that

$$\kappa\frac{dt(m,\kappa)}{d\kappa} = \gamma_{\varphi^2}t(m,\kappa). \tag{2.30}$$

An important feature of the above is that the determination of $\Gamma^{(2)}$ by integrating $\Gamma^{(2,1)}$ allows us to bypass the need to determine m_B^2 perturbatively.

In terms of the renormalized vertex functions, the conditions (2.19-2.22) are equivalent to

$$\partial_{p^2}\Gamma^{(2)}(p,t(\kappa,\kappa),\lambda,L,\kappa)\Big|_{p^2=0} = 1 \tag{2.31}$$

$$\Gamma^{(2,1)}(0,t(\kappa,\kappa),\lambda,L,\kappa) = 1 \tag{2.32}$$

$$\Gamma^{(4)}(0,t(\kappa,\kappa),\lambda,L,\kappa) = \lambda. \tag{2.33}$$

$$\Gamma^{(2)}(0,t(\kappa,\kappa),\lambda,L,\kappa) = \kappa^2. \tag{2.34}$$

We could have replaced (2.32) by the condition

$$t(\kappa,\kappa) = \kappa^2. \tag{2.35}$$

This condition together with (2.34) determines a multiplicative renormalization of t_B, and of φ^2 insertions via a renormalization function Z_t. The two renormalization functions Z_t and Z_{φ^2} are different, the latter being determined by (2.32). The quantity $\gamma_t = -\frac{d\ln Z_t}{d\ln\kappa}$ is an analog of γ_{φ^2}, however, the problem with implementing a condition such as (2.35) in perturbation theory is that the resulting Z_t involves diagrams with massless propagators, some of which are strictly infinite even after the introduction of an ultraviolet cutoff.

Defining an effective exponent $\nu_{eff} = \frac{d\ln \xi_L^{-1}}{d\ln t}$, one finds

$$\nu_{eff} = (2-\gamma_t)^{-1} = \frac{\int_0^{\xi_L^{-1}} \frac{dx}{x}(2-\gamma_\varphi) e^{\int_{\xi_L^{-1}}^{x} \frac{dy}{y}(2-\gamma_{\varphi^2})}}{(2-\gamma_\varphi)}. \tag{2.36}$$

One can also define what we term a floating exponent, $\nu_f = (2-\gamma_{\varphi^2})^{-1}$. As near a fixed point γ_φ and γ_{φ^2} go to constants we can see from (2.36) that $\gamma_t \to \gamma_{\varphi^2}$, hence both the true effective exponent and the floating exponent interpolate between the same two asymptotic values. This may not be evident perturbatively. One can think of the floating exponents evaluated in environmentally friendly RG improved perturbation theory as approximations to the true effective exponents [2]. Another way of thinking about them is from the point of view of a redefined temperature variable in the following way: if one defines $t' = tf(t)$ and $\nu_f = \frac{d\ln \xi_L^{-1}}{d\ln t'} = (2-\gamma_{\varphi^2})^{-1}$ one finds that

$$\frac{d\ln f}{d\ln t} + 1 = \frac{(2-\gamma_{\varphi^2})}{(2-\gamma_\varphi)} \int_0^{\xi_L^{-1}} \frac{dx}{x}(2-\gamma_\varphi) e^{\int_{\xi_L^{-1}}^{x} \frac{dy}{y}(2-\gamma_{\varphi^2})}. \tag{2.37}$$

Near a fixed point $f \to 1$ hence $t' \to t$.

The solution of the RG equation (2.26) for the specific heat is

$$\begin{aligned}\Gamma^{(0,2)}(t(m,\kappa),\lambda(\kappa),L,\kappa) =& e^{2\int_1^\rho \frac{dx}{x}\gamma_{\varphi^2}} \Gamma^{(0,2)}(t(m,\rho\kappa),\lambda(\rho\kappa),L,\kappa\rho) \\ &- \int_1^\rho \frac{dx}{x} B^{(2)}(x) e^{2\int_1^x \frac{dy}{y}\gamma_{\varphi^2}}.\end{aligned} \tag{2.38}$$

Reparametrization invariance is now manifest in the fact that the left hand side of (2.38) is independent of ρ, the latter being just an arbitrary rescaling of κ.

We will now discuss some possible normalization conditions for $\Gamma^{(0,2)}$, thus specifying $A^{(2)}(\kappa)$. One possible choice is the normalization condition

$$\Gamma^{(0,2)}(t(\kappa,\kappa),\lambda(\kappa),L,\kappa) = 0 \tag{2.39}$$

which is equivalent to

$$A^{(2)}(\kappa) = -Z_{\varphi^2}^2 \Gamma_B^{(0,2)}(t_B(\kappa),\lambda_B,L). \tag{2.40}$$

The advantage of this condition is that all the "physics", in the sense of the effects of all fluctuations, is now purely in the inhomogeneous term. The normalization condition (2.39), however, is natural as the $\Gamma^{(0,2)}$ does indeed vanish in the mean field regime, or at least goes to a constant which can be chosen to be zero. Neglecting the inhomogeneous term $\Gamma^{(0,2)}$ being zero is then an invariant statement with respect to RG transformations.

A methodology which avoids some of the pitfalls of additive renormalization is to relate $\Gamma^{(0,2)}$ to the correlation function $\Gamma^{(0,3)}$, the advantage of this approach being that the latter is multiplicatively renormalizable in $d < 6$. We have the analog of (2.28)

$$\Gamma^{(0,2)}(t) - \Gamma^{(0,2)}(t_i) = \int_{t_i}^{t} \Gamma^{(0,3)}(t')dt'. \tag{2.41}$$

Using the relation

$$\Gamma^{(0,3)}(t(m,\kappa),\lambda(\kappa),L,\kappa) = e^{3\int_1^\rho \frac{dx}{x}\gamma_{\varphi^2}}\Gamma^{(0,3)}(t(m,\rho\kappa),\lambda(\rho\kappa),L,\kappa\rho) \tag{2.42}$$

and the relation between the correlation length and the temperature (2.29) one finds

$$\Gamma^{(0,2)}(t(m,\kappa)) = \kappa^{d-4}\int_{\infty}^{m}\frac{dx}{x}(2-\gamma_\varphi)e^{\int_1^x(2\gamma_{\varphi^2}-4+d)\frac{dy}{y}}\bar{\Gamma}^{(0,3)}(x) \tag{2.43}$$

where we have normalized $\Gamma^{(0,2)}$ to vanish in the mean field limit and

$$\bar{\Gamma}^{(0,3)}(m) = \frac{\Gamma^{(0,3)}\Gamma^{(2)^3}}{\Gamma^{(2,1)^3}m^d}. \tag{2.44}$$

It is not difficult to show that in fact this expression is exactly the same as that obtained from the additive renormalization prescription with the normalization condition (2.39) at $t(\infty,\infty)$.

§ 3. Perturbative calculations

We begin this section by analysing the β function for the coupling, as we will perturbatively expand all other functions in terms of the solution of this equation . In terms of the floating

coupling h [1], chosen to be the leading term in the perturbative series for γ_λ, one finds, for $\rho\frac{dh}{d\rho} = \beta(h,z)$, to one loop

$$\beta(h,z) = -\varepsilon(z)h + h^2. \tag{3.1}$$

The function ε, in an obvious diagrammatic notation, is

$$\varepsilon(z) = \frac{6\kappa^4 \bigcirc}{\bigcirc} - 2\,, \tag{3.2}$$

depends on d and $z = \rho\kappa L$ but is independent of N. We take the solution of (3.1)

$$h(z) = \frac{e^{-\int_{z_0}^{z}\varepsilon(x)\frac{dx}{x}}}{h_0^{-1} - \int_{z_0}^{z} e^{-\int_{z_0}^{x}\varepsilon(y)\frac{dy}{y}}\frac{dx}{x}} \tag{3.3}$$

as our perturbation parameter.

After solving the equation we specify the arbitrary scale ρ to be $\rho = \frac{1}{\kappa\xi_L}$ and relate it to temperature via (2.29) whereupon z becomes L/ξ_L. In (3.3) the initial coupling is then taken to be at a "microscopic" scale κ. For $d < 4$ this microscopic scale can be sent to infinity and a universal floating coupling, the separatrix solution $h(z) = -\frac{4z^2 \bigcirc}{\bigcirc}$ obtained [2]. Of course, if one is interested in corrections to scaling, as is usually the case in comparing with experimental data, then κ should be left finite and fitted to the data.

For periodic boundary conditions one finds

$$\varepsilon(z) = 5 - d - (7-d)\frac{\sum_{n=-\infty}^{\infty}\frac{4\pi^2n^2}{z^2}\left(1+\frac{4\pi^2n^2}{z^2}\right)^{\frac{d-9}{2}}}{\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{z^2}\right)^{\frac{d-7}{2}}} \tag{3.4}$$

and the separatrix coupling

$$h(z) = (5-d)\frac{\sum_{n=-\infty}^{\infty}(1+\frac{4\pi^2n^2}{z^2})^{\frac{(d-7)}{2}}}{\sum_{n=-\infty}^{\infty}(1+\frac{4\pi^2n^2}{z^2})^{\frac{(d-5)}{2}}}. \tag{3.5}$$

For $d = 3$ the results are particularly simple

$$h(z) = 1 + \frac{z}{\sinh z} \tag{3.6}$$

$$\varepsilon(z) = 1 + \frac{z^2 \coth(\frac{z}{2})}{\sinh z + z} \tag{3.7}$$

where, of course, we are now restricted to $N = 1$.

We present here the corresponding results for Dirichlet and antiperiodic boundary conditions. For Dirichlet boundary conditions

$$\varepsilon(z) = 5 - d - (7-d)\frac{\sum_{n=1}^{\infty} \frac{\pi^2(n^2-1)}{z^2}\left(1 + \frac{\pi^2(n^2-1)}{z^2}\right)^{\frac{d-9}{2}}}{\sum_{n=1}^{\infty}\left(1 + \frac{\pi^2(n^2-1)}{z^2}\right)^{\frac{d-7}{2}}} \tag{3.8}$$

and for the separatrix coupling

$$h(z) = (5-d)\frac{\sum_{n=1}^{\infty}(1 + \frac{\pi^2(n^2-1)}{z^2})^{\frac{(d-7)}{2}}}{\sum_{n=1}^{\infty}(1 + \frac{\pi^2(n^2-1)}{z^2})^{\frac{(d-5)}{2}}}. \tag{3.9}$$

For antiperiodic boundary conditions one finds

$$\varepsilon(z) = 5 - d - (7-d)\frac{\sum_{n=-\infty}^{\infty} \frac{\pi^2 n(n+1)}{z^2}\left(1 + \frac{\pi^2 n(n+1)}{z^2}\right)^{\frac{d-9}{2}}}{\sum_{n=-\infty}^{\infty}\left(1 + \frac{\pi^2 n(n+1)}{z^2}\right)^{\frac{d-7}{2}}} \tag{3.10}$$

and finally the separatrix coupling

$$h(z) = (5-d)\frac{\sum_{n=-\infty}^{\infty}(1 + \frac{\pi^2 n(n+1)}{z^2})^{\frac{(d-7)}{2}}}{\sum_{n=-\infty}^{\infty}(1 + \frac{\pi^2 n(n+1)}{z^2})^{\frac{(d-5)}{2}}}. \tag{3.11}$$

For $d = 3$ the results are once again very simple. For the Dirichlet case

$$\varepsilon(y) = 1 + \frac{3\pi^2}{y^2} + 2(1 + \frac{\pi^2}{y^2})\frac{(\frac{y^2}{\sinh^2 y} - \frac{\tanh y}{y})}{(1 + \frac{2y}{\sinh 2y} - 2\frac{\tanh y}{y})} \tag{3.12}$$

where $y = (z^2 - \pi^2)^{\frac{1}{2}}$. Even though $y(z)$ has a branch point $h(z)$ is analytic in z. The separatrix coupling is

$$h(y) = (1 + \frac{\pi^2}{y^2})\frac{(1 + \frac{2y}{\sinh 2y} - \frac{2\tanh y}{y})}{(1 - \frac{\tanh y}{y})}. \tag{3.13}$$

The corresponding results for antiperiodic boundary conditions are

$$\varepsilon(y) = 1 + \frac{3\pi^2}{y^2} - \frac{(y^2 + \pi^2)\tanh(y/2)}{(\sinh y - y)} \tag{3.14}$$

and

$$h(y) = (1 + \frac{\pi^2}{y^2})(1 - \frac{y}{\sinh y}). \tag{3.15}$$

The Wilson function γ_{φ^2} is given by

$$\gamma_{\varphi^2}(h, z) = \frac{(N+2)}{(N+8)}h \tag{3.16}$$

whilst $\gamma_\varphi = 0$ to one loop. Two loop Padé resummed expressions for the Wilson functions and the floating coupling, for the case of periodic boundary conditions, can be found in [2,5]. Substituting any of the above floating couplings into (3.16) yields γ_{φ^2} for the three different types of boundary condition. As mentioned corrections to scaling can easily be included. For example for $d = 3$ and periodic boundary conditions

$$h^{-1}(z) = \frac{z\sinh(\frac{z}{2})^2}{\sinh z + z}(\frac{1}{h(z_0)}\frac{\sinh z_0 + z_0}{z_0\sinh(\frac{z_0}{2})^2} - 2\frac{\coth(\frac{z_0}{2})}{z_0}) + \frac{\sinh z}{\sinh z + z}. \tag{3.17}$$

Turning now to $\Gamma^{(0,2)}$, up to two loop order and once again in an obvious diagrammatic notation (note that we have made the diagrams dimensionless by pulling out an overall scale) $\Gamma_B^{(0,2)}$ is given by

$$\Gamma_B^{(0,2)} = -\frac{N}{2}(\rho\kappa)^{d-4}\Big[\bigcirc - \lambda_B\kappa^{d-4}\frac{(N+2)}{6}\bigcirc^2\Big] \tag{3.18}$$

the two loop graph with "tadpole" having been absorbed into the one loop propagator by the replacement of t_B with $\rho\kappa$ using (2.22). Implementing the normalization condition (2.39) one finds that

$$B^{(2)} = -2N\kappa^2 \bigcirc|_{n.p} \tag{3.19}$$

where the subscript denotes that the diagram is evaluated at the normalization point. Thus we see that the one and two loop expressions for $B^{(2)}$ in terms of renormalized quantities are identical. Explicitly to one loop for periodic boundary conditions one finds

$$B^{(2)} = -\frac{N}{L}\frac{\Gamma(\frac{7-d}{2})(\rho\kappa)^{d-5}}{(2\pi)^{\frac{d-1}{2}}}\sum_{n=-\infty}^{\infty}(1+\frac{4\pi^2n^2}{L^2\kappa^2\rho^2})^{\frac{d-7}{2}}. \tag{3.20}$$

The $\Gamma^{(0,2)}$ is thus found by substituting (3.19) and (3.16) into (2.38) to obtain

$$\begin{aligned}\Gamma^{(0,2)}(t,\lambda,L,\kappa) =& \frac{N}{2L\kappa}\frac{\Gamma(\frac{7-d}{2})\kappa^{d-4}}{(2\pi)^{\frac{d-1}{2}}} \\ & \int_1^{\rho}\frac{dx}{x}x^{d-5}\sum_{n=-\infty}^{\infty}(1+\frac{4\pi^2n^2}{L^2\kappa^2x^2})^{\frac{d-7}{2}}e^{2\int_1^x\left(\frac{N+2}{N+8}\right)h\frac{dy}{y}}\end{aligned} \tag{3.21}$$

where the arbitrary scale ρ, as before, is associated directly with the inverse correlation length. Thus we calculate the specific heat and other physical quantities directly in terms of the finite size correlation length. Equation (2.29) relating ξ_L to L and t provides a parametric representation of physical quantities in terms of t.

In the limit $\rho \to 0$ only the $n = 0$ term in the sum is important and one finds

$$\Gamma^{(0,2)} \to -\frac{N(N+8)}{2(4-N)L\kappa}\frac{\Gamma(\frac{5-d}{2})\kappa^{d-4}}{(2\pi)^{\frac{d-1}{2}}}\rho^{d-5+2(5-d)\left(\frac{N+2}{N+8}\right)}. \tag{3.22}$$

In the same limit one finds $\rho \to (\frac{t}{\kappa^2})^{\nu_{d-1}}$, $\nu_{d-1} = (2-\left(\frac{N+2}{N+8}\right)(5-d))^{-1}$ being the $d-1$ dimensional correlation length exponent. Hence

$$\Gamma^{(0,2)} \to -\frac{N(N+8)}{2(4-N)L\kappa}\frac{\Gamma(\frac{5-d}{2})\kappa^{d-4}}{(2\pi)^{\frac{d-1}{2}}}(\frac{t}{\kappa^2})^{-\alpha_{d-1}} \tag{3.23}$$

where $\alpha_{d-1} = \frac{5-d-2(5-d)\left(\frac{N+2}{N+8}\right)}{2-\left(\frac{N+2}{N+8}\right)(5-d)}$ is the $d-1$ dimensional specific heat exponent. Similarly, in the limit $L\kappa\rho \to \infty$, $\rho \to 0$ the sum can be converted to an integral and one finds that

$$\Gamma^{(0,2)} \to -\frac{N(N+8)}{2(4-N)}\frac{\Gamma(\frac{4-d}{2})\kappa^{d-4}}{(2\pi)^{d/2}}\rho^{d-4+2(4-d)\left(\frac{N+2}{N+8}\right)} \tag{3.24}$$

and $\rho \to (\frac{t}{\kappa^2})^{\nu_d}$ where ν_d is the d dimensional correlation length exponent. In the bulk limit

$$\Gamma^{(0,2)} \to -\frac{N(N+8)}{2(4-N)}\frac{\Gamma(\frac{4-d}{2})\kappa^{d-4}}{(2\pi)^{d/2}}(\frac{t}{\kappa^2})^{-\alpha_d} \tag{3.25}$$

where $\alpha_d = \frac{4-d-2(4-d)\left(\frac{N+2}{N+8}\right)}{2-\left(\frac{N+2}{N+8}\right)(4-d)}$ is the d dimensional specific heat exponent. Thus we see that the specific heat crosses over precisely between the expected d and $d-1$ dimensional asymptotic forms.

Note that the amplitude of $\Gamma^{(0,2)}$ in the above expressions appears to diverge at $N=4$ this is an artifact of the one loop approximation. What actually happens is that for d between two and four there is some value of N for which $\alpha(N,d)=0$, at this value of N and d we expect the specific heat to have a logarithmic dependence on t. For $N=1$ this occurs at $d=2$, however, at one loop the value appears to be independent of d and occurs at $N=4$, which is the relevant value for $d=4$.

A plot of the specific heat as a function of correlation length is shown in Fig. 1 for a three dimensional Ising film with periodic boundary conditions. The effective specific heat exponent defined as

$$\alpha_{eff} = -\frac{d\ln C}{d\ln t} \tag{3.26}$$

is plotted in Fig. 2 for the same model. Note that in this approximation the asymptotic two dimensional value of α_{eff} is 0.5 as opposed to the exact value of zero, obtained from the solution of the two dimensional Ising model. This is a weakness of the perturbative approach which effects the specific heat exponent in a particularly acute manner. In the next section by matching to the known asymptotic exponents of the model we investigate

the more realistic behaviour. In the case of a four dimensional $O(N)$ film, in the limit $\frac{L}{\xi_L} \to \infty$, $\xi_L \to \infty$, one finds that

$$\Gamma^{(0,2)} \to -\frac{N}{16\pi^2}\left(\frac{N+8}{4-N}\right)\left|\frac{1}{2}\ln\frac{t}{\kappa^2}\right|^{\frac{4-N}{N+8}} \tag{3.27}$$

in accordance with known results. Fig.'s 3 and 4 show plots of the specific heat and α_{eff} for the four dimensional Ising film. Note the presence in the figures of logarithmic tails at the four dimensional end as described by (3.27). Fig. 5 shows a comparison of α_{eff} for a three dimensional Ising film with Dirichlet and antiperiodic boundary conditions. Additionally the result for the Gaussian model is ploted in Fig. 6 with periodic boundary conditions.

§ 4. Crossover to Logarithmic Behaviour in a Three Dimensional Ising Film.

In this section we will consider the crossover between three and two dimensions for an Ising model in a way that is capable of accessing the logarithmic behaviour characteristic of the two dimensional specific heat. For the two dimensional Ising model $\alpha = 2 - \nu d = 0$. The consequent logarithmic behaviour of the specific heat is thus due to a competition between ν and d. For $d = 2$ the correlation length exponent $\nu = 1$, hence $\alpha = 0$. Now for a three dimensional Ising film with periodic boundary conditions, at one loop the crossover is governed by the floating coupling $h = 1 + \frac{z}{\sinh z}$. This implies a crossover for ν_{eff} between 1/6 and 1/3. By far the biggest error involved in evaluating crossover functions is associated with the values of the asymptotic exponents themselves. With this in mind one is inclined to try to match the scaling function to the asymptotic exponents. This can very simply be done in the case at hand by writing $h = A + \frac{Bz}{\sinh z}$ where now the constants A and B will be determined by demanding that as $z \to 0$, $\nu_{eff} \to 1$ and that as $L \to \infty$, $z \to 0$ one finds $\nu_{eff} \to 0.630$. The values 1 and 0.630 are the exact two dimensional and three dimensional 6-loop Borel resummed [13] exponents respectively. Thus one finds that $A = 1.238$ and $B = 1.762$.

In Fig. 7 we plot α_{eff} as a function of $\ln z$ by substituting our ansatz for h into (3.21). Note the logarithmic tail as the two dimensional critical region is approached. More interestingly, there is a pronounced bump in the curve which is absent in the one-loop approximation. This arises due to a competition between the effects of ν_{eff} and the effective dimensionality d_{eff}[2]. The bump remains even if one uses a completely different interpolating function such as $h = A + \frac{Bz}{1+z}$, though its amplitude and width vary somewhat. In Fig. 8 we plot analogous results for the case of Dirichlet and antiperiodic boundary conditions. Once again the bump is clearly present. In the case of Dirichlet conditions however there is also a dip before the bump is reached. Based on previous experience of the behaviour of effective exponents with Dirichlet boundary conditions [2] this is not totally unexpected. In Fig.'s 9 and 10 we have used instead of the universal floating coupling the coupling (3.17). There is now a double crossover; firstly between mean field theory and the three dimensional asymptotic exponent and then to the asymptotic behaviour of the two dimensional exponent. In Fig. 9 we plot the result for the case where we do not match to the exact two dimensional exponent and in Fig. 10 the result with matching. The asymptotic three dimensional regime would most probably be much narrower than that shown. This can be very easily modeled by adjusting the initial condition for the RG flow. In the case at hand, we have, for the sake of clarity, and to emphasize the double crossover, left it large. It is clear from the figure how the effective exponent would be modified as the well developed three dimensional universal regime is narrowed.

§ 5. Conclusions

In this paper, using environmentally friendly renormalization, we have treated the finite size crossover of the specific heat of an $O(N)$ model in a d-dimensional film geometry. For $N > 1$ we considered $3 < d \leq 4$, and for $N = 1$, $3 \leq d \leq 4$. We derived expressions for the specific heat and an effective critical exponent α_{eff} that were completely regular across

the entire crossover, the expansion parameter for the perturbative series being the floating coupling h. We considered periodic, Dirichlet and antiperiodic boundary conditions.

For the crossover from three to two dimensions of an Ising film we saw that one loop answers in the asymptotic two dimensional regime were quite poor. As is known, generally speaking, perturbation theory becomes more unreliable as one goes to lower dimensions. For the specific heat the problem is particularly acute as the one loop effective specific heat exponent was seen to be monotonically increasing whereas, as we know from the solution of the exact two-dimensional Ising model, the two dimensional specific heat exponent is strictly less than the three dimensional one. Hence we could say that the one loop approximation is failing to capture a qualitative feature of the crossover in this case. To circumvent this problem, and in the knowledge that the dominant source of error in calculating scaling functions is the uncertainty in the asymptotic critical exponents, we took a more pragmatic line by making an ansatz for the floating coupling constant so as to be able to asymptotically match the "known" two and three dimensional correlation length exponents. By so doing we were able to access in a very simple way the crossover between power law and logarithmic behaviour in the asymptotic regime, finding that the resultant crossover curve had a very interesting bump. We also analyzed the crossover to mean field theory thereby accessing a double crossover governed by three different fixed points. Our global, environmentally friendly RG captured all of these fixed points in one uniform approximation scheme.

The crossover between three and two dimensions for $N > 1$, and in particular for $N = 2$, are potential problems that could be analyzed using the techniques of this paper. The latter being a problem of longstanding interest for experiments with liquid helium confined to a film geometry [6,14]. We hope to return to these issues in the future.

Acknowledgements.

We would like to thank Prof. Michael Fisher for his interest and several very useful discussions especially relating to the matching of asymptotic exponents. CRS was supported by an EU Human Capital and Mobility Fellowship.

References.

[1] D. O'Connor and C.R. Stephens, *Nucl. Phys.* **B360** (1991) 297; *J. Phys.* **A25** (1992) 101.

[2] D. O'Connor and C.R. Stephens, *Int. J. Mod. Phys.* **A9** (1994) 2805.

[3] D. O'Connor and C.R. Stephens, *Proc. Roy. Soc.* **444A**, (1994), 287.

[4] F. Freire, D. O'Connor and C.R. Stephens, *J. Stat. Phys.* **74**, (1994) 219.

[5] D. O'Connor and C.R. Stephens, *Phys. Rev. Lett.* **72**, (1994) 506.

[6] T.P. Chen and F.M. Gasparini, *Phys. Rev. Lett.* **40**, (1978) 331;
F.M. Gasparini, G. Agnolet and J.D. Reppy, *Phys. Rev.* **B 29**, (1984) 128.

[7] K. Binder, *Z. Phys.* **B 43**, (1981) 119.

[8] J. Rudnick, H. Guo and D. Jasnow, *J. Stat. Phys.* **41**, (1985) 353.

[9] W. Huhn and V. Dohm, *Phys. Rev. Lett.* **61**, (1988) 1368.

[10] M. Krech and S. Dietrich, *Phys. Rev.* **A46**, (1992) 1886; *Phys. Rev.* **A46**, (1992) 1922.

[11] A.E. Ferdinand and M.E. Fisher, *Phys. Rev.* **185**, (1969) 832.

[12] J. A. Hertz, *Phys. Rev.* **B14**, (1976) 1165.

[13] G.A. Baker, B.G. Nickel and D.I. Meiron, *Phys. Rev.* **B17**, (1978) 1365.

[14] I. Rhee, F.M. Gasparini and D.J. Bishop, *Phys. Rev. Lett.* **63**, (1989) 410.

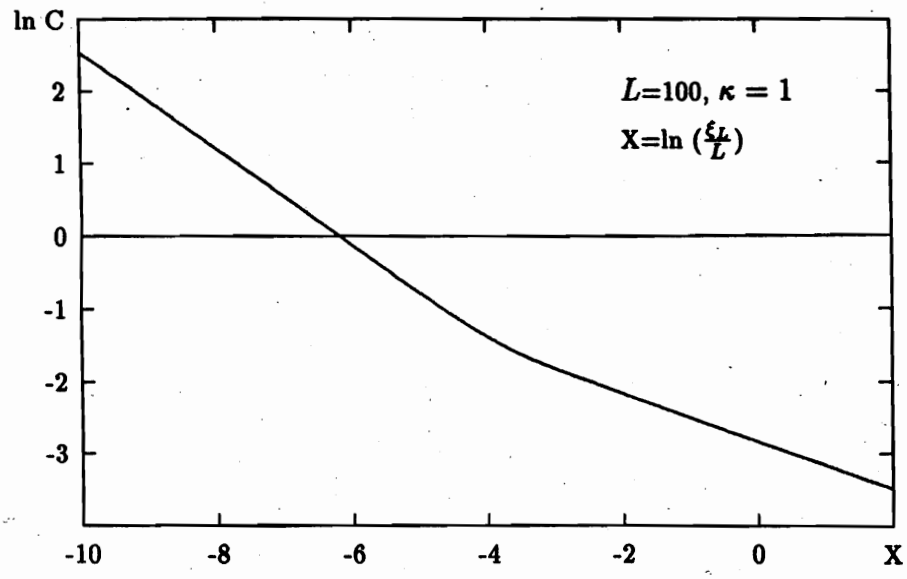

Fig.1 Logarithmic plot of the specific heat C against $\frac{\xi_L}{L}$ for a three dimensional layered Ising model with periodic boundary conditions.

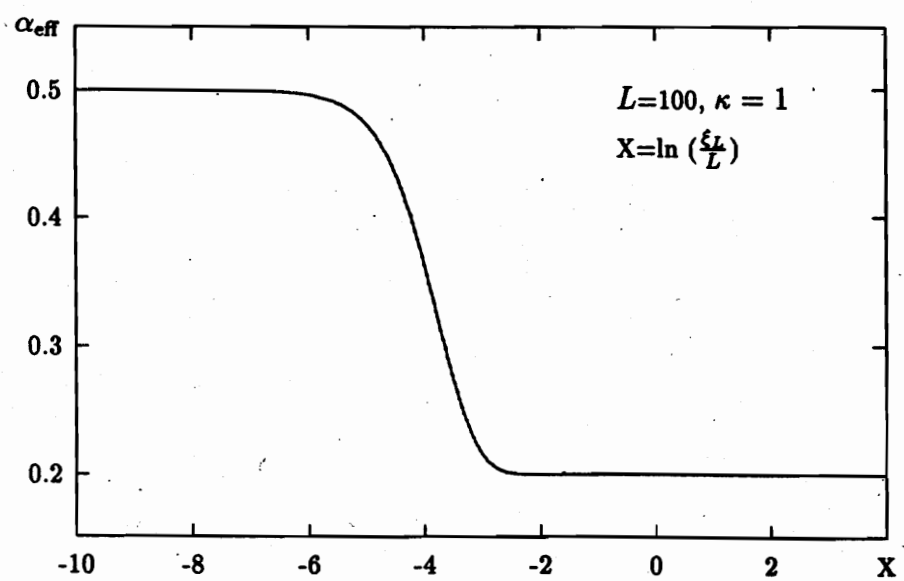

Fig.2 The effective specific heat exponent α_{eff} defined in (3.26) for a three dimensional layered Ising model with periodic boundary conditions.

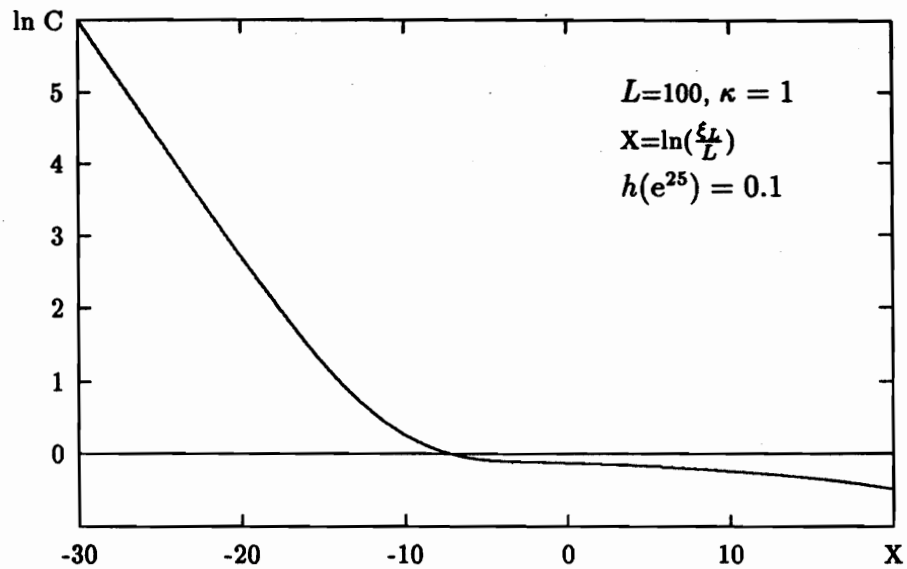

Fig.3 Logarithmic plot of the specific heat C against $\frac{\xi_L}{L}$ for a four dimensional layered Ising model with periodic boundary conditions.

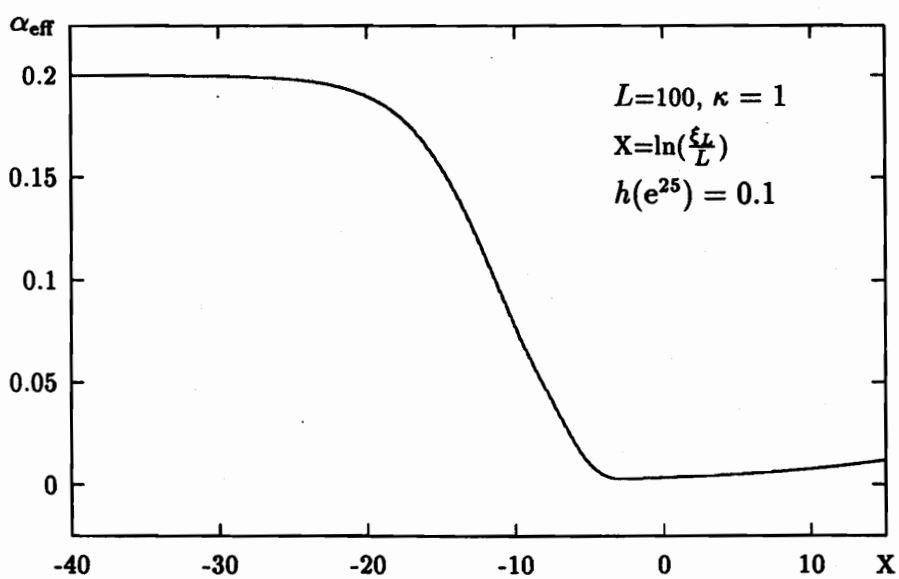

Fig.4 The effective specific heat exponent α_{eff} defined in (3.26) for a four dimensional layered Ising model with periodic boundary conditions.

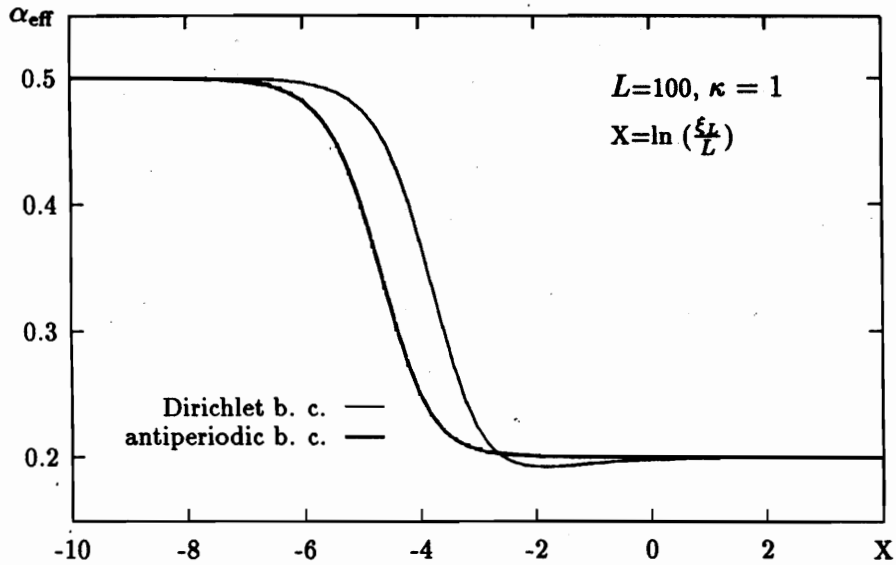

Fig.5 The effective specific heat exponent $\alpha_{\rm eff}$ defined in (3.26) for a three dimensional layered Ising model with both Dirichlet and antiperiodic boundary conditions.

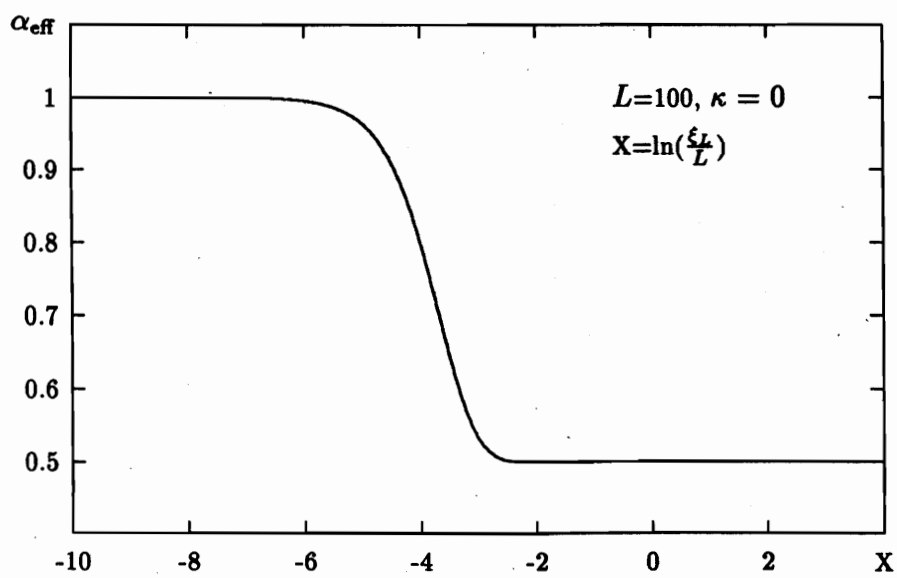

Fig.6 The effective specific heat exponent $\alpha_{\rm eff}$ defined in (3.26) for a three dimensional layered Gaussian model with periodic boundary conditions.

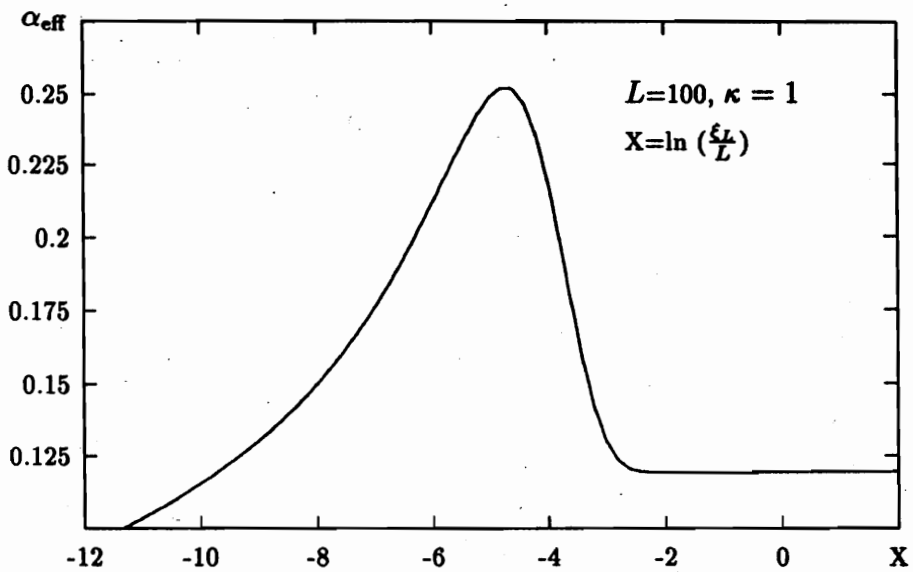

Fig.7 The effective specific heat exponent $\alpha_{\rm eff}$ defined in (3.26) for a three dimensional layered Ising model when $h(z)$ is fitted to the exact two dimensional and three dimensional 6-loop Borel resummed values of ν with periodic boundary conditions.

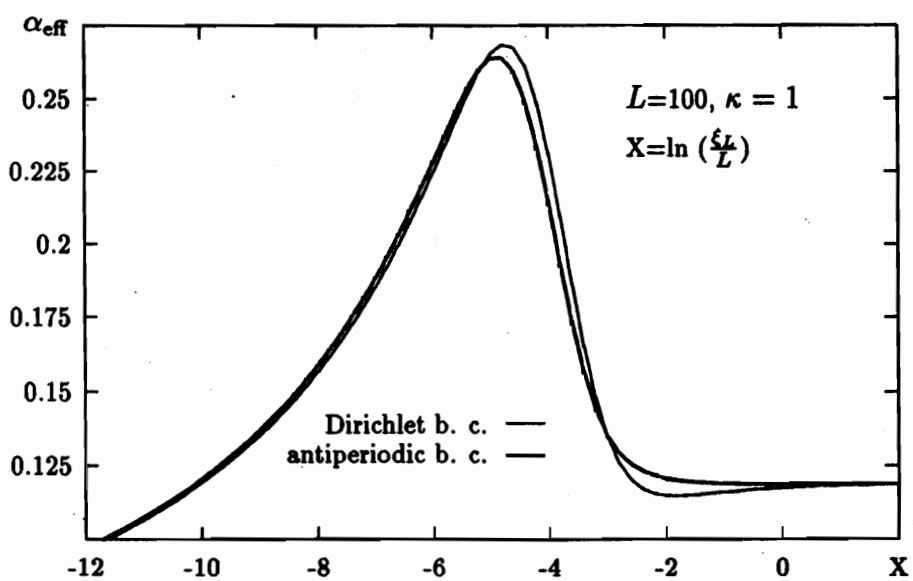

Fig.8 The effective specific heat exponent $\alpha_{\rm eff}$ defined in (3.26) for a three dimensional layered Ising model when $h(z)$ is fitted to the exact two dimensional and three dimensional 6-loop Borel resummed values of ν with Dirichlet and antiperiodic boundary conditions.

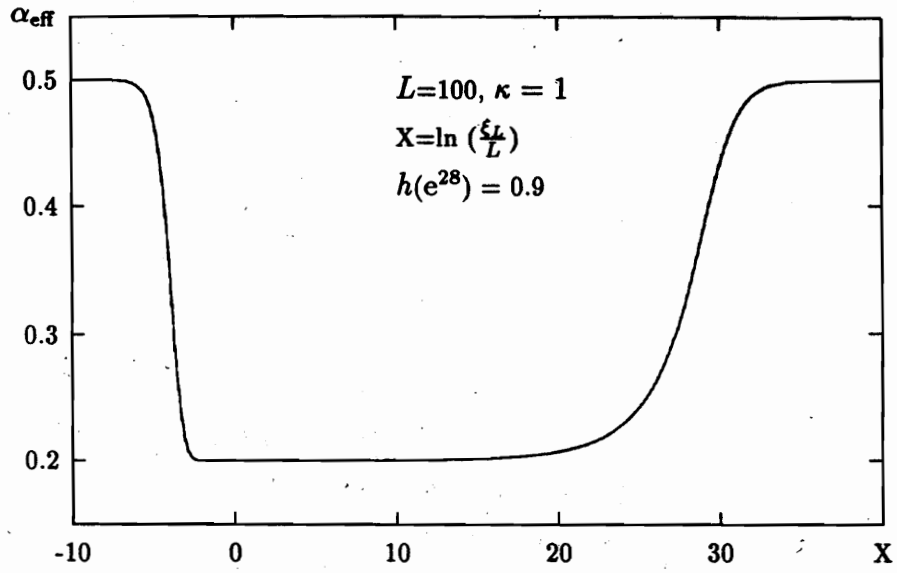

Fig.9 The effective specific heat exponent $\alpha_{\rm eff}$ defined in (3.26) for a three dimensional layered Ising model with periodic boundary conditions. The non-universal part of $h(z)$ is kept.

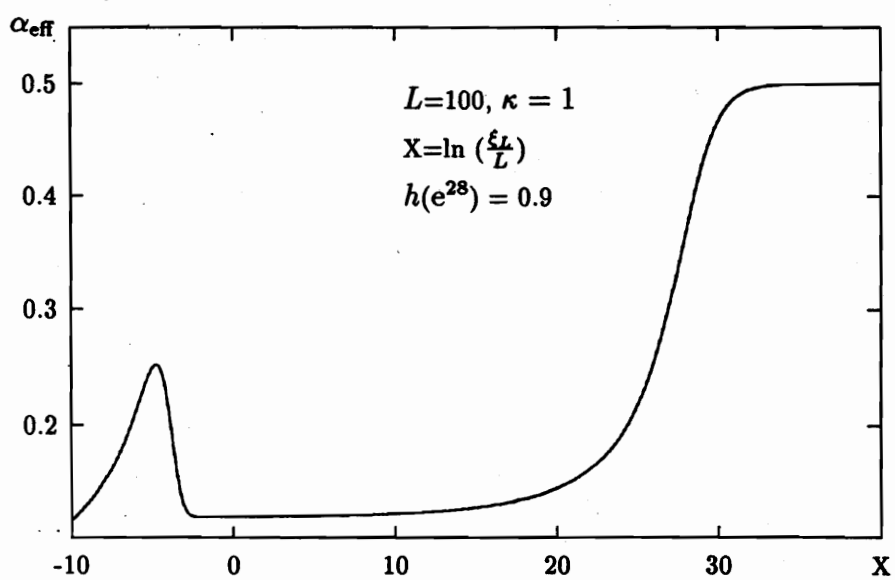

Fig.10 The effective specific heat exponent $\alpha_{\rm eff}$ defined in (3.26) for a three dimensional layered Ising model where $h(z)$ is fitted to the exact two dimensional and three dimensional 6-loop Borel resummed values of ν with periodic boundary conditions.
The non-universal part of $h(z)$ is kept.