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The ground state wavefunctions of a conformally invariant scalar field in different spacetime backgrounds.

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Abstract

In this paper we carefully construct the ground state wavefunction of a conformally invariant scalar field as an Euclidean path integral over real field configurations that vanish at the past end point of Euclidean time. We consider two Euclidean spacetime backgrounds. The compact without boundary, i.e, a 4-sphere, and the asymptotically flat, i.e, a wormhole spacetime. Finally, we show that the analytic continuation of these states to the corresponding Lorentzian spacetime satisfy the time-dependent Schrödinger equation with proper choice of the pre-factor for the wavefunction.

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1 Introduction

In one of our previous papers we studied the classical and quantum evolution of a conformally invariant scalar field in a compact background, i.e, on a 4-sphere without boundary. We first constructed the quantum states as a Euclidean functional integral and then we analytically continued these states to the Lorentzian section of the spacetime. The analytically continued states were found to satisfy the time-dependent Schrödinger equation for a conformally invariant scalar field (from now on CISF) in a de Sitter background. We therefore obtain the Hartle-Hawking wavefunctions for matter perturbtions derived from a CISF in a de Sitter background[3,5].

The central idea was to obtain the quantum states (QSs) of a CISF in a de Sitter background from the analytic continuation of the qts of a CISF in a compact 4-sphere without boundary. These states were constructed from a Euclidean functional integral over real field configurations that vanish at the past end point of the Euclidean time ($a = \tau = 0$), i.e, over regular field configurations.

In another paper we studied the classical and quantum evolution of cosmological perturbations generated from a massive scalar field [1]. We found an approximate solution for the equation of motion of each mode from which we evaluate its wavefunction and the classical density perturbations coming from the quantum fluctuations in each mode.

In this paper we study the equation of motion of cosmological perturbations generated from a conformally coupled scalar field in different cosmological backgrounds.

The problem of the quantum evolution of cosmological perturbations has been studied in the context of the inflationary universe models [2] and of perturbative minisuperspace models [1, 3]. However, in all the previous work the solutions for the equations of motion were either too complicated to be written down in a simple closed form or they were just approximate solutions that hold in the adiabatic limit.

In this paper the emphasis is on obtaining a simple and exact solutions

which hold for all time evolution. As usual, the price one has to pay is that of studying simple models which may not be very realistic. The models which we will be studying are conformally invariant. It is precisely this symmetry which allows one to obtain exact solutions [4,5].

Moreover, we compare the construction of the ground state wavefunctions of a CISF for different background spacetime. Then we explicitly calculate these wavefunctions for each perturbation mode. The resulting wavefunctions are analytically continued into the real Lorentzian regime. The analytically continued states have the following two interesting properties:

1- The initial states at (t=0) are the ground states of minimum excitations.

2- These wavefunctions, when analytically continued, evolve according to the time-dependent Schrödinger equations in the corresponding Lorentzian background (in the first case the de Sitter background and in the second case the Friedmann background).

In Sec.2 we discuss the construction, by using Euclidean path integral of the ground state wavefunction of a massless scalar field in a flat Minkowski background. In Sec.3 we consider how to generalize the construction to the curved backgrounds. Then in Sec.4 we apply the construction to find the Hartle-Hawking wavefunctions in de Sitter spacetime from those for the compact 4-sphere without boundary.In Sec.5, we consider the asymptotically flat Euclidean spacetime of the wormhole. Also in this case we find explicitly the ground state wavefunctions by functionally integrating over real field configurations that vanish in the far past Euclidean time. In Sec.6 we show explicitly that the analyticlly continued ground states satisfy the time dependent Schrödinger equations on the Lorentzian background obtained from the analytic continuation of the corresponding Euclidean backgrounds. In Sec.7 we conclude the paper by making some physical comments on the results obtained.

2 The flat Minkowski background

In this section we will consider as an example the flat Minkowski spacetime background. In this case the analog of the conformally invariant scalar field is the massless scalar field with action:

$$S = 1/2 \int (\partial_\mu \phi \partial^\mu \phi) d^4 x. \tag{1}$$

The Euclidean definition of the ground state wavefunction is:

$$\psi_0(\phi, \tau) =< \phi, \tau/0, -\infty >= \int_{\phi(-\infty)=0}^{\phi(\tau)} D[\phi] e^{(-I(\phi,\tau))}. \tag{2}$$

We break the ϕ field into its fourier components

$$\Phi(X, \tau) = \int \frac{d^3 k}{(2\pi)^{3/2}} \phi_k(\tau) exp(iK.X). \tag{3}$$

If we substitute this expression in the action we get:

$$I = 1/2 \int \frac{d^3 k}{(2\pi)^{3/2}} \int (\dot{\phi}_k^2 + |K|^2 \phi_k^2) d\tau. \tag{4}$$

The equation of motion for each kth mode is:

$$\ddot{\phi}_k - |K|^2 \phi_k = 0. \tag{5}$$

This equation has two solutions:

$$A_- exp(|K|\tau) + A_+ exp(-|K|\tau). \tag{6}$$

Now the above definition of the ground state wavefunction for the field corresponds to the following definition of the ground state wavefunction of each mode:

$$\psi_k(\phi_k; \tau) = \int D[\phi_k] exp(-I(\phi_k)), \tag{7}$$

where $I(\phi_k)$ is

$$I = 1/2 \int (\dot{\phi}_k^2 + |K|^2 \phi_k^2) d\tau. \tag{8}$$

Clearly in order to obtain the ground state we have to functionally integrate over quantum fluctuations around the classical Euclidean solution that vanish in the asymptotic Euclidean past $\tau = -\infty$, i.e, we have to include only quantum fluctuations about the classical negative frequency mode A_-, because it is the only mode mode that has $\phi_k(-\infty) = 0$. Now, the classical

Euclidean action evaluated for this mode, when integrated by parts, it takes the following form:

$$I_K^{ext} = 1/2 \int \phi_k(-\ddot{\phi}_k + |K|^2\phi_k)d\tau + 1/2\phi_k\dot{\phi}_k|_{-\infty}^{+\tau}. \tag{9}$$

The first term vanishes by virtue of the Euclidean equation of motion; the lower limit of the second term vanishes because of the boundary condition satisfied by the negative frequency solution. We therefore get the following simple formula for the extremizing action:

$$I_k^{ext} = 1/2\frac{\dot{\phi}_k^{cl}}{\phi_k^{cl}}\phi_k^2. \tag{10}$$

From which we obtain the ground state wavefunction of the Kth mode

$$\psi_k(\phi_k, \tau) = C_k(\tau)exp(-\frac{1}{2}|K|\phi_k^2) \tag{11}$$

3 The curved backgrounds

Firstly, we define the ground state wavefunctions in the following way as a Euclidean path integral:

i)-The asymptotically flat case:

$$\psi_0(\phi, \tau) =< \phi, \tau/0, -\infty >= \int_{\phi(-\infty)=0}^{\phi(\tau)} D[\phi]e^{(-I(\phi,\tau))}; \tag{12}$$

ii)- The compact background:

$$\psi_0(\phi, \tau) =< \phi, \tau/0, 0 >= \int_{\phi(0)=0}^{\phi(\tau)} D[\phi]e^{(-I(\phi,\tau))}. \tag{13}$$

Secondly, we consider a specific model for the matter field, that is, a conformally invariant scalar field with an action of

$$S = 1/2 \int (\partial_\mu\phi\partial^\mu\phi - \frac{1}{6}R\phi^2)a^3 d\Omega_3 dt. \tag{14}$$

The Lorentzian background is

$$dS^2 = -dt^2 + a^2(t)d\Omega_3^2. \tag{15}$$

And the Euclidean background metric is

$$dS^2 = d\tau^2 + a^2(\tau)d\Omega_3^2, \tag{16}$$

where $a(\tau)$ is

$$a^2(\tau) = a_0^2 + \alpha\tau^2 \tag{17}$$

for the asymptotically flat (the wormhole) case; and

$$a(\tau) = H^{-1}sin(H\tau) \tag{18}$$

for the compact background (4-sphere). Now we may expand the field $\phi(X,\tau)$ in terms of harmonics on S^3,

$$\phi(X,\tau) = \sum_{n=1}^{\infty} f_n(\tau)Q_n(X) \tag{19}$$

The wavefunction of each individual mode is found from the general definition of the ground state wavefunctions presented in the introduction. The asymptotically flat case is

$$\psi_0(f_n,a) = C(a)exp[-I^{ext}(f_n,a)] \qquad a \geq a_0, \tag{20}$$

where I^{ext} is the Euclidean action along the regular mode solution in the Euclidean section [3]. Again we repeat the steps which we have taken to obtain Equations (9) and (10) to get the following simple formula for the classical Euclidean action,

$$I^{ext} = \frac{a^3}{2}\frac{\frac{dg_n}{d\tau}}{g_n}f_n^2 \qquad a \geq a_0, \tag{21}$$

where g_n obeys the classical Euclidean equation of motion on the wormhole background. That is the action of the classical Euclidean solution between the two end point values (f_n,τ) and $(0,-\infty)$. For the compact case,

$$\psi_0(f_n,a) = N(a)exp[-I^{ext}(f_n,a)] \qquad Ha \leq 1, \tag{22}$$

where I^{ext} is the Euclidean action along the regular mode solution in the Euclidean section [3].

$$I^{ext} = \frac{a^3}{2}\frac{\frac{dg_n}{d\tau}}{g_n}f_n^2 \qquad Ha \leq 1, \tag{23}$$

where g_n is the regular solution of the Euclidean equation of motion for the 4-sphere background. We remark that in sec.6 we will show that the analytically continued wavefunctions obey the Schrödinger equation with a given choice of the pre-factors, and that with this choice, the wavefunctions are only approximately normalised with the normalisation constant equal to the pre-factor only in the case of zero real part of the mode functions.

4 Conformally coupled scalar field model for cosmological perturbations on a de Sitter background

The action for a conformally coupled massless scalar field is given in Equation (14). If the cosmological background has S^3 spatial sections, then we may expand the field ϕ in terms of harmonics on the 3-sphere,

$$\phi(t;\xi,\theta,\Phi) = \sum_{nlm} f_{nlm}(t) Q^{nlm}(\xi,\theta,\Phi), \tag{24}$$

where ξ, θ, Φ are the three angle coordinates in the 3-sphere, and Q^{nlm} are the scalar harmonics on the 3-sphere. From now on, we will drop the (lm) indices. The equation of motion for each mode is

$$\frac{1}{a^3}(a^3 \dot{f}_n)^{\cdot} + [\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{n^2}{a^2}] f_n = 0. \tag{25}$$

Consider the conformal transformations,

$$dt = a d\theta \quad and \quad F_n = a f_n. \tag{26}$$

Such transformations map the expanding universe into a static universe and at the same time transform the equation of motion for the perturbations generated from the matter field ϕ into that of a simple harmonic oscillator with constant frequency w_n. As in ref.[5], the time evolution of the physical perturbations may be obtained by solving the equations of motion for the transformed modes F_n and using the inverse of the conformal transformation to obtain solutions for the equations of motion of the physical mode f_n:

$$g_n(t) = \frac{A}{a} Exp[i w_n \int \frac{dt}{a(t)}] + \frac{B}{a} Exp[-i w_n \int \frac{dt}{a(t)}]. \tag{27}$$

Clearly, both of these modes appear to be singular at (a = 0), however, one must remember two things: (1) these modes are Lorentzian modes; and (2) in the exponential factor we have $\frac{1}{a}$. For ($w_n > 1$), which is the case for all inhomogeneous modes, the regular mode function remains regular (it vanishes at a=0) even if multiplied by a factor of $\frac{1}{a}$. We shall write the solution explicitly for the closed de Sitter background. It is written in terms of the de Sitter scale factor ($aH = cosh(Ht)$ for $Ha \geq 1$),

$$g_n(a) = \alpha \frac{[1 + i\sqrt{(Ha)^2 - 1}]^{w_n}}{(Ha)^{w_n+1}} + \beta \frac{(Ha)^{w_n-1}}{[1 + i\sqrt{(Ha)^2 - 1}]^{w_n}}. \tag{28}$$

Thus the mode which is regular at a=0 is

$$g_n(a) = \beta \frac{(Ha)^{w_n-1}}{[1 + \sqrt{1 - (Ha)^2}]^{w_n}} \quad Ha \leq 1, \tag{29}$$

$$g_n(a) = \beta \frac{(Ha)^{w_n-1}}{[1 + i\sqrt{(Ha)^2 - 1}]^{w_n}} \quad Ha \geq 1. \tag{30}$$

It is clear that in the case of the homogeneous mode $w_n = n = 1$ the regular part of the mode remains constant for small values of the scale factor. Now we can evaluate the quantum state of the mode f_n. In the Lorentzian region the quantum states of these modes evolve according to the time dependent Schrödinger equation [1, 3, 5]. This will be shown in Sec.6. We will also show that we may obtain initial condition for the evolution of these quantum states by using Hartle-Hawking boundary conditions which allow for only regular matter configurations in the definition of the quantum state as a functional integral ($\psi_0(f_n,\ a) = < f_n,\ a/\ 0,\ 0 >$) [1, 3, 5]. We therefore get Equation (22). Notice that all the dependence on the end point f_n is in I^{ext}, i.e, in the Euclidean action along the regular mode solution in the Euclidean section Equation (23). In the Lorentzian region $\psi(f_n,\ Ha \geq 1)$ is the analytic continuation of $\psi_0(f_n, a)$.

In order to evaluate $\psi(f_n,\ Ha \geq 1)$, we may evaluate the value $\frac{\dot{g}_n}{g_n}$ in the region $Ha \geq 1$, which gives the result

$$\frac{\dot{g}_n}{g_n} = -(1 - i\frac{w_n}{\sqrt{(Ha)^2 - 1}})\frac{\dot{a}}{a}. \tag{31}$$

5 Conformally coupled scalar field theory in an asymptotically flat spacetime

In this section we consider the Euclidean definition of the ground state wavefunction of a conformally invariant scalar field in the asymptotically flat Euclidean backgrounds, i.e, the spacetime of the Hawking wormhole (see figure 2). Occasionally, we will comment on the analogy with the compact bacground discussed in the last section.

We again use the Euclidean functional integral to construct the ground state wavefunctions, and we find an analogy between the two cases.

The surprising result, as will be verified soon, is that the quantum states constructed as Euclidean path integral over a very specific class of histories are found to satisfy two properties:

1- They are ground states;

2- They satisfy the corresponding Schrödinger equation in the given spacetime background, i.e, the Lorentzian background obtained by analytic continuation from the Euclidean background.

Let us Construct the ground state. We define the ground state wavefunctions in the following way as a Euclidean path integral in the asymptotically flat case

$$\psi_0(\phi,\tau) =< \phi,\tau/0,-\infty >= \int_{\phi(-\infty)=0}^{\phi(\tau)} D[\phi]e^{(-I(\phi,\tau))}. \tag{32}$$

This is the analog of Equation (13) for the compact background. Now we consider a specific model for the matter field, that is, a conformally invariant scalar field with an action as in Equation (14). And the Lorentzian background metric is

$$dS^2 = -dt^2 + a(t)^2 d\Omega_3^2, \tag{33}$$

while the Euclidean background is

$$dS^2 = d\tau^2 + a(\tau)^2 d\Omega_3^2, \tag{34}$$

where$a(\tau)$ is

$$a^2(\tau) = a_0^2 + \alpha\tau^2 \tag{35}$$

for the asymptotically flat (the wormhole) case and

$$a(\tau) = H^{-1} sin(H\tau) \tag{36}$$

for the compact background (4-sphere). Now we may expand the field $\phi(X,\tau)$ in terms of harmonics on S^3 as in Equation (19).

The mode equation of motion is

$$\frac{1}{a^3}(a^3\dot{g}_n)^{\cdot} + (\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 - \frac{n^2}{a^2})g_n = 0, \tag{37}$$

where the dots denote derivative with respect to the Euclidean time τ. The general solution is

$$g_n(\tau) = \frac{A_+}{a} Exp(n\int_{\tau_0}^{\tau} \frac{d\tau}{a(\tau)}) + \frac{A_-}{a} Exp(-n\int_{\tau_0}^{\tau} \frac{d\tau}{a(\tau)}). \tag{38}$$

it holds for any $a(\tau)$, but one has to choose the proper past end point of the Euclidean time (for example $\tau_0 = -\infty$ in the asymptotically flat case and $\tau_0 = 0$ in the compact case).

In the asymptotically flat case,

$$g_n(\tau) = \frac{A_+}{a}[\frac{a}{a_0} + \frac{\sqrt{a^2 - a_0^2}}{a_0}]^{\frac{n}{\sqrt{\alpha}}} + \frac{A_-}{a}[\frac{a}{a_0} + \frac{\sqrt{a^2 - a_0^2}}{a_0}]^{\frac{-n}{\sqrt{\alpha}}}. \tag{39}$$

Clearly this equation is the Euclidean analog of Equation (28) for the compact case.

These two explicit solutions reveal a lot of facts to us. The most evident is the understanding of what is intended when one says that the ground state wavefunction is constructed by allowing only regular real Euclidean field configurations or field configurations that vanish at $\tau = -\infty$ in the asymptotically flat case and field configurations that vanish at the degenerate three geometry ($\tau = a = 0$) in the case of compact background without boundary. The real meaning is that the functional integral should be taken over quantum fluctuations about the classical Euclidean solution which has a vanishing inhomogeneous spatial field configuration in the past Euclidean end point of the time. A careful look shows that in the asymptotically flat

case, also the regular homogeneous mode starts with a vanshing amplitude at $\tau = -\infty$. Morover, it is clear that the two mode functions are real and one of them is regular (vanishes) at $\tau = -\infty$ while the other one is singular. This means that one should use negative frequency modes inorder to evaluate the wavefunction, that is, one should use the modes that satisfy:

$$g_n(-\infty) = 0, \quad A.F; \quad g_n(0) = 0, \quad \textit{for the compact background.} \tag{40}$$

Thus for the asymptotically flat spacetime we may calculate $\frac{\dot{g}_n}{g_n}$ for the regular Euclidean mode

$$\frac{\dot{g}_n}{g_n} = \frac{n}{a} + \frac{\sqrt{a^2 - a_0^2}}{a^2}. \tag{41}$$

6 Analytic continuation of the wavefunctions to the Lorentzian section of the spacetime

The Lorentzian version of the wormhole metric is

$$ds^2 = -N^2 dt^2 + a^2 d\Omega_3^2 \tag{42}$$

where $a^2(t) = a_0^2 - \alpha t^2$. It is a Friedmann universe that starts in a maximum radius a_0 at t=0 and collapses to a singularity at $t = \frac{a_0}{\sqrt{\alpha}}$. On the other hand, the Lorentzian spacetime corresponding to the 4-sphere is the de Sitter hyperboloid with scale factor $a(t) = H^{-1} cosh Ht$. This spacetime is joined to the 4-sphere at the point where the 4-spere has a maximum radius[1]. Therefore, in the real Lorentzian time, this spacetime starts with a minimum radius H^{-1} and expands to infinite radius. So to some extent these two examples are contrary to each other because the first example represents a universe that starts with infinite radius and collapses to a singularity, while the second case it represents a spacetime starting to expand from zero radius in a regular manner until it reaches an infinite radius. Now we shall consider

[1]The analytic continuation from the Euclidean to the Lorentzian sections or vice versa should respect the continuity of both the intrinsic and extrinsic curvatures at the surface where the signature changes

the analytic continuation of the quantum states to the Lorentzian section of the spacetime.

It is clear that if we want to calculate the quantum state at the corresponding Lorentzian spacetime, we should analytically continue the scale factor to values less than a_0, The analytically continued states have the following form,

$$\psi(f_n, t) = C(a) Exp(i\frac{a^3}{2}\frac{\dot{g}_n}{g_n} f_n^2), \tag{43}$$

where a is the scale factor of the Lorentzian spacetime and g_n are the Lorentzian modes obtained by analytically continuing the regular Euclidean mode. Clearly this mode satisfies the Lorentzian equation of motion (25) where the dot now refers to the derivative with respect to Lorentzian time t and a = a(t) is the scale factor of the Lorentzian section. Now we evaluate $\frac{\dot{g}_n}{g_n}$ for the Lorentzian mode, the mode obtained by analytically continuing the regular Euclidean mode. For the asymptotically flat case we get:

$$\frac{\dot{g}_n}{g_n} = \frac{i}{\sqrt{\alpha}}[\frac{n}{a} + i\frac{\sqrt{a_0^2 - a^2}}{a^2}] \qquad a \leq a_0, \tag{44}$$

while for the compact case we get Equation(31). Therefore, the quantum state of a conformally invariant scalar field in the Friedmann universe obtained by analytically continuing the quantum state in the Euclidean wormhole background is

$$\psi(f_n, a) = C(a) exp\left(-\frac{a^3}{2\sqrt{\alpha}}[\frac{w_n}{a} - i\frac{\sqrt{a_0^2 - a^2}}{a^2}]f_n^2\right) \quad a \leq a_0. \tag{45}$$

For the compact case we obtain the quantum state of a CISF in de Sitter spacetime by analytically continuing the states obtained for the 4-sphere:

$$\psi(f_n, a) = N(a) exp\left(-\frac{a^3}{2}[\frac{w_n}{a} - i\frac{\sqrt{(Ha)^2 - 1}}{a}]f_n^2\right) \quad Ha \geq 1. \tag{46}$$

It is an important step to show that these states evolve according to the time-dependent Schrödinger equation for the CISF in the corresponding background. This can be shown easily by directly substituting the explicit expressions obtained for the wavefunction in the relevent Schrödinger equations

[1,2]:

$$i\partial_t\psi_n(f_n,t) = \frac{1}{2a^3}\left(-\partial^2_{f_n} + a^6 W_n^2(a) f_n^2\right)\psi(f_n,t) \qquad Ha \geq 1, \tag{47}$$

where $W_n^2(a) = \frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{w_n^2}{a^2}$ we first write ψ_n as:

$$\psi_n(f_n,a) = C(a)Exp(iS(f_n,a)), \tag{48}$$

where $S(f_n,a) = 1/2\frac{\dot{g}_n}{g_n}f_n^2$. So from the left hand side of Schrödiger equation we get

$$i\dot{\psi} = [i\frac{\dot{C}}{C} - \dot{S}]\psi, \tag{49}$$

where

$$\dot{S} = \frac{a^3}{2}f_n^2[3\frac{\dot{a}}{a}\frac{\dot{g}_n}{g_n} + \frac{\ddot{g}_n}{g_n} - (\frac{\dot{g}_n}{g_n})^2], \tag{50}$$

and

$$-\frac{\partial^2}{\partial f_n^2}\psi = (-ia^3\frac{\dot{g}_n}{g_n} + (a^3\frac{\dot{g}_n}{g_n})^2)\psi. \tag{51}$$

Therefore we get from the imaginary part of the Schrödinger equation:

$$\frac{\dot{C}}{C} = -1/2\frac{\dot{g}_n}{g_n}, \tag{52}$$

while from the real part:

$$-\frac{a^3}{2}[3\frac{\dot{a}}{a}\frac{\dot{g}_n}{g_n} + \frac{\ddot{g}_n}{g_n}] = 1/2a^3W_n^2(a) \tag{53}$$

Clearly the second equation is the equation of motion for the perturbation mode f_n in the cosmological background that has a scale factor $a(t)$. The wavefunctions in both cases satisfy the SEs, provided we choose the pre-factor to satisfy Equation (52) which has the solution,

$$C = C_0\sqrt{\frac{1}{g_n}}. \tag{54}$$

A very important point to note is the vanishing of the real part of $\frac{\dot{g}_n}{g_n}$ at the minimum radius, i.e., at the 3-geometry where the transition from Euclidean to Lorentzian signature takes place. This means that in the Lorentzian space-time the proper initial condition for the wavefunction $\psi(f_n,t)$ at (t=0) is exactly the ground state wavefunction. In fact, similar thing happens in the case of compact background in which the real part of $\frac{\dot{g}_n}{g_n}$ also vanishes at the

maximum radius of the 4-sphere. So in both cases the proper initial condition at (t=0) is the ground state wavefunction.

From the individual mode wavefunctions, we may construct the ground state wavefunctional of the field ϕ, which is the product of all the modes' wavefunctions and satisfies the functional Schrödinger equation on the given background,

$$\psi(\phi,t) = C(t)\Pi_n Exp(\frac{ia^3}{2}\frac{\dot{g}_n}{g_n}f_n^2). \tag{55}$$

7 Conclusions

In this section we comment on the results obtained in this paper with an emphasis on the compact or de Sitter case. Therefore we emphasise the physical meaning of our results. In the de Sitter case, our result shows clearly how initial conditions can be obtained for the evolution of the perturbations at the horizon crossing from the quantum fluctuations. Explicitly, the quantum state of the inhomogeneous f_n mode obeying the regularity boundary condition is given in Equation (46). The condition for the perturbations to behave classically is that $\psi_n(f_n, a)$ oscillates rapidly giving rise to a family of classical histories $g_n^{cl}(t)$, i.e.,

$$|Re(\frac{\dot{g}_n}{g_n})| >> |Im(\frac{\dot{g}_n}{g_n})| \tag{56}$$

This is equivalent to ($\frac{w_n}{a} << \frac{\dot{a}}{a}$), which amounts to the fact that the perturbations are outside the Hubble radius. Clearly, the dispersion of these quantum states evolves trivially with time (it represents an adiabatic redshift of the frequency of the mode with the expansion of the background); and according to the methods of reference [1] this model is not good for studying the birth of the seeds for the origin of the large scale structure, i.e., the birth of classical density perturbations from quantum fluctuations.

Two features of our result deserve comments. The first is the behaviour of the quantum state as a function of the physical frequency with the scale factor. This feature is very clear and explicit from Equation (46). The result shows how the quantum state behaves as a function of the physical wavelength of the nth mode $\lambda_n = \frac{a}{n}$, and that as the wavelength is being

stretched to values larger than the Hubble radius, the quantum state starts becoming a rapidly oscillating function of f_n, therefore exhibiting explicit classical behaviours. And second in the asymptotically flat case the frequency of the mode blue shifts as the universe collapses.

However, also in the asymptotically flat or wormhole case, one would like to understand the behaviour of the quantum state in the Lorentzian section of the spacetime, i.e, Equation(45). This Lorentzian spacetime is not very physical; it represents a Friedmann universe that starts from small maximum radius and collapses to form a singularity. However, it is possible to use the result in another way. By studying the behaviour of the quantum state in the two asymptotically flat regions of the wormhole spacetime, it is possible to study the scattering of these states with the spacetime curvature in the Euclidean section. One may prepare a state in the asymptotic past in the Euclidean section and see how it scatters with the spacetime curvature near the minimum radius by examining its behaviour in the asymptotic future.

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Figure 1: The background spacetime; in the case of de Sitter background the Euclidean section is half of a 4-sphere, we study the behavior of the perturbations on the euclidean section in order to impose the regularity boundary condition at a =0

Figure 2: The background spacetime; in the case of the asymptotically flat spacetime of the wormhole the boundary conditions are imposed at $a = \infty$ in the past asymptotic region. The Lorentzian spacetime corresponding to this background is a Friedmann universe which starts from maximum radius a_0 at t=0 and collapses into a singularity at t=t_0.

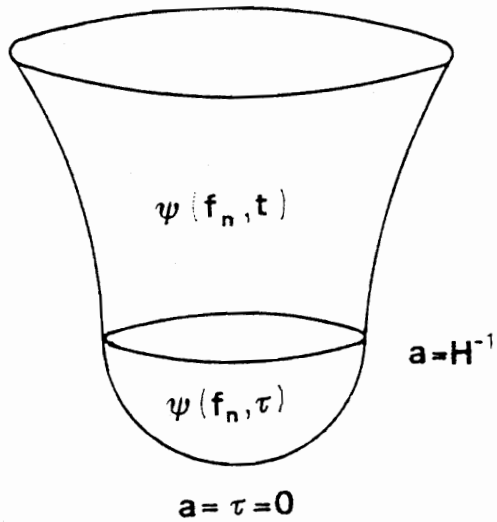

Fig.1

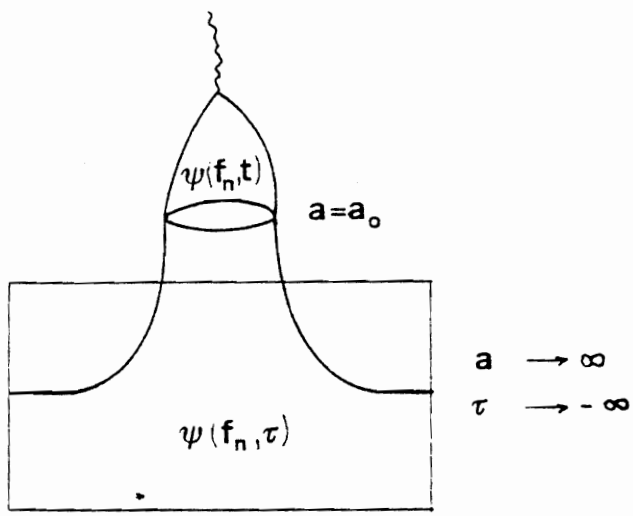

Fig.2