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Quarks on a Lattice

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Abstract

We explore alternate solutions of Dirac equation on a lattice. In this sector we explore the characteristics of the states and argue that the excitation spectrum is different.

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Fermions on a lattice is a subject of wide interest. In high energy physics it is increasingly being used to understand the world of hadrons[1]. The lattice version of quantum chromodynamics, QCD, has allowed calculations into areas of hadron physics previously inaccessible[2-8]. In condensed matter physics the fermions are mostly on a lattice, and are the basis of much of the theoretical calculations.

The states on a lattice are the Bloch functions. For a system of many particles a determinant of these states are formed. The excitation spectrum, usually consists of single particles over the fermionic sea. It turns out, however, that there are other ways of constructing multiparticle states that have different excitation spectrum. We deal with one such possibility in this work.

We work in the lattice version of Dirac equation. On a lattice this leads to a hopping system, which is well studied[9]. It can be diagonalised, and for the simplest hopping the result is band states made of bloch waves. There is, however, an infinite dimensional[10,11] global symmetry hidden in the hopping system that allows us to construct other solutions to the hopping problem. These sectors of solutions have fermions all paired, unlike the band states, which have single particle spectrum.

To proceed, let us start with a simple hamiltonian and illustrate all the ideas. At that stage we take up the more complicated hopping scenario in staggered fermion systems and show how the simple ideas generalize.

In section I a simple hopping of fermions is used to arrive at the infinite dimensional algebra. States are constructed that are based on this algebra. We take up the hopping scenario in staggered fermions[12] in section II. The algebra and the states are illustrated. It turns out one can construct another independent set of generators, and thus there is a degeneracy to these states. In section III we discuss the energies of these states and explore some possible excitations.

Section I : Alternate Solutions of Simple Hopping

Let us place on each site of a lattice a single component fermi field a_n [n is the site index] satisfying

$$\{a_n , a_m\} = 0$$
$$\{a_n^+ , a_m\} = \delta_{nm} \qquad \ldots (1)$$

The dynamics of hopping of these fermions is given by the hamiltonian:

$$H = -\sum(a_n^+ a_{n+1} + a_{n+1}^+ a_n) \qquad \ldots (2)$$

The equations of motion are :

$$\dot{a}_n = -(a_{n+1} + a_{n-1}) \qquad \ldots (3)$$

The Hamiltonian diagonalizes if we go to the bloch basis:

$$|k\rangle = \sum a_k(n) |n\rangle , \qquad \ldots (4)$$

where $|n\rangle$ are Wannier states , k are bloch momenta, with the relation :

$$a_k(n+1) = e^{ika} a_k(n) \qquad \ldots (5)$$

following from Bloch's theorem.

To pass to another sector for this hamiltonian we construct[13] the generator

$$e_1 = \sum_{\substack{ij \\ \eta \text{ as in} \\ \text{fig.1}}} T_{ij}^+(\eta) \qquad \text{where} \ldots (6)$$

i and j are nearest neighbor site indices; η is defined as.

$$T_{ij}^+ = a_i^+ a_j^+ , \quad T_{ij}^+(\eta) = T_{ij}^+(\pm) = \pm T_{ij}^+ . \qquad \ldots (7)$$

It is straightforward to show that this generator commutes with the hamiltonian (2). Similarly, the generator,

$$e_{-1} = e_1^+ \qquad \ldots (8)$$

also commutes with (2).

From the generators e_1 , e_{-1} we can construct the algebra of invariance of (2). To construct the other

generators of this closed algebra we calculate the commutator of e_1 and e_{-1}, and find it contains two invariant generators:

$$h_1 = \sum a_n^+ a_n \, , \qquad \ldots (9)$$

which is the global number of fermions, and a translation generator given by :

$$h_2 = \sum a_n^+ a_{n+2} \qquad \ldots (10)$$

We can now get new invariant generators by calculating the commutators of h_1 and h_2 with e_1 and e_{-1}. We find, the commutator of e_1 and h_2 is an invariant generator e_2 given below. Clearly the algebra closes on periodic boundary conditions.

Briefly then the generators of invariance of (2) may be grouped as follows :

(a) The h type generators :

$$h_1 = \sum a_n^+ a_n \, ;$$
$$h_2 = \sum a_n^+ a_{n+2} \, ,$$
$$h_3 = \sum a_n^+ a_{n+4} \quad \ldots . \text{ etc.} \qquad \ldots (11)$$

(B) The $e_\pm$ type generators :

$$e_1 = \sum T_{ij}^+ (\eta) \, ; \; e_{-1} = e_1^+ \, ,$$

$$e_2 = T_{14}^+ (+) + T_{25}^+ (-) + T_{36}^+ (+) + \ldots \, ; \; e_{-2} = e_2^+$$

$$\ldots (12)$$

The set of generators h commute with one another. This may be taken as the maximally commuting set. Let us work in the basis of the global number, h_1. In this basis there is a state $|0\rangle$, which is devoid of all fermions. Since all the other generators of the set h operating on $|0\rangle$ leave it invariant, it is an appropriate custodial state. To construct[14-20] the solutions of (2) we follow the procedure for construction of the coherent states. Thus [21] if Z are a set of complex numbers, the solutions of (2) are .

$$|Z\rangle = (\exp \sum_i Z_i e_i - h.c.)) \, |0\rangle \qquad \ldots (13)$$

It is important to note that the fiducial state has been chosen as the eigenstate of h_1 with eigenvalue of zero. The commutator of $e_\pm$ with h_1 is noteworthy, namely,

$$[h_1 , e_\pm] = \pm 2 e_\pm \qquad \ldots (14)$$

Thus, the generators $e_\pm$ change global number in much the same way a^+ and a do for the local number. In that sense $|Z\rangle$ are states that have the number of fermions fluctuating about some average value, somewhat analogous to the coherent states of optics.[21] The set $|Z\rangle$ are all degenerate solutions of (2). It is also worth noting that unlike the band states, which are single particle solutions of (2), the $|Z\rangle$ solutions have the fermions all paired up. Further, because of the η assignment in figure 1 the size of the unit cell has been doubled, halving the Brillouin zone. This might be useful in solving the species doubling problem.

Section II : Hopping of Staggered Fermions

We now explore the Dirac equation to see if it has similar solutions on a lattice. We work in the hamiltonian formalism, and for simplicity assume the fermions to be massless. We work in a two dimensional lattice, i.e., in two space dimensions. The fermions have two components, and the continuum hamiltonian of interest is[22] :

$$H = i \; \psi^{+} \alpha \cdot \nabla \psi \qquad \ldots (15)$$

where

$$\left\{ \alpha_i \, , \, \alpha_j \right\} = 2\delta_{ij} \qquad \ldots (16)$$

We choose the representations of α-matrix in two dimensions[23]

$$\alpha_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; \; \alpha_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} . \qquad \ldots (17)$$

The hamiltonian on a two dimensional lattice may be cast as

$$H = i \sum_{j,\mu} \psi^{+}(j) \; \alpha_{\mu} \{ \psi(j+\mu) - \psi(j-\mu) \} \qquad \ldots (18)$$

where $j = (n_1 \, , \, n_2)$.

This hamiltonian may be 'diagonalised' , if we define two independent objects, ϕ and ϕ^{+} as[24]:

$$\psi = \alpha_1^{|n_1|} \; \alpha_2^{|n_2|} \; \phi \qquad \ldots (19)$$

$$\psi^{+} = \phi^{+} \alpha_1^{|n_1|} \; \alpha_2^{|n_2|} \qquad \ldots (20)$$

The hamiltonian written in terms of ϕ and ϕ^{+} is

$$H = i \sum_{j,\mu} \phi^{+}(j) (-1)^{f(n_1, n_2)} [\phi(j+\mu) - \phi(j-\mu)] \qquad \ldots (21)$$

where

$$(-1)^{f(n_1, n_2)} = +1 \quad \text{in the direction } n_1 \text{ at the point } j . \qquad \ldots (22)$$

and

$$(-1)^{f(n_1, n_2)} = (-1)^{n_1} \quad \text{in the } n_2 \text{ direction at the point } j. \qquad \ldots (23)$$

If we write this out for a single plaquette, we get for the plaquette abhg in fig. 2.

$$H_{abhg} = -i \{ \phi^{+}(a) \phi(b) - \phi^{+}(b) \phi(a) + \phi^{+}(a) \phi(g) - \phi^{+}(g) \phi(a) + \phi^{+}(g) \phi(h) - \phi^{+}(h) \phi(g) - \phi^{+}(b) \phi(h) + \phi^{+}(h) \phi(b) \} ; \qquad \ldots (24)$$

$$H = H_{abhg} + \text{Other Plaquettes.}$$

Let us now construct a coherentised global pairing operator,

$$\bar{e}_1 = \sum_{ij} \bar{T}^{+}_{ij}(\beta) \qquad \ldots (25)$$

where

$$\bar{T}^{+}_{ij} = \phi^{+}(i)\ \phi^{+}(j) \qquad \ldots (26)$$

and

$$\bar{T}^{+}_{ij}(\beta) = \bar{T}^{+}_{ij}(\pm) = \pm\bar{T}^{+}_{ij} \qquad \ldots (27)$$

If β in equation (25) is chosen as shown in figure (2), $\bar{e}_1$ commutes with (24).

Similarly, $\bar{e}_{-1}$ obtained from $\bar{e}_1$ as :

$$\bar{e}_{-1} = \bar{e}^{+}_{1} \qquad \ldots (28)$$

also is a symmetry generator of (24).

From $\bar{e}_1$ and $\bar{e}_{-1}$, much like before, we can construct the closed algebra of invariance of (24). To illustrate again, if we take the commutator of $\bar{e}_1$ and $\bar{e}_{-1}$ we obtain two invariant generators, $\bar{h}_1$ and $\bar{h}_2$ given as,

$$\bar{h}_1 = \sum_i \phi^{+}_i\ \phi_i \text{ , the global number operator,} \qquad \ldots (29)$$

$$\bar{h}_2 = \phi^{+}(a)\ \phi(c) + \phi^{+}(b)\ \phi(d) + \phi^{+}(n)\ \phi(g) + \ldots \qquad \ldots (30)$$

which is a translation, From $\bar{h}_2$ and $\bar{e}_1$ we obtain, another invariant operator, $\bar{e}_2$, as :

$$\bar{e}_2 = -\phi^{+}(a)\ \phi^{+}(d) - \phi^{+}(b)\phi^{+}\ e' + \phi^{+}(m)\ \phi^{+}(g) + \phi^{+}(n)\phi^{+}(O) + \ldots \qquad \ldots (31)$$

The algebra closes for periodic boundary conditions.

Is it possible to change the assignment β in fig (2) to generate other invariant operators that are independent of the generators of the above algebra? It turns out there is just one other possible assignment that leads to independent invariant generators. The proof of it is as follows. It is clear once phases are assigned on a plaquette the β values of the entire lattice are fixed. We, therefore, confine ourselves to the possible assignments on a single plaquette. In figure 3 we show four possible β assignments that lead to a commuting configuration. However, 3A and 3B are not independent, they are related by an overall sign. Similarly 3C and 3D differ by an overall sign. Thus, we have only two independent configurations. Therefore, much like the set $\bar{e}$, $\bar{h}$, there is another set $\bar{E}$ and $\bar{H}$ which form an independent closed algebra. We have states of (24) given by,

$$|z\rangle = \left\{\exp \sum_{\alpha} \left(z_\alpha\ \bar{e}_\alpha - h.c\right)\right\} 10 \qquad \ldots (32)$$

which are exact eigensolutions of the hamiltonian. Replacing $\bar{e}$ by $\bar{E}$ another set of states are obtained. Note that all

these solutions are degernate with $|0\rangle$. In this sector the fermions [see (25) and (26)] are all paired up.

Section III : The Spectrum

The states $|Z\rangle$, as is clear, are degenerate with the state $|0\rangle$ for the hamiltonians we have considered. Since the usual band states, up to half filling, are lower in energy than $|0\rangle$, the states $|Z\rangle$ lie on the top edge of a half-filled band.

There is a aspect of the spectrum of excitations of this sector of hopping that is worth pointing out. For simplicity let us get back to the first example (2) and observe fig.1 again. Since the sign η, between the points 1 and 2 and between 7 and 8 are both positive, the periodic boundary conditions cannot be implemented. Had we started with seven points, instead of eight, there would have been no problem. Since points on the boundary are sufficiently far away to alter physics, we conclude that the case when two identical signs, η, sit on neighboring links, be it on the boundary, or somewhere in the middle, it must be in the spectrum of the system. Thus, the case of two neighboring identical signs is an excited state of the system. It is a phase defect excitation.

The situation with respect to the second example, (24), is more or less identical. The two β configurations of fig 3 could form a boundary wall. This once again gives

rise to a pure phase excitation. The energy of such a pure phase excitation we have not determined, but the presence of these and their mixing with other scalar objects is likely to be a part of physics if it arises from these sectors.

So far we have discussed the case of pure hopping. In presence of interactions do these $|Z\rangle$ states survive? Many a times interactions tend to favor pairing of particles, and in these instances, the $|Z\rangle$ basis could be better than the band ones. We have shown recently that in the Hubbard model, with repulsive on site interaction of strength U, the spin singlet coherent $|Z\rangle$ states are solutions. They lie in the middle of the spin-density wave gap[25]. Whether the $|Z\rangle$ basis of the staggered fermions survives interactions remains under investigation.

When self energy, i.e. mass terms, are included, these terms are proportional to the generator h_1. Since, in general the generators e_β and h_1 do not commute, $|Z\rangle$ are solutions with a special meaning. Since the expectation value of h_1 is a constant in the $|Z\rangle$ states, the $|Z\rangle$ are coherent state solutions[26] of the hamiltonian[27]. It is in this sense that the $|Z\rangle$ are states of the Hubbard model.

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References

1. P.Roy, Theory of lepton-Hadron Processes at High-Energies, Clarendon Press, Oxford.
   R.C.Hwa, Quark Gluon Plasma, World Scientific (1990).
2. K.G.Wilson, Phys.Rev. **D15** (1977) 1128.
   See also
   A.M.Polyakov, Phys.Lett **B59** (1975) 79, 82.
   F.Wegner, J.Math.Phys. **12** (1971) 2259.
3. M.Creutz, Phys.Rev. **D15** (1977) 1128.
   M.Creutz, Phys.Rev. **D21** (1980) 2308.
   M.Creutz, Phys.Rev.Lett. **45** (1980) 313.
4. L.McLerran and B.Svetitsky, Phys.Lett. **98B** (1981) 195.
   J.Engels, F.Karsch, I.Montvay and H.Satz, Nucl.Phys. **B205** (1982) 545.
   J.Kogut, M.Stone, H.Wyld, W.Gibbs, J.Shingemitsu, S.Shenker and D.Sinclair, Phys.Rev.Lett., **50** (1983) 393.
5. M.Creutz, Quarks, Gluons and Lattices, Cambridge University Press, Cambridge 1983.
6. S.J.Hands and M.Teper, Nucl.Phys. **B347** (1990) 819.
7. I.Bender, R.Horsley and W.Wetzel, Nucl.Phys. **B349** (1991).
8. A.Hasenfratz and T.Neuhaus, Phys.Lett. **B220** (1989) 435.
9. C.Kittel, Introduction to Solid State Physics, Wiley (New York).
10. J.Chakrabarti and R.Chaudhury, Mod.Phys.Lett. **B5** (1991) 1525.
11. J.Chakrabarti and F.H.M.Faisal, Mod.Phys.Lett. **B5** (1991) 1025.
12. J.Kogut and L.Susskind, Phys.Rev. **D11** (1975) 395.
    T.Banks, S.Raby, L.Susskind, J.Kogut, D.R.T.Jones, P.N.Scharbach and D.K.Sinclair, Phys.Rev. **D15** (1977) 1111.
13. J.Chakrabarti and F.H.M.Faisal, Phys.Lett. **A145** (1990) 46.
14. R.J.Glauber, Phys.Rev. **130** (1963) 2529, ibid **131** (1963) 2766.
15. E.C.G.Sudarshan, Phys.Rev.Lett. **10** (1963) 277.
16. J.R.Klauder, J.Math.Phys. **4** (1963) 1055, 1058.
17. J.R.Klauder, B.S.Skagerstam, Coherent States, Applications in Physics and Mathematical Physics, World Scientific, Singapore.
18. D.Bhaumik, K.Bhaumik and B.Dutta Roy, J.Phys.A : Math.Gen. **9** (1976) 1507.
19. A.Perelomov, Commun.Math.Phys. **26** (1972) 222.
20. R.Gilmore, Ann.Phys, (NY) **74** (1972) 391.
21. W.M.Zhang, D.H.Feng and R.Gilmore, Rev.Mod.Phys. **62** (1990) 867.
22. J.Sakurai, Advanced Quantum Mechanics, Addison-Wesley; J.D.Bjorken and S.D.Drell, Relativistic Quantum Fields, McGraw Hill (New York).
23. J.Kogut, Rev.Mod.Phys. **55** (1983) 775.
24. N.Kawamoto and J.Smit, Nucl.Phys. **B192** (1981) 100.
25. J.Chakrabarti, Spin-Singlet Coherent solutions of the Hubbard Model, Mod.Phys.Lett. **B7** (1993) 813.
26. Similar operator algebra has been considered by others before. We present here a short list. P.Carruthers and M.Nieto, Rev.Mod.Phys. **40** (1968) 411.
    A.O.Barut and L.Girardello, Commun.Math.Phys. **21** (1971) 41.
    E.Onofri, J.Math.Phys. **16** (1975) 1087.
    M.Hongoh, J.Math.Phys. **18** (1977) 2081.

For a more complete list see
A.M.Perelomov, Generalized Coherent States and Their Applications (Springer, Berlin) 1986.

27. It is to be noted the projections of $|z\rangle$ onto states of definite global number are solutions of z, and are also states of h_1. When self energy (mass) terms are included these projected states are exact eigenstates.

Figure Captions :

1. A configuration of signs, η, that make e_1 commute with the hamiltonian (2).

2. Pairs of fermions are created on the links of a lattice. The signs associated with the lines indicate the choice of β on the links of the line. If β is chosen as shown, an invariant operator $\hat{e}$ results.

3. The possible independent β assignments on a single plaquette. Note that (A) and (B) differ by an overall sign; so do (C) and (D).

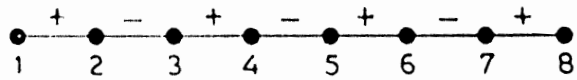

Fig.1

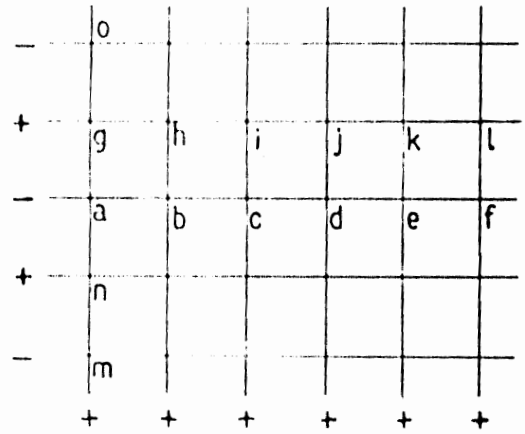

Fig.2

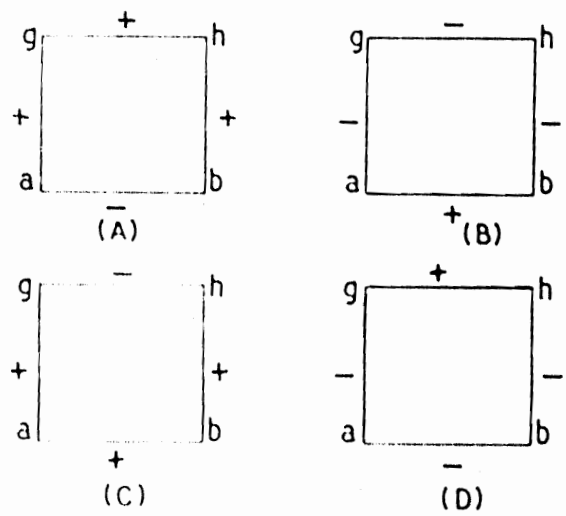

Fig.3