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# NONCOMMUTATIVE DIFFERENTIAL CALCULUS, GAUGE THEORY AND GRAVITATION

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**Abstract**

We introduce a model of "dynamical geometry". It involves ideas of noncommutative differential geometry and aims at a derivation of field equations from a single universal dynamical law. The basic structure is a deformed differential calculus on the commutative algebra of smooth functions on a manifold. The deformation is "dynamical" in the sense that the differential of a function has an additional part which involves a *second* order differential operator. With it we develop a generalization of gauge theory and a generalized tensor calculus related to gravity.

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## Introduction

According to a wellknown result of algebraic geometry, commutative algebras correspond to topological spaces (see [1] for the corresponding results in the case of $C^*$-algebras). This means that the algebra can be realized as the algebra of functions on such a space. Manifolds are in this sense just commutative algebras and a differential structure – as well as other notions of differential geometry – may be regarded as an algebraic structure on such a commutative algebra. Quantization may be viewed as replacing a commutative algebra corresponding to some classical phase space by some noncommutative algebra.

Noncommutative differential geometry [2] generalizes the calculus of differential forms on manifolds to arbitrary (not necessarily commutative) algebras. Such a differential calculus then provides means of introducing geometrical structures on these noncommutative algebras. In particular, there are corresponding applications [3, 4, 5] to "quantum groups" which are Hopf algebras. By now there are numerous applications of noncommutative differential geometry in physics (see, for example, [6, 7, 8, 9, 10, 11, 12, 13]).

Even if the algebra itself is commutative, there are in general consistent differential calculi which deviate from the classical differential calculus.[1] In this paper we present an example which exhibits surprising relations with the dynamical laws of gauge theories and gravitation.

The differential calculus which we study in this work is formally obtained from the canonical commutation relations (CCRs) of quantum mechanics. A deeper analysis of its relation to quantum mechanics is contained in section 3. In the CCRs

$$[x^\mu, p^\nu] = i\,\hbar\, g^{\mu\nu} \tag{1.1}$$

we express the momenta as

$$p^\nu = \frac{\hat{d}x^\nu}{\hat{d}T} \tag{1.2}$$

with some "quantum" exterior derivative operator $\hat{d}$ and an "evolution parameter" $T$. The appearance of the extra "time" parameter $T$ provokes some discussion and section 2 is reserved for this purpose. The CCRs are now transformed into

$$[x^\mu, \hat{d}x^\nu] = i\,\hbar\, g^{\mu\nu}\hat{d}T \tag{1.3}$$

(assuming that $\hat{d}T$ commutes with $x^\mu$). These are commutation relations between the algebra elements $x^\mu$ and the "quantum differentials" $\hat{d}x^\nu$. The rules can be completed

[1] See [14] for examples of consistent deformations of the classical differential calculus on a Lie group.

to a consistent differential calculus (section 4). This differential calculus (and thus in particular (1.3) ) is covariant with respect to general coordinate transformations (see section 6).

In section 2 we reanalyse the notion of time in physics and give arguments in favour of two (at least a priori) independent concepts of time. We are led to the idea of a universal dynamical law from which the specific field equations for physical fields can be derived. Section 3 then explains, how these ideas can be realized and how noncommutative differential geometry enters the stage. The corresponding mathematics is developed in sections 4-7. In section 4 we introduce the differential calculus for a flat metric and show in particular that the Klein-Gordon equation (in the proper time formulation [15]) naturally arises from it. Section 5 is concerned with charged fields and a generalization of (Yang-Mills) gauge theory. Covariant derivatives are introduced which involve the *deformed* exterior derivative $\hat{d}$ and generalized connections. In a similar way, generalized linear connections and covariant derivatives are defined in section 6 and contact is made with gravity. In section 7 we show how geodesic motion and the motion of bosonic strings can be described in our framework. Some conclusions and further remarks are collected in section 8.

From a purely mathematical point of view we think our work gives a version of noncommutative[2] gauge theory and noncommutative tensor calculus which according to our knowledge has not been discovered previously. The reader who is just interested in these mathematical structures should skip the following two sections and start with section 4.

The reader who quickly wants to grasp the physical ideas underlying our work should read sections 2 and 3 first. We have to stress, however, that the material presented in these sections is partly of a rather sketchy nature and far from a fully elaborated and completed picture.

## 2 From Time to Time

In order to describe the evolution of a physical system one first has to determine a space of possible configurations which the system can take during its process of changing. A possible motion corresponds to a sequence of configurations. If the configuration space carries a topological structure, this sequence may be a continuous curve. There are many ways of parametrizing this (oriented) curve and such a parameter gives an *intrinsic* notion

[2]The algebra itself is commutative in our case, but differentials do not commute with algebra elements.



of time peculiar to this specific motion.

In classical mechanics the changes we are interested in are the motions of particles. All possible particle motions are assumed to take place in the same three-dimensional space which is therefore a common configuration space for all mechanical systems. There is then a natural choice of intrinsic time for each particle, namely the length of the particle's trajectory between two subsequent positions (measured with the Euclidean space metric). Although this is a *universal* prescription, i.e. the same prescription for all particles, it does not yield a *common* notion of time for all particles (as will be explained in the following paragraph). The problem of mechanics was to find a general dynamical law which when restricted to a specific mechanical system correctly describes its motion. This necessitated the comparison of different motions and the reduction of the intrinsic times to a common notion of time. Newtonian mechanics achieved this by introducing the concept of *inertial mass*.

To illustrate this let us recall one of the most fundamental observations in classical mechanics. In a fixed inertial frame of reference[3] we consider two particles at rest having (almost) the same position, say A. We then push them simultaneously in the same way. Assuming free motion (i.e. neglecting other forces) the configuration space may be taken to be the straight line passing through A in the direction of the motion. In order to compare the two motions we have to compare the positions of the two particles as they move. In general the particles will not pass through a point, say B, simultaneously. The traversed length $\Delta x$ is therefore not a *common* time parameter, although it is defined in the same way for both particles. Only after rescalings by factors characteristic to the particles we obtain a time variable common to both processes. If $p$ is the momentum which the push transmits to the two particles, then $\Delta t_1/m_1 = \Delta x/p = \Delta t_2/m_2$ so that $\Delta t_1 \neq \Delta t_2$ if $m_1 \neq m_2$. The variable $t$ can now take care of the fact that the two particles do not simultaneously pass through the same point. The proportionality factor between the intrinsic time parameter of a particle and the common time variable $t$ is essentially the inertial mass. Of fundamental importance in this analysis is the identification of the configuration spaces for the two particles.

The common time parameter $t$ extends the spatial frame of reference to a spatiotemporal reference frame.

In the above analysis we still have to discuss the dependence on the frame of reference. Since motion is only relative we do have to consider an arbitrary inertial frame. Newton got out of this problem by postulating that the common time variable is independent of the frame, it is an *absolute time*. It is striking that this time parameter turned out to

[3]We mean a *spatial* frame of reference.

be universally applicable to all mechanical systems. A basic ingredient of the notion of absolute time is the hypothesis that the three-dimensional (Euclidean) space is a common configuration space for all mechanical systems.

The criticism of absolute time put forward by Poincaré and Einstein in the beginning of this century only corrected this picture in the following sense. The common time parameter $t$ is no longer regarded as "absolute" but now depends on the (spatial) frame of reference used in its construction. The origin and interpretation of inertial mass as the ratio of two notions of time is not affected.

The demand of a description independent of the choice of inertial frames led to the concept of (Minkowski) space-time. Instead of the time parameter associated with an inertial frame we can introduce the proper time $\tau$ to parametrize the motion of a particle. The proper time is a parameter intrinsic to the *world line* of the particle but not intrinsic to the process, i.e. the particle's motion. It is related to the intrinsic time $T$ of the particle (in suitable units) by $T = \tau/m$ where $m$ is its inertial mass.

The time $t$ of a Lorentz frame corresponds to the physical notion of time which we use to describe experiments. Although $t$ is a parameter common to all processes, it does not share the universality of $T$.[4] Using $T$ instead of $t$ to describe the dynamics opens the possibility to formulate a *universal dynamical law* (cf. Stueckelberg's equation discussed below). This suggests to regard $T$ as a universal time parameter which is (at least "off shell") independent from the relativistic notion of time.[5] Such an enlarged space-time (or rather: space-time-time) picture has been studied by various authors with different motivations. Below we will briefly review some of the arguments and results.

In 1941 Stueckelberg [16] suggested to describe the relativistic motion of a point particle in a gravitational and an electromagnetic field by

$$\frac{d^2x^\mu}{dT^2} = -\Gamma^\mu{}_{\kappa\lambda}\,\frac{dx^\kappa}{dT}\,\frac{dx^\lambda}{dT} + e\,F^\mu{}_\nu\,\frac{dx^\nu}{dT} \tag{2.1}$$

in terms of a parameter $T$ without reference to the mass. As a consequence of this equation,

$$m^2 := -g_{\mu\nu}\,\frac{dx^\mu}{dT}\,\frac{dx^\nu}{dT} \tag{2.2}$$

[4]Since $t$ is a parameter with which the evolution of all processes can be measured simultaneously, it may also get the attribute "universal". However, the dynamical law formulated in terms of $t$ is different for different particles (if their inertial masses are different) whereas it takes a universal form when expressed in terms of $T$. In this sense $T$ is "dynamically universal".

[5]Canonical quantization of a free particle with mass $m$ replaces the classical momentum $p = m\dot{x}$ by the operator $(\hbar/i)\partial/\partial x$. The latter does, however, not depend on $m$. The correspondence becomes more understandable if we can regard $T = t/m$ as a universal parameter.



is a constant of motion.[6] If $m^2 > 0$, then

$$m^2 = \left(\frac{d\tau}{dT}\right)^2 \tag{2.3}$$

where $\tau$ is the proper time of the particle.[7] Using

$$d\tau = \pm m\,dT \tag{2.4}$$

to eliminate $T$ in favour of the proper time in the equation of motion, we recover the more familiar formula

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu{}_{\kappa\lambda}\,\frac{dx^\kappa}{d\tau}\,\frac{dx^\lambda}{d\tau} \pm \frac{e}{m}\,F^\mu{}_\nu\,\frac{dx^\nu}{d\tau}\,. \tag{2.5}$$

The sign ambiguity corresponds to the two possible signs of the electric charge. Its dependence on the relative orientation of $\tau$ and $T$ already hints towards Stueckelberg's interpretation of antiparticles as ordinary particles going backwards in time. Deviating from Einstein's theory, Stueckelberg admitted worldlines (parametrized by $T$) with segments going backwards in relativistic time (with increasing intrinsic time $T$). The turning points are related to pair creation and annihilation.

In this picture which has later been rediscovered by Feynman [17], without the use of an independent time parameter, the relativistic time no longer describes the evolution of phenomena. Nambu [18] therefore suggested to "introduce some parameter with which the four-dimensional world is going to shape itself" and revived Fock's [15] proper time formalism.

Fock [15] also showed that the concept of an independent proper time parameter is a very convenient tool to solve wave equations. He suggested to consider a five-dimensional analogue of the Schrödinger equation. For free particles (for simplicity) this reads

$$i\hbar\,\frac{\partial}{\partial\tau}\phi = -\frac{\hbar^2}{2m}(\Box - \frac{m^2}{\hbar^2})\,\phi \equiv H\,\phi\,. \tag{2.6}$$

This equation is easily solved (more generally for any operator $H$ which does not depend on $\tau$). Integration over $\tau$,

$$\Psi = \int \phi\,d\tau\,, \tag{2.7}$$

[6]In 1937 Fock [15] already suggested to regard the proper time as an independent parameter. The new aspect in Stueckelberg's work is the absorption of the mass into this parameter.

[7]We use units such that the velocity of light is $c = 1$.

leads to a solution of the four-dimensional Klein-Gordon equation with mass $m$ (if $\phi$ satisfies certain boundary conditions [15]). In this way one recovers the computationally so convenient integral representations of Green's functions [19].

The mass term in (2.6) can be absorbed by a redefinition of $\phi$,

$$\phi \mapsto \exp(-i\frac{m\tau}{2\hbar})\,\phi\,. \tag{2.8}$$

Then

$$i\hbar\frac{\partial}{\partial\tau}\phi = -\frac{\hbar^2}{2m}\Box\phi \tag{2.9}$$

from which the Klein-Gordon equation $(\Box - m^2/\hbar^2)\phi = 0$ is obtained for "stationary states" (with respect to $\tau$), i.e. as the eigenvalue equation for the operator on the r.h.s. of (2.9).

Stueckelberg considered instead of (2.6) the "universal" equation

$$i\hbar\frac{\partial}{\partial T}\phi = -\frac{\hbar^2}{2}\Box\phi\,. \tag{2.10}$$

The transformation (2.8) of Fock's equation depends on a specific physical system (since it depends on the mass $m$ of the particle). In Stueckelberg's theory a transformation like (2.8) would formally correspond to

$$\phi \mapsto \exp(-i\frac{m^2}{2\hbar}T)\,\phi \tag{2.11}$$

(making the identification $T = \tau/m$). But in contrast to Fock's equation (2.6), respectively (2.9), his equation (2.10) does not make any reference to the notion of mass and therefore a transformation like (2.11) could at best be regarded as a kind of gauge transformation. Indeed, in this way we may build a bridge between the two formulations of Fock and Stueckelberg. Let us therefore promote (2.10) to an equation which is invariant with respect to the (more general) gauge transformation

$$\phi \mapsto e^{i\alpha(T)}\,\phi \tag{2.12}$$

where $\alpha$ is an arbitrary (differentiable) function of $T$. This requires the introduction of a gauge potential $M$ such that

$$D_T := \frac{\partial}{\partial T} + \frac{i}{2\hbar}M \tag{2.13}$$

transforms under (2.12) as a covariant derivative. Instead of (2.10) we now consider the gauge invariant equation

$$i\hbar\,D_T\,\phi = -\frac{\hbar^2}{2}\Box\phi\,. \tag{2.14}$$



If we break the gauge invariance (for example, by restricting $\phi$ to be $T$-independent), then $M$ becomes a scalar. In the special case where this scalar is constant, it becomes the ordinary mass term when shifted to the r.h.s. of (2.14). The resulting equation is then of the form of (2.6) and for stationary states (now with respect to $T$) we obtain the Klein-Gordon equation.

The proper time formulation of relativistic classical and quantum dynamics due to Fock and Stueckelberg has been further developed and investigated by several authors [20, 21, 22, 23, 24, 25] (see also [26] and the references cited there).

Horwitz [26] defines a universal time parameter $\tau$ operationally as the reading on an ideal inertial clock, i.e. a clock in free fall.[8] Then

$$d\tau^2 = -g_{\mu\nu}\,dx^\mu\,dx^\nu \tag{2.15}$$

for timelike geodesics. In the ordinary spacetime picture, this leads to an inconsistency. For example, let us imagine a gravitational lens experiment where two timelike geodesics (freely falling clocks) start at one point in spacetime (with $\tau = 0$) and meet again at another point. In general, the clocks will then show different times $\tau_1$ and $\tau_2$, respectively, at the meeting point. The postulated universality of $\tau$ therefore requires to consider a spacetime point with different values of $\tau$ as different events, visualized by enlarging the spacetime by a fifth dimension (corresponding to the parameter $\tau$).

Such an enlarged spacetime as the arena for relativistic quantum mechanics has also been argued for on the basis of a group theoretical analysis [27].

Arguments in favour of an extra time parameter have also been given in the context of quantum gravity [28].[9] Ordinary quantum-mechanical theories assume that measurements are made at instants of time. In generally covariant theories only histories (in space-time) have physical significance. The time coordinate merely labels a spacelike hypersurface. Without a deep revision of either quantum mechanics or general relativity (or both) it seems that one can only combine these theories by introducing an additional time variable external to space-time.

[8] In this case Horwitz's notion of universal time coincides with our notion of intrinsic time. Due to the weak equivalence principle intrinsic time and proper time coincide for a particle in free fall (up to a conventional factor which does not depend on the particle).

[9] For other discussions of the notion of time in the context of quantum gravity see the various contributions in [29].

# 3 Towards Universal Dynamics

It is often said that in physics we have three basic categories: space, time and matter. Actually there is in addition a more basic one, the four-dimensional manifold $\mathcal{M}$. According to special relativity, space and time are a single inseparable entity expressed by a space-time metric on $\mathcal{M}$. General relativity relates matter to the curvature of the space-time metric and in some sense merges space-time and matter to a single entity.

Matter fields as well as the space-time metric may be viewed as (multiplets of) functions on $\mathcal{M}$, i.e. as elements of the abelian algebra of functions on $\mathcal{M}$.[10] We may think of this algebra as an abstract commutative algebra $\mathcal{A}$ (without reference to $\mathcal{M}$). The manifold $\mathcal{M}$ can then be reconstructed from $\mathcal{A}$.[11] All what can be said about $\mathcal{M}$ can be expressed in terms of the algebra $\mathcal{A}$. The interesting point here is that in such a description all four categories mentioned above are merged into a single entity, the algebra $\mathcal{A}$ ! The imagination that matter fields cannot be thought of without the space-time manifold and vice versa simply becomes a mathematical fact in this picture.

The "space-time algebra" $\mathcal{A}$ is much more than the collection of physically relevant fields since it contains *all* (smooth) functions on $\mathcal{M}$. It is the entirety of all dynamical processes, real or virtual, with any type of initial "matter" configuration, realistic or not. Real processes must be distinguished by some principle, like the principle of least action.

There is nothing "exterior" to a space-time manifold. It is thus determined by the relations between its parts. How do we describe such relations between different parts of space-time ? A classical field on $\mathcal{M}$ is a collection of $\mathbb{C}$-valued functions $\psi^r(x^\mu)$ on $\mathcal{M}$ where the coordinates $x^\mu$ label the points of $\mathcal{M}$. Since the $x^\mu$ are themselves functions on $\mathcal{M}$, the fields $\psi^r(x)$ can be viewed as a *relation* between elements $\psi^r$ and $x^\mu$ of the algebra $\mathcal{A}$. In the algebraic picture of space-time the elements of $\mathcal{A}$ themselves are without a meaning. Only relations between them may receive physical significance. We should be familiar with this point of view from quantum mechanics. A wave function $\psi(x)$ is interpreted as the correlation $\langle x|\psi\rangle$ between two states $|x\rangle$ and $|\psi\rangle$. Physically relevant are only transition probabilities like $|\langle x|\psi\rangle|^2$ which are relations between the two states, but not the states themselves.

[10] More generally, matter fields should be regarded as cross sections of bundles over $\mathcal{M}$. This slightly complicates our picture but does not make an essential difference.

[11] This is a wellknown result in algebraic geometry, in the context of "commutative algebra". One often finds it specialized to $C^*$-algebras as the "Gelfand isomorphism theorem". It says that every commutative $C^*$-algebra $\mathcal{A}$ can be written as $\mathcal{A} = \mathcal{C}(\mathcal{M})$ (continuous $\mathbb{C}$-valued functions on $\mathcal{M}$ which vanish at infinity) for some uniquely determined locally compact space $\mathcal{M}$, the "spectrum" of $\mathcal{A}$ (see [1], for example). Here we consider the algebra $\mathcal{C}^\infty(\mathcal{M})$ which already takes care of the differential manifold structure of $\mathcal{M}$.



From the discussion in the previous section we are led to the fascinating idea that all the apparently rather different laws of motion for physical fields (like the electromagnetic field, Yang Mills, gravity) could have its origin in some universal dynamics of the space-time algebra $\mathcal{A}$. In the following we will suggest how this idea could be pursued.

We assume that the universal dynamical process of the algebra $\mathcal{A}$ can be described as a flow $\Phi_T : \mathcal{A} \to \mathcal{A}$ which satisfies

$$\frac{d}{dT}\Phi_T(f) = A\,\Phi_T(f) \tag{3.1}$$

where $A$ is a linear operator. There is an important difference in the interpretation of (3.1) as compared with the interpretation of dynamical systems we are used to deal with. In the latter case, one always considers some structure, e.g. a matter distribution, *on* a manifold. The manifold thus serves as a background on which the dynamics takes place. In our algebraic framework, however, we do not have (a priori) a distinction between manifold and matter any more. Any transformation of the algebra $\mathcal{A}$, like $\Phi_T$ above, is then without a *physical* meaning since it is a transformation *of*[12] the algebra instead of a transformation of something *on* the manifold. Only when we select a certain subset of $\mathcal{A}$ (declaring the elements to be "matter fields") (3.1) can describe nontrivial dynamics. A possible selection principle is the stationarity of $\Phi_T f$, i.e.

$$\frac{d}{dT}\Phi_T f = 0\,. \tag{3.2}$$

Special examples for $\Phi_T$ are obtained from flows on the manifold $\mathcal{M}$ which are generated by vector fields on $\mathcal{M}$. If we want the theory to be generally covariant, we cannot imagine how such a flow could induce nontrivial dynamics (see also the discussion below). In the algebraic description of the manifold there are, however, linear maps $\mathcal{A} \to \mathcal{A}$ which do not correspond to maps $\mathcal{M} \to \mathcal{M}$ and those are more promising for our purpose. From Gelfand's theorem we know that the characters (homomorphisms $\mathcal{A} \to \mathbb{C}$)of an abelian $C^*$-algebra $\mathcal{A}$ constitute the space $\mathcal{M}$ on which the elements of the algebra can be realized as functions. The set of characters is a subset of the dual $\mathcal{A}^*$ of the algebra which carries a linear structure. If the flow $\Phi_T$ is not an automorphism of $\mathcal{A}$, then its transpose defined by

$$(\Phi_T^*\chi)(f) := \chi(\Phi_T f) \qquad \forall \chi \in \mathcal{A}^*,\ f \in \mathcal{A} \tag{3.3}$$

[12] Let us explain in which sense (3.1) describes *mathematically* dynamics *of* the algebra. The natural configuration space associated with $\mathcal{A}$ is the space $\mathrm{C}(\mathcal{A})$ of bijective maps $\mathcal{A} \to \mathcal{A}$. We then have to understand the flow $\Phi_T$ as a mapping $\varphi_T : \mathrm{C}(\mathcal{A}) \to \mathrm{C}(\mathcal{A})$. The linear operator $A$ in (3.1) induces on $\mathrm{C}(\mathcal{A})$ an operator $\mathrm{A}$ via $(\mathrm{A}\phi)(f) := A(\phi f)$. The dynamical law (3.1) then corresponds to the equation $\partial_T\varphi_T(Id) = \mathrm{A}(\varphi_T(Id))$ with $\varphi_T(Id) = \Phi_T$ where $Id$ is the identity map on $\mathcal{A}$.

sends characters to elements of $\mathcal{A}^*$ which are not characters, in general, but at best linear combinations of characters. This means that a point of $\mathcal{M}$ flows into a "linear combination of points", an expression which does not make sense in ordinary geometrical terms. But if we think of the points as states, we have the familiar picture of quantum mechanics.

We will not just postulate (3.1) as a dynamical law. The assumption of universality suggests to build it into a fundamental and universal structure on the manifold, respectively the algebra $\mathcal{A}$. This is the differential calculus.

If $A$ is a vector field, say $P$, then $\Phi_T$ is a diffeomorphism of $\mathcal{M}$. Let $X^\mu$ be coordinate functions on $\mathcal{M}$. We introduce comoving coordinates $x^\mu := \Phi_T X^\mu$. Let $f \in \Phi(\mathcal{A})$ mean that there is a function $F$ on $\mathcal{M}$ such that $f(x^\mu, T) = \Phi_T F(X^\mu)$. It is easily verified that $f$ does actually not explicitly depend on $T$. We understand as the "total differential" on $\Phi(\mathcal{A})$ the expression

$$\hat{d}f = dT\,(Pf) + dx^\mu\,\partial_\mu f \tag{3.4}$$

where $\partial_\mu = \partial/\partial x^\mu$. In particular, using $Pf = P^\mu \partial_\mu$, we have

$$\hat{d}x^\mu = dT\,P^\mu + dx^\mu\,. \tag{3.5}$$

Expressing $\hat{d}f$ in terms of $\hat{d}x^\mu$, we obtain

$$\hat{d}f = \hat{d}x^\mu\,\partial_\mu f\,. \tag{3.6}$$

We see that in terms of the $\hat{d}$-differential calculus the dynamics is absorbed and in this sense trivial.

If, however, $A$ in (3.1) is a second order differential operator of the form $A = g^{\mu\nu}\,\partial_\mu\partial_\nu$, then we are led to

$$\hat{d}f = dT\,Af + \hat{d}x^\mu\,\partial_\mu f \tag{3.7}$$

(note that $\hat{d}x^\mu = dx^\mu$ since $Ax^\mu = 0$). In this case the flow $\Phi_T$ does not correspond to a diffeomorphism. The operator $\hat{d}$ defined above is *not* an "exterior derivative" since it does not satisfy the Leibniz rule. In the following we will see, however, that this can be corrected by assuming a nontrivial commutation relation between elements of $\mathcal{A}$ and their $\hat{d}$-differentials.

Let us embed the algebra $\mathcal{A}$ into the Heisenberg algebra. We think of the latter as generated by $T, x^\mu$ and $p_\nu$ satisfying canonical commutation relations with $x^\mu$,

$$[\,x^\mu\,,\,p_\nu\,] = i\hbar\,\delta^\mu_\nu\,. \tag{3.8}$$



The Heisenberg equation of motion is

$$\frac{d}{dT}f = \frac{1}{i\hbar}[\,f\,,\,H\,] \tag{3.9}$$

since $f \in \mathcal{A}$ has no explicit dependence on $T$. With it we associate a differential calculus via

$$\hat{d}f = dT\,\frac{1}{i\hbar}[\,f\,,\,H\,] + dx^\mu\,\partial_\mu f\,. \tag{3.10}$$

With the Hamiltonian

$$H = \frac{1}{2}g^{\mu\nu}\,p_\mu p_\nu \tag{3.11}$$

we find

$$\hat{d}x^\mu = dT\,p^\mu + dx^\mu \tag{3.12}$$

and therefore

$$\hat{d}f = dT\,\frac{i\hbar}{2}\,g^{\mu\nu}\,\partial_\mu\partial_\nu\,f + \hat{d}x^\mu\,\partial_\mu f\,. \tag{3.13}$$

From (3.12) and the canonical commutation relations we find

$$[\,x^\mu\,,\,\hat{d}x^\nu\,] = i\hbar\,dT\,g^{\mu\nu}\,. \tag{3.14}$$

This nontrivial commutation relation between algebra elements and differentials is crucial for $\hat{d}$ to have the properties of an exterior derivative. (3.14) does not refer to the Heisenberg algebra any more, but only to the algebra $\mathcal{A}$. Instead of working with the Heisenberg algebra we may now restrict ourselves to the algebra $\mathcal{A}$ and introduce dynamics by postulating (3.14) which leads to a "deformed differential calculus".

At the macroscopic level when $\hbar$ is negligible all noncommutativities disappear and we are back to the classical differential calculus. In the microscopic domain we may use the representation theory of quantum mechanics to introduce states. From (3.14) an uncertainty relation between coordinates and their differentials should result, though it is not obvious how it has to be formulated and understood. Nevertheless it is plausible that the lower bound in the uncertainty inequality will be proportional to $dT$, which shows that an uncertainty in the measurement of $x^\mu$ and $\hat{d}x^\nu$ will appear only during evolution, i.e. it is a dynamical effect. Pursuing these ideas may lead us to a new version of quantum mechanics which does not live on the assumption that measurements are made at a definite instant of time. This assumption seems to be the origin of fundamental problems in canonical quantum gravity [28].

# 4 The Differential Calculus

Let $x^\mu$, $\mu = 0,1,2,3$ be Lorentz coordinates on the Minkowski spacetime $(\mathbb{R}^4, \eta)$.[13] For the metric $\eta$ we choose the signature $(-,+,+,+)$. Let $T$ be a real variable independent of $x^\mu$, which we call *evolution parameter* or *dynamical time*.[14] These quantities together with the constant function 1 generate by usual multiplication a complex, associative, commutative algebra $\mathcal{A}_T$ with unity 1. The defining relations are thus

$$[T, x^\mu] = 0 \quad , \quad [x^\mu, x^\nu] = 0 \tag{4.1}$$

together with the properties of the unity.

A differential calculus on $\mathcal{A}_T$ is a $\mathbb{Z}$-graded algebra

$$\bigwedge(\mathcal{A}_T) = \bigoplus_{r\in\mathbb{Z}} \overset{r}{\bigwedge}(\mathcal{A}_T) \quad , \quad \overset{0}{\bigwedge}(\mathcal{A}_T) = \mathcal{A}_T \quad , \quad \overset{r}{\bigwedge}(\mathcal{A}_T) = \{0\} \quad \forall r < 0$$

together with an operator $\hat{d} : \bigwedge^r(\mathcal{A}_T) \to \bigwedge^{r+1}(\mathcal{A}_T)$ which satisfies

$$\hat{d}1 = 0 \quad , \quad \hat{d}^2 = 0 \tag{4.2}$$

and the Leibniz rule

$$\hat{d}(\omega_r\,\omega) = (\hat{d}\omega_r)\,\omega + (-1)^r\,\omega_r\,\hat{d}\omega \tag{4.3}$$

where $\omega_r \in \bigwedge^r(\mathcal{A}_T)$, $\omega \in \bigwedge(\mathcal{A}_T)$. The elements of $\bigwedge^r(\mathcal{A}_T)$ will be called $r$-forms.

An endomorphism $\Phi : \mathcal{A}_T \to \mathcal{A}_T$ of the algebra $\mathcal{A}_T$ extends to an endomorphism of $\bigwedge(\mathcal{A}_T)$ if we demand

$$\Phi(\hat{d}\omega) = \hat{d}(\Phi\omega)\,. \tag{4.4}$$

The complex conjugation extends to an involution on $\mathcal{A}_T$ with

$$T^* = T \quad , \quad (x^\mu)^* = x^\mu \tag{4.5}$$

and to a graded anti-involution on $\bigwedge(\mathcal{A}_T)$ via

$$(\hat{d}\omega)^* = \hat{d}(\omega^*) \quad , \quad (\omega_r\,\omega_s)^* = (-1)^{rs}\,\omega_s^*\,\omega_r^*\,. \tag{4.6}$$

[13] The restriction to four dimensions is not essential. Most of our formulae are valid in an arbitrary number of dimensions.

[14] In contrast to $x^0/c = t$, which we call *relativistic* time.



For our differential calculus we still have to specify commutation relations between 1-forms and the elements of $\mathcal{A}_T$. Motivated by our considerations in the previous sections we choose[15]

$$[T, \hat{d}T] = 0 \quad , \quad [T, \hat{d}x^\mu] = 0 \quad , \quad [x^\mu, \hat{d}T] = 0 \tag{4.7}$$

and

$$[x^\mu, \hat{d}x^\nu] = i\,\hbar\,\hat{d}T\,\eta^{\mu\nu}\,. \tag{4.8}$$

We assume that $\bigwedge^1(\mathcal{A}_T)$ as a right $\mathcal{A}_T$-module is free with basis $\hat{d}T, \hat{d}x^\mu$, $\mu = 0,1,2,3$. This allows us to define "partial derivatives" of elements of $\mathcal{A}_T$ by

$$\hat{d}f = \hat{d}T\,\hat{\partial}_T f + \hat{d}x^\mu\,\hat{\partial}_\mu f \qquad (\forall f \in \mathcal{A}_T)\,. \tag{4.9}$$

In particular, this implies

$$\hat{\partial}_\nu x^\mu = \delta^\mu_\nu \quad , \quad \hat{\partial}_T x^\mu = 0 \quad , \quad \hat{\partial}_\nu T = 0 \quad , \quad \hat{\partial}_T T = 1\,. \tag{4.10}$$

From (4.1), (4.7) and (4.8) we obtain

$$[f, \hat{d}g] = [f, \hat{d}x^\mu]\,\hat{\partial}_\mu g = [x^\mu, \hat{d}f]\,\hat{\partial}_\mu g = [x^\mu, \hat{d}x^\nu]\,(\hat{\partial}_\nu f)\,(\hat{\partial}_\mu g) \tag{4.11}$$

for elements $f, g \in \mathcal{A}_T$, and therefore

$$[f, \hat{d}g] = i\,\hbar\,\hat{d}T\,\eta^{\mu\nu}\,(\hat{\partial}_\mu f)\,(\hat{\partial}_\nu g)\,. \tag{4.12}$$

It may appear suprising that the l.h.s. of the above equation contains only one derivative whereas there are two on the r.h.s.. One should remember, however, that a commutator acts like a derivation.

$\bigwedge^1(\mathcal{A}_T)$ is also a free *left* $\mathcal{A}_T$-module (with the basis $\hat{d}T, \hat{d}x^\mu$). In fact, using (4.7), (4.12) and (4.10) the equation $f\,\hat{d}T + f_\mu\,\hat{d}x^\mu = 0$ is turned into $\hat{d}T\,(f + i\hbar\,\eta^{\mu\nu}\,\hat{\partial}_\mu f_\nu) + \hat{d}x^\mu f_\mu = 0$, which implies $f = 0$, $f_\mu = 0$. This allows us to also define "*left* partial derivatives" by

$$\hat{d}f = \overset{\leftarrow}{\partial}_T f\,\hat{d}T + \overset{\leftarrow}{\partial}_\mu f\,\hat{d}x^\mu\,. \tag{4.13}$$

These are related to the *right* partial derivatives by

$$\overset{\leftarrow}{\partial}_\mu = \hat{\partial}_\mu \quad , \quad \overset{\leftarrow}{\partial}_T = \hat{\partial}_T + i\hbar\,\eta^{\mu\nu}\,\hat{\partial}_\mu\hat{\partial}_\nu\,. \tag{4.14}$$

[15] In four space-time dimensions these relations are invariant under the "relativistic Galilei group" $G_5 = (T^4 \times T^1) \wedge (T^{\prime 4} \wedge SO(3,1))$ where $\times$ means a direct and $\wedge$ a semidirect product. $T^4$ are the space-time translations, $T^{\prime 4}$ the $T$-boosts $(x^\mu \mapsto x^\mu + b^\mu T)$, and $T^1$ the $T$-translation group $(T \mapsto T + \theta)$. See also [27].

Acting with $\hat{d}$ on (4.7) and (4.8), we obtain the following 1-form commutation relations:

$$(\hat{d}T)^2 = 0 \quad , \quad \hat{d}T\,\hat{d}x^\mu + \hat{d}x^\mu\,\hat{d}T = 0 \quad , \quad \hat{d}x^\mu\,\hat{d}x^\nu + \hat{d}x^\nu\,\hat{d}x^\mu = 0 \tag{4.15}$$

which imply the usual wedge product between forms of grade greater than zero.[16]

Consistency of the differential calculus requires to satisfy the equations which follow from the following procedures.

1. Apply the operator $\hat{d}$ to the defining relations (4.1) of the algebra and use (4.7) and (4.8) to commute all differentials to one side.
2. Use (4.7) and (4.8) to commute the 1-forms $\hat{d}T$ and $\hat{d}x^\mu$ through the defining relations (4.1) of the algebra.[17]
3. Apply the conjugation to (4.7) and (4.8), and use (4.5), (4.6).
4. Commute the generators $T$ and $x^\mu$ of the algebra $\mathcal{A}_T$ through the 1-form commutation relations (4.15) using again (4.7) and (4.8).

It is easily verified that our differential calulus is indeed consistent.

Using the Leibniz rule (4.3) for $\hat{d}$ in the form

$$\hat{d}(fg) = (\hat{d}f)\,g + f\,\hat{d}g = (\hat{d}f)\,g + (\hat{d}g)\,f + [f, \hat{d}g] \tag{4.16}$$

together with (4.9) and (4.12), we find the corresponding rules which the partial derivatives have to obey:

$$\begin{aligned} \hat{\partial}_\mu(fg) &= (\hat{\partial}_\mu f)\,g + f\,\hat{\partial}_\mu g \\ \hat{\partial}_T(fg) &= (\hat{\partial}_T f)\,g + f\,\hat{\partial}_T g + i\hbar\,\eta^{\mu\nu}\,(\hat{\partial}_\mu f)(\hat{\partial}_\nu g)\,. \end{aligned} \tag{4.17}$$

Hence $\hat{\partial}_\mu$ is a derivation, but $\hat{\partial}_T$ acts according to a more complicated rule. From the last equation we find

$$\hat{\partial}_T = \frac{i\hbar}{2}\,\eta^{\mu\nu}\,\hat{\partial}_\mu\,\hat{\partial}_\nu + \delta \tag{4.18}$$

where $\delta$ is an arbitrary derivation. The nilpotency of $\hat{d}$ together with (4.9) and the 1-form commutation relations (4.15) imply that the partial derivatives $\hat{\partial}_T, \hat{\partial}_\mu$ commute with one another. Thus

$$[\delta, \hat{\partial}_\mu] = 0 \quad , \quad [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0\,. \tag{4.19}$$

[16] Replacing $\eta^{\mu\nu}$ by a metric which is not constant results in a deformation of the usual wedge product, see section 6.

[17] It is helpful to use the Jacobi identity here.



Restricting our considerations to those elements of the algebra $\mathcal{A}_T$ which are differentiable (in the ordinary sense), we have a realization of the operators $\hat{\partial}_\mu$, $\hat{\partial}_T$ by

$$\hat{\partial}_\mu = \partial_\mu \quad , \quad \hat{\partial}_T = \partial_T + \frac{i\hbar}{2}\partial^2\,. \tag{4.20}$$

Ordinary partial derivatives are written without a superscript and $\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the usual d'Alembertian. The operators defined in (4.20) indeed satisfy (4.17) and (4.19).

The realization (4.20) is actually unique. According to theorem 4.2.16 in [30] each derivation of $\mathcal{A}_T = C^\infty(\mathcal{N})$ for a finite-dimensional $C^\infty$-manifold $\mathcal{N}$ can be represented as the Lie derivative with respect to some vector field. In our case, $\mathcal{N} = \mathbb{R}^5$ and (4.19) tells us that the vector fields corresponding to the derivations $\hat{\partial}_\mu$ and $\delta$ commute. As a consequence there are, at least locally, coordinates $\tilde{x}^\mu, \tilde{T}$ on $\mathcal{N}$ such that $\hat{\partial}_\mu = \partial/\partial\tilde{x}^\mu, \delta = \partial/\partial\tilde{T}$. But then $\tilde{x}^\mu = x^\mu$ and $\tilde{T} = T$ according to (4.10).

We would like to stress the following points here. The partial derivative ("quantum" first order derivative) associated with the evolution parameter is actually a second order differential operator. Furthermore, a function need not depend on $T$ in order to have a non-trivial behaviour with respect to the action of $\hat{\partial}_T$. Therefore it is of particular interest to restrict our attention to $\mathcal{A} = \mathcal{A}_T/I_T$, where $I_T$ is the two-sided ideal of $\mathcal{A}_T$ generated by $T$. On $\mathcal{A}$ we have

$$\hat{\partial}_T = \frac{i\hbar}{2}\,\partial^2 \tag{4.21}$$

and from (4.14) also

$$\hat{\delta}_T = -\frac{i\hbar}{2}\,\partial^2\,. \tag{4.22}$$

We define the flow of $\hat{\partial}_T$ on $\mathcal{A}$ to be a one-parameter semigroup of $\mathbb{C}$-linear mappings $\Phi_s : \mathcal{A} \to \mathcal{A}$, differentiable with respect to $s$, such that $\Phi_0 = \mathrm{id}_{\mathcal{A}}$ and

$$\frac{d}{ds}\Phi_s f = \Phi_s(\hat{\partial}_T f) \qquad (f \in \mathcal{A}) \tag{4.23}$$

Differentiating the semigroup property $\Phi_{s'}\Phi_s = \Phi_{s'+s}$ with respect to $s'$ and setting $s' = 0$ afterwards, we obtain

$$\hat{\partial}_T\Phi_s = \Phi_s\hat{\partial}_T \tag{4.24}$$

which means that $\hat{\partial}_T$ commutes with its flow. Since $\hat{\partial}_T$ is not a derivation on $\mathcal{A}$, $\Phi_s$ cannot be an automorphism of the algebra, i.e. $\Phi_s$ is not induced by a diffeomorphism of $\mathbb{R}^4$ (cf theorem 4.2.36 in [30]). To obtain the flow of a derivation $X$, one first has to find

the integral curves $x^\mu(s)$ of $X$, i.e. one has to solve (4.23) for the generators of $\mathcal{A}$. The behaviour of any other function $f(x)$ along the flow then follows from the automorphism property of the flow, $(\Phi_s f)(x) = f(x(s))$. The last formula no longer holds in the case of $\hat{\partial}_T$. Here one has to solve (4.23) for each function separately.[18] Setting $f_s := \Phi_s f$ we obtain from (4.21), (4.23) and (4.24)

$$i\hbar \frac{\partial}{\partial s} f_s = -\frac{\hbar^2}{2} \eta^{\mu\nu} \partial_\mu \partial_\nu f_s \tag{4.25}$$

which is the extension of the Klein-Gordon equation in the proper time formulation of quantum mechanics [15].

The reader may wonder why we have introduced the parameter $T$ in the first place since finally we withdraw it by turning to the algebra $\mathcal{A}$. Indeed, we could have started with the algebra generated by the $x^\mu$ only and replace the differential $\hat{d}T$ by a 1-form, say $\tau$. It is, however, essential for our construction that the 1-form $\tau$ which then appears on the r.h.s. of (4.8) is independent from the 1-forms $\hat{d}x^\mu$. But this hints at the existence of some "hidden dimension", i.e. a parameter $T$.

Let $(\Lambda(\mathcal{A}), \hat{d})$ denote the differential calculus obtained by application of the rules stated in the beginning of this section to $\mathcal{A}$ instead of $\mathcal{A}_T$. Furthermore, let $\mathcal{T}$ be the two-sided differential ideal of $\Lambda(\mathcal{A})$ generated by $\hat{d}T$. The algebra $\Lambda(\mathcal{A})/\mathcal{T}$ with the differential operator $d$, obtained from $\hat{d}$ by projection, can be identified with the classical differential calculus on $\mathbb{R}^4$. We therefore denote it as $\Lambda_{cl}(\mathcal{A})$. In the following we simply write $dT$ instead of $\hat{d}T$ (although $\hat{d}T$ must not necessarily be represented by an ordinary differential). Since according to (4.8) all deviations from the classical differential calculus are proportional to $dT$, we may then also omit the hat from other differentials whenever they appear multiplied with $dT$ in some expression.

# 5 Charged Fields and Gauge Theory

If a multicomponent field transforms according to a representation of a Lie algebra $G$, the differential $\hat{d}\psi$ is not gauge covariant. Introducing a (generalized) connection 1-form we can construct a covariant exterior derivative like in the case of ordinary gauge theories. One has to be cautious, however, because of the noncommutativity between functions and (generalized) differentials.

[18] For $f = x^\mu$ equation (4.23) implies $d(\Phi_s x^\mu)/ds = 0$, so that $\Phi_s x^\mu = x^\mu$. But for $f = x^\mu x^\nu$ we have $d\Phi_s(x^\mu x^\nu)/ds = i\hbar\,\eta^{\mu\nu}$ which leads to $\Phi_s(x^\mu x^\nu) = i\hbar\,\eta^{\mu\nu} s + x^\mu x^\nu$.



## 5.1 Generalized covariant derivatives

Let $\mathcal{V}$ be an m-dimensional $\Lambda(\mathcal{A})$-bimodule, $G$ a Lie group with Lie algebra $\mathcal{G}$ and

$$\begin{aligned} \pi : G &\to \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \\ g &\mapsto U(g) \end{aligned}$$

a representation of $G$ on $\mathcal{V}$. By writing $\psi \in \mathcal{V}$ we understand the tupel of components of an element of $\mathcal{V}$ with respect to some fixed basis. We will simply write $U$ instead of $U(g)$ which is represented by an $m \times m$-matrix with entries in $\mathcal{A}$ (with respect to the fixed basis of $\mathcal{V}$). Furthermore, we consider a $\mathcal{G}$-valued gauge potential $A_\mu$ on $\mathcal{V}$ which obeys the usual gauge transformation law

$$\partial_\mu U = U\,A_\mu - A'_\mu\,U\,. \tag{5.1}$$

An unfamiliar feature of our noncommutative calculus is the fact that the differentials $\hat{d}x^\mu$ are not gauge invariant.[19] Indeed, using (4.12) we obtain

$$U\,\hat{d}x^\mu\,U^{-1} = i\hbar\,dT\,\eta^{\mu\nu}\,(\partial_\nu U)\,U^{-1} + \hat{d}x^\mu = i\hbar\,dT\,\eta^{\mu\nu}\,(U\,A_\nu\,U^{-1} - A'_\nu) + \hat{d}x^\mu\,. \tag{5.2}$$

On the other hand, this equation shows that the 1-forms

$$\hat{\mathcal{D}}x^\mu := \hat{d}x^\mu - i\hbar\,dT\,\eta^{\mu\nu}\,A_\nu \tag{5.3}$$

are covariant, i.e.,

$$\hat{\mathcal{D}}'x^\mu = U\,\hat{\mathcal{D}}x^\mu\,U^{-1}\,. \tag{5.4}$$

In order to obtain $G$-covariant equations we should use $\hat{\mathcal{D}}x^\mu$ as a basis of $\mathcal{V}$-valued differential forms. Note that $\hat{\mathcal{D}}x^\mu \cdots \hat{\mathcal{D}}x^\nu$ is totally antisymmetric. We extend the operator $\hat{d}$ to a (generalized) exterior covariant derivative[20]

$$\hat{\mathcal{D}}\psi := \hat{d}\psi + \hat{A}\psi \tag{5.5}$$

with

$$\hat{A} := \frac{i\hbar}{2}\,dT\,A_T + \hat{d}x^\mu\,A_\mu\,. \tag{5.6}$$

[19] $x^\mu$ are Lorents coordinates, so that the metric is $\eta = \mathrm{diag}(-1,1,1,1)$.

[20] Although this looks like a familiar formula, the reader should notice that instead of the ordinary exterior derivative $d$ the r.h.s. involves $\hat{d}$ which, in its $dT$ part, corresponds to a second rather than first order differential operator.

The unknown part $A_T$ is restricted by the condition that $\hat{\mathcal{D}}\psi$ has to be $G$-covariant. This means that $\hat{A}$ must transform like a connection form,

$$\hat{d}U = U\,\hat{A} - \hat{A}'\,U\,. \tag{5.7}$$

Now we decompose the l.h.s. of this equation according to (4.9). On the r.h.s. we insert the decomposition (5.6) for $\hat{A}$ and use (4.12) to commute differentials to the left. Eliminating all derivatives of $U$ with the help of (5.1) then leads to

$$(A'_T - \partial^\mu A'_\mu + A'^\mu A'_\mu)\,U = U\,(A_T - \partial^\mu A_\mu + A^\mu A_\mu)\,. \tag{5.8}$$

This shows that

$$A_T = \eta^{\mu\nu}\,(\partial_\mu A_\nu - A_\mu A_\nu) + M \tag{5.9}$$

with an undetermined part $M$ which transforms covariantly according to the adjoint representation, i.e. $M' = UMU^{-1}$. Using this result and (4.20), we find

$$\begin{aligned}\hat{\mathcal{D}}\psi &= \frac{i\hbar}{2}\,dT\,(\partial^2\psi + A_T\psi) + \hat{d}x^\mu\,\nabla_\mu\psi \\ &= \frac{i\hbar}{2}\,dT\,(\partial^2\psi + A_T\psi + 2\,\eta^{\mu\nu}\,A_\mu\,\nabla_\nu\psi) + \hat{\mathcal{D}}x^\mu\,\nabla_\mu\psi\end{aligned} \tag{5.10}$$

where we have introduced the ordinary covariant derivative $\nabla_\mu\psi = \partial_\mu\psi + A_\mu\psi$. Substituting our expression for $A_T$, we obtain

$$\hat{\mathcal{D}}\psi = \frac{i\hbar}{2}\,dT\,(\nabla^2\psi + M\psi) + \hat{\mathcal{D}}x^\mu\,\nabla_\mu\psi \tag{5.11}$$

where $\nabla^2\psi = \eta^{\mu\nu}\,\nabla_\mu\nabla_\nu\psi$.

Since $\hat{\mathcal{D}}x^\mu$ carries inner indices, its exterior covariant derivative is not trivial. We find

$$\hat{\mathcal{D}}\hat{\mathcal{D}}x^\mu = \hat{d}\hat{\mathcal{D}}x^\mu + \hat{A}\,\hat{\mathcal{D}}x^\mu + \hat{\mathcal{D}}x^\mu\,\hat{A} = i\hbar\,dT\,dx^\nu\,F_\nu{}^\mu \tag{5.12}$$

where

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu] \tag{5.13}$$

is the ordinary (Yang-Mills) field strength. The generalized field strength

$$\hat{F} := \hat{d}\hat{A} + \hat{A}^2 \tag{5.14}$$

of $\hat{A}$ transforms according to the adjoint representation. This is also obvious from the relation

$$\hat{\mathcal{D}}^2\psi = \hat{F}\,\psi\,. \tag{5.15}$$



Evaluation of (5.14) yields

$$\hat{F} = \frac{i\hbar}{2}\,dT\,(\mathcal{D}^*F - \mathcal{D}M) + \hat{\mathcal{F}} \tag{5.16}$$

where we have introduced

$$\hat{\mathcal{F}} := \frac{1}{2}\,\hat{\mathcal{D}}x^\mu\,\hat{\mathcal{D}}x^\nu\,F_{\mu\nu} \tag{5.17}$$

and $\mathcal{D}^*$ is the adjoint of the ordinary exterior covariant derivative $\mathcal{D}$, so that

$$\mathcal{D}^*F = dx^\mu\,\nabla^\nu F_{\nu\mu}\,. \tag{5.18}$$

The expression for $\hat{F}$ thus contains the "Yang-Mills operator" in its "evolution part". $\hat{F}$ satisfies the Bianchi-identity $\hat{\mathcal{D}}\hat{F} = 0$. In terms of $\hat{\mathcal{F}}$ this reads

$$\hat{\mathcal{D}}\hat{\mathcal{F}} = \frac{i\hbar}{2}\,dT\,(\mathcal{D}\mathcal{D}^*F - [F, M])\,. \tag{5.19}$$

Let $\epsilon_{\mu\nu\rho\sigma}$ denote the totally antisymmetric tensor with $\epsilon_{0123} = 1$. We introduce the 3-form

$$\hat{\epsilon}_\mu = \frac{1}{3!}\,\epsilon_{\mu\nu\rho\sigma}\,\hat{\mathcal{D}}x^\nu\hat{\mathcal{D}}x^\rho\hat{\mathcal{D}}x^\sigma\,. \tag{5.20}$$

Using (5.12) we then obtain

$$\hat{\mathcal{D}}\hat{\mathcal{D}}\hat{\epsilon}_\mu = -i\hbar\,dT\,\epsilon\,\nabla_\nu F^\nu{}_\mu \tag{5.21}$$

where $\epsilon$ is the classical volume form on $\mathbb{R}^4$. The Yang-Mills equation in vacuum can therefore be expressed as $\hat{\mathcal{D}}\hat{\mathcal{D}}\hat{\epsilon}_\mu = 0$. To derive (5.21) we can also use the Ricci identity in the form

$$\hat{\mathcal{D}}\hat{\mathcal{D}}\hat{\epsilon}_\mu = [\hat{F}, \hat{\epsilon}_\mu] \tag{5.22}$$

and insert (5.16). Note that in this calculation the origin of the dynamical Yang-Mills term is *not* the corresponding term in (5.16).

*Remark:* (5.21) allows us to regard the Yang-Mills equations as integrability conditions of the equation

$$\hat{\mathcal{D}}\hat{\epsilon}_\mu = B_\mu \tag{5.23}$$

where $B_\mu$ stands for any expression annihilated by $\hat{\mathcal{D}}$. Solving this equation, we automatically have a solution of the vacuum Yang-Mills equations. One choice for $B_\mu$ is

$B_\mu = \lambda\, dT\, dx_\mu\, F$ where $F = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$ and $\lambda$ is a constant. Then consistency restricts $\lambda$ and (5.23) turns out to be the (anti-) self-duality condition for the Yang-Mills field strength $F$. □

Elements $\phi$ of the dual module $\mathcal{V}^*$ transform according to $\phi \to \phi U^{-1}$. The corresponding covariant derivative is

$$\begin{aligned} \hat{\mathcal{D}}\phi &= \hat{d}\phi - \phi\hat{A} \\ &= -\frac{i\hbar}{2}\, dT\,(\nabla^2\phi + \phi M) + (\nabla_\mu\phi)\,\hat{\mathcal{D}}x^\mu\,. \end{aligned} \tag{5.24}$$

If we write $\phi\psi$ for the duality contraction, then we have

$$\hat{d}(\phi\psi) = (\hat{\mathcal{D}}\phi)\,\psi + \phi\,\hat{\mathcal{D}}\psi\,. \tag{5.25}$$

Let a conjugation $^\dagger : \mathcal{V} \to \mathcal{V}^*$ be given such that for $\omega \in \bigwedge(\mathcal{A})$ and $\psi \in \mathcal{V}$ we have $(\omega\psi)^\dagger = \psi^\dagger\omega^*$ and $(\hat{d}\psi)^\dagger = \hat{d}\psi^\dagger$. If $\pi$ is unitary, i.e. $U^{-1} = U^\dagger$, then we also demand

$$(\hat{\mathcal{D}}\psi)^\dagger = \hat{\mathcal{D}}\psi^\dagger\,. \tag{5.26}$$

Evaluation of this equation leads to the usual condition, that the connection is antihermitian, i.e. ${A_\mu}^\dagger = -A_\mu$, and the additional condition

$$M^\dagger = M\,. \tag{5.27}$$

Then, in particular, $(\hat{\mathcal{D}}x^\mu)^\dagger = \hat{\mathcal{D}}x^\mu$.

## 5.2 Extension to tensor products

In the following we extend the calculus introduced in the previous subsection to tensor products. If we try to extend the covariant exterior derivative $\hat{\mathcal{D}}$ to a tensor product $\mathcal{V}\otimes\mathcal{V}$ we have to face the following problem. If $\psi_1 \otimes \psi_2$ transforms as $U \otimes U$, then this is not so for $(\hat{\mathcal{D}}\psi_1) \otimes \psi_2$. The reason is that $\hat{\mathcal{D}}\psi_1$ contains the differential $\hat{d}x^\mu$ which does not commute with the $U$ coming from the transformation of $\psi_2$ (note that the tensor product is over $\bigwedge(\mathcal{A})$). Indeed, we have

$$\begin{aligned} (\hat{\mathcal{D}}\psi_1)' \otimes \psi_2' &= U\,\hat{\mathcal{D}}\psi_1 \otimes U\,\psi_2 \\ &= U \otimes U\,(\hat{\mathcal{D}}\psi_1 \otimes \psi_2 - i\hbar\, dT\, \nabla_\mu\psi_1 \otimes U^{-1}\partial^\mu U\,\psi_2)\,. \end{aligned} \tag{5.28}$$

With the help of (5.1) this equation can be rewritten as

$$(\hat{\mathcal{D}}\psi_1)' \otimes \psi_2' - i\hbar\, dT\,(\nabla_\mu\psi_1)' \otimes A'^\mu\,\psi_2' = U \otimes U\,(\hat{\mathcal{D}}\psi_1 \otimes \psi_2 - i\hbar\, dT\, \nabla_\mu\psi_1 \otimes A^\mu\,\psi_2) \tag{5.29}$$



from which we find the following definition of $\hat{\mathcal{D}}$ on the tensor product:

$$\hat{\mathcal{D}}(\psi_1 \otimes \psi_2) = (\hat{\mathcal{D}}\psi_1) \otimes \psi_2 - i\hbar\, dT\, \nabla_\mu\psi_1 \otimes A^\mu\,\psi_2 + \psi_1 \otimes \hat{\mathcal{D}}\psi_2\,. \tag{5.30}$$

The second term on the r.h.s. is the correction to the Leibniz rule which is necessary to achieve the transformation law

$$\hat{\mathcal{D}}(\psi_1 \otimes \psi_2)' = (U \otimes U)\,\hat{\mathcal{D}}(\psi_1 \otimes \psi_2)\,. \tag{5.31}$$

The general formula for the covariant derivative is

$$\begin{aligned} \hat{\mathcal{D}}(\psi_1 \otimes \cdots \otimes \psi_r) &= \hat{\mathcal{D}}\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_r \\ &\quad -i\hbar\, dT \sum_{k=2}^{r} \nabla_\mu\psi_1 \otimes \psi_2 \otimes \cdots \otimes A^\mu\psi_k \otimes \cdots \otimes \psi_r \\ &\quad +\psi_1 \otimes \hat{\mathcal{D}}\psi_2 \otimes \cdots \otimes \psi_r \\ &\quad -i\hbar\, dT \sum_{k=3}^{r} \psi_1 \otimes \nabla_\mu\psi_2 \otimes \cdots \otimes A^\mu\psi_k \otimes \cdots \otimes \psi_r \\ &\quad +\ldots + \psi_1 \otimes \cdots \otimes \psi_{r-1} \otimes \hat{\mathcal{D}}\psi_r\,. \end{aligned} \tag{5.32}$$

The last formula can also be obtained iteratively from the following construction. Let $\mathcal{V}_i$, $i = 1, 2$ be $\bigwedge(\mathcal{A})$-bimodules with representations $\pi_i$ of $G$, and $\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2$ the tensor product over $\bigwedge(\mathcal{A})$ which carries the representation $\pi = \pi_1 \otimes \pi_2$ of $G$. Let $A_{(1)\mu}$ and $A_{(2)\mu}$ denote the gauge potential of a connection in the representation $\pi_1$ and $\pi_2$, respectively. We define the generalized exterior covariant derivative on $\mathcal{V}$ by

$$\hat{\mathcal{D}} = \hat{d} + \hat{A}_\pi \tag{5.33}$$

with

$$\begin{aligned} \hat{A}_\pi &= \frac{i\hbar}{2}\, dT\,(A_{(1)T} \otimes I_{(2)} + I_{(1)} \otimes A_{(2)T} - 2\,\eta^{\mu\nu}\, A_{(1)\mu} \otimes A_{(2)\nu}) \\ &\quad +\hat{d}x^\mu\,(A_{(1)\mu} \otimes I_{(2)} + I_{(1)} \otimes A_{(2)\mu}) \end{aligned} \tag{5.34}$$

where $I_{(i)}$ is the identity on $\mathcal{V}_i$ and $A_{(i)T}$ is defined as in (5.9). It is easily verified that for $\mathcal{V}_1 = \mathcal{V}_2$ this definition is equivalent to (5.30).

We also have the tensor product $\mathcal{V} \otimes \mathcal{V}^*$. Its elements transform according to the adjoint representation. Hence

$$\hat{\mathcal{D}}(\psi \otimes \phi) = \hat{d}(\psi \otimes \phi) + \hat{A}\psi \otimes \phi - \psi \otimes \phi\hat{A} \tag{5.35}$$

and the usual Leibniz rule $\hat{\mathcal{D}}(\psi \otimes \phi) = \hat{\mathcal{D}}\psi \otimes \phi + \psi \otimes \hat{\mathcal{D}}\phi$ holds in this case.

# 6 Gravitation

In this section we generalize the differential calculus introduced in section 4 by replacing the flat Minkowski metric by an arbitrary metric $g_{\mu\nu}$. The resulting structure is covariant with respect to general coordinate transformations. We generalize the notion of a linear connection and covariant derivative. This allows us to construct covariant generalized differential forms, the coefficients of which must then be ordinary tensors.

## 6.1 Covariant differential calculus

Let $\mathcal{A}$ be the algebra of $C^\infty$-functions on a differential manifold $\mathcal{M}$ on which we have a metric $g$. If $x^\mu$ are local cordinates on $\mathcal{M}$, then we replace (4.8) by

$$[x^\mu, \hat{d}x^\nu] = i\hbar\, dT\, g^{\mu\nu} \,. \tag{6.1}$$

This equation is covariant under changes of coordinates (corresponding to differentiable automorphisms of the algebra $\mathcal{A}$ which leave the evolution parameter $T$ invariant):

$$\begin{aligned}
[x'^\mu, \hat{d}x'^\nu] &= [x'^\mu, \hat{d}x^\lambda\, \partial_\lambda x'^\nu + dT\, \hat{\partial}_T x'^\nu] \\
&= [x'^\mu, \hat{d}x^\lambda]\, \partial_\lambda x'^\nu \\
&= [x^\lambda, \hat{d}x'^\mu]\, \partial_\lambda x'^\nu \\
&= [x^\lambda, \hat{d}x^\kappa]\, (\partial_\kappa x'^\mu)(\partial_\lambda x'^\nu) \\
&= i\hbar\, dT\, g^{\kappa\lambda}\, (\partial_\kappa x'^\mu)(\partial_\lambda x'^\nu) \\
&= i\hbar\, dT\, g'^{\mu\lambda} \,.
\end{aligned} \tag{6.2}$$

Generalized partial derivatives are again introduced via equation (4.9). Repeating the arguments given in section 4, it follows that

$$\hat{\partial}_\mu = \partial_\mu \quad , \quad \hat{\partial}_T = \frac{i\hbar}{2} g^{\mu\nu}\, \partial_\mu \partial_\nu + \partial_T \tag{6.3}$$

where $\partial_\mu$ and $\partial_T$ are the ordinary partial derivatives with respect to $x^\mu$ and $T$, respectively. If $f$ and $h$ are two differentiable functions on $\mathcal{M}$, then

$$[f, \hat{d}h] = i\hbar\, dT\, g^{\mu\nu} \partial_\mu f\, \partial_\nu h \tag{6.4}$$

as a consequence of (6.1).

This extension of the differential calculus of section 4 is consistent. The only modification of the rules we had in section 4 (apart from the substitution $\eta \mapsto g$) appears in



the 1-form commutation relations and, as a consequence, in the commutation relations for the partial derivatives.

Differentiating (6.1), we obtain

$$(dT)^2 = 0 \quad , \quad dT\, \hat{d}x^\mu + \hat{d}x^\mu\, dT = 0 \tag{6.5}$$

and

$$\hat{d}x^\mu\, \hat{d}x^\nu + \hat{d}x^\nu\, \hat{d}x^\mu = -i\hbar\, dT\, dx^\rho\, \partial_\rho g^{\mu\nu} \,. \tag{6.6}$$

If the metric is not flat, then the wedge product between the generalized differentials $\hat{d}x^\mu$ is no longer the classical one.

Let $\Gamma^\mu{}_{\nu\rho}$ be a $g$-compatible linear connection, i.e.[21]

$$\partial_\rho g^{\mu\nu} = -g^{\lambda(\mu}\Gamma^{\nu)}{}_{\lambda\rho} \tag{6.7}$$

with torsion

$$Q^\mu{}_{\rho\sigma} := -\Gamma^\mu{}_{[\rho\sigma]} \,. \tag{6.8}$$

We will use the abbreviations $\partial^2 = g^{\mu\nu}\partial_\mu\partial_\nu$ and $\Gamma^\mu = g^{\rho\sigma}\, \Gamma^\mu{}_{\rho\sigma}$ in the following. If $\tilde{\Gamma}^\mu{}_{\rho\sigma}$ denotes the Levi-Civita connection (characterized by vanishing torsion), then we have

$$\Gamma^\mu = \tilde{\Gamma}^\mu - Q^\mu \tag{6.9}$$

where $Q^\mu := g^{\mu\nu}\, Q^\rho{}_{\rho\nu}$ is the torsion vector. Using the transformation law

$$\partial_\alpha(\partial_\beta x'^\mu) = \Gamma^\gamma{}_{\beta\alpha}\, (\partial_\gamma x'^\mu) - (\partial_\beta x'^\nu)\, (\partial_\alpha x'^\rho)\, \Gamma'^\mu{}_{\nu\rho} \tag{6.10}$$

we find

$$\hat{d}x'^\mu = \frac{i\hbar}{2}\, dT\, \partial^2 x'^\mu + \hat{d}x^\alpha\, (\partial_\alpha x'^\mu) = \frac{i\hbar}{2}\, dT\, (\Gamma^\alpha(\partial_\alpha x'^\mu) - \Gamma'^\mu) + \hat{d}x^\alpha(\partial_\alpha x'^\mu) \,. \tag{6.11}$$

The differentials $\hat{d}x^\mu$ are therefore not covariant. But the last equation shows that the 1-forms

$$\hat{D}x^\mu := \hat{d}x^\mu + \frac{i\hbar}{2}\, dT\, \Gamma^\mu \tag{6.12}$$

are right-covariant, i.e.

$$\hat{D}' x'^\mu = (\hat{D}x^\alpha)\, \partial_\alpha x'^\mu \,. \tag{6.13}$$

[21] (Anti-) symmetrisation is understood without factors, e.g. $\Gamma^\mu_{[\rho\sigma]} = \Gamma^\mu_{\rho\sigma} - \Gamma^\mu_{\sigma\rho}$.

As a consequence, $(\hat{D}x^\mu)^*$ is left-covariant. If torsion vanishes, $\hat{D}x^\mu$ and $\hat{d}x^\mu$ coincide in the harmonic gauge $\tilde{\Gamma}^\mu = 0$.

In order to define tensor-valued forms we have to distinguish between right- and left-covariant forms. In what follows all equations will consist of either only left- or only right-covariant forms. If we express a (generalized) differential form in terms of the basic 1-forms $\hat{D}x^\mu$ or $(\hat{D}x^\mu)^*$ their position decides about the covariance properties. For example, if $\alpha_\mu$ transforms covariantly, then $\hat{D}x^\mu\,\alpha_\mu$ and $\alpha_\mu\,(\hat{D}x^\mu)^*$ are both scalar one forms, but $\alpha_\mu\,\hat{D}x^\mu$ and $(\hat{D}x^\mu)^*\,\alpha_\mu$ are not. We speak of "left forms" if the basic 1-forms are to the left of all other factors. We have a "right form" if the $(\hat{D}x^\mu)^*$ are to the right of all other factors. Now left forms are right covariant and right forms are left covariant.[22]

Let us write the differential of a function $f$ in terms of the basic 1-forms:

$$\begin{aligned}\hat{d}f &= \frac{i\hbar}{2}\,dT\,\partial^2 f + \hat{d}x^\mu\,\partial_\mu f = \frac{i\hbar}{2}\,dT\,\partial^2 f + (\hat{D}x^\mu - \frac{i\hbar}{2}\,dT\,\Gamma^\mu)\,\partial_\mu f \\ &= \frac{i\hbar}{2}\,dT\,\nabla^2 f + \hat{D}x^\mu\,\partial_\mu f \end{aligned} \tag{6.14}$$

where $\nabla^2 := g^{\mu\nu}\nabla_\mu\nabla_\nu$. Equation (6.14) does not depend on the choice of a (metric compatible) connection, i.e. on the torsion. This is obvious from the fact that $\hat{d}$ involves the metric only. The same holds for any expression which is obtained by application of $\hat{d}$ to a scalar form.

For a contravariant 0-form $Y^\mu$ we define the right-covariant exterior derivative by[23]

$$\hat{D}Y^\mu := \hat{d}Y^\mu + Y^\nu\,{}_\nu\hat{\Gamma}^\mu \tag{6.15}$$

where

$${}_\nu\hat{\Gamma}^\mu := \frac{i\hbar}{2}\,dT\,{}_\nu\Lambda^\mu + \hat{d}x^\rho\,\Gamma^\mu{}_{\nu\rho}\,. \tag{6.16}$$

$\hat{D}Y^\mu$ is indeed right-covariant if the (generalized) connection form obeys the following transformation rule:

$$\hat{d}(\partial_\alpha x'^\mu) = {}_\alpha\hat{\Gamma}^\beta\,(\partial_\beta x'^\mu) - (\partial_\alpha x'^\nu)\,{}_\nu\hat{\Gamma}'^\mu\,. \tag{6.17}$$

This equation now restricts the part of the generalized connection which is proportional to $dT$ to

$${}_\nu\Lambda^\mu = g^{\rho\sigma}\,(\partial_\rho\Gamma^\mu{}_{\nu\sigma} + \Gamma^\mu{}_{\lambda\rho}\,\Gamma^\lambda{}_{\nu\sigma}) + M^\mu{}_\nu \tag{6.18}$$

[22] It is also possible to construct left-covariant left forms and right-covariant right forms, but this requires a rather complicated notation.

[23] In order to obtain a right-covariant exterior derivative we must write $\hat{\Gamma}$ to the right of $Y$ in this definition. It does not work the other way.



where the undetermined part $M^\mu{}_\nu$ has to transform tensorially. Substituting this result into (6.15) we obtain

$$\hat{D}Y^\mu = \frac{i\hbar}{2}\,dT\,(\nabla^2 Y^\mu + M^\mu{}_\nu\,Y^\nu) + \hat{D}x^\nu\,\nabla_\nu Y^\mu\,. \tag{6.19}$$

Similarly, for a covariant 0-form $\alpha_\mu$ we set

$$\hat{D}\alpha_\mu := \hat{d}\alpha_\mu - \alpha_\nu\,{}^\nu\hat{\Gamma}_\mu\,. \tag{6.20}$$

$\hat{D}\alpha_\mu$ is right-covariant if the connection ${}^\nu\hat{\Gamma}_\mu$ transforms as follows:

$$\hat{d}(\partial_\alpha x'^\mu) = (\partial_\beta x'^\mu)\,{}^\beta\hat{\Gamma}_\alpha - {}^\mu\hat{\Gamma}'_\nu\,(\partial_\alpha x'^\nu)\,. \tag{6.21}$$

This formula restricts the generalized connection ${}^\nu\hat{\Gamma}_\mu$ to the form

$${}^\nu\hat{\Gamma}_\mu = \frac{i\hbar}{2}\,dT\,{}^\nu\Lambda_\mu + \hat{d}x^\rho\,\Gamma^\nu{}_{\mu\rho} \tag{6.22}$$

with

$${}^\nu\Lambda_\mu := g^{\rho\sigma}\,(\partial_\rho\Gamma^\nu{}_{\mu\sigma} - \Gamma^\nu{}_{\lambda\rho}\,\Gamma^\lambda{}_{\mu\sigma}) + \tilde{M}^\nu{}_\mu \tag{6.23}$$

where $\tilde{M}$ transforms tensorially. Inserting this result into (6.20), we find

$$\hat{D}\alpha_\mu = \frac{i\hbar}{2}\,dT\,(\nabla^2\alpha_\mu - \alpha_\nu\,\tilde{M}^\nu{}_\mu) + \hat{D}x^\nu\,\nabla_\nu\alpha_\mu\,. \tag{6.24}$$

Both $\hat{D}Y^\mu$ and $\hat{D}\alpha_\mu$ are right-covariant left-forms. In the next subsection we show that the tensors $\tilde{M}$ and $M$ have to be identified.

## 6.2 Tensor calculus and right-covariant derivatives

If we try to extend the covariant derivative to tensor products like $Y^\mu Z^\nu$, we have to deal with the problem that $Y^\mu\hat{D}Z^\nu$ is not right-covariant (whereas $(\hat{D}Y^\mu)\,Z^\nu$ is). We therefore have to introduce a correction in a similar way as in section 5.2 where we treated the case of gauge theories. The generalized Leibniz rule for $\hat{D}$ then reads

$$\hat{D}(Y^\mu Z^\nu) = (\hat{D}Y^\mu)\,Z^\nu + Y^\rho\,{}_\rho(\hat{D}Z^\nu)^\mu \tag{6.25}$$

where

$${}_\rho(\hat{D}Z^\nu)^\mu := \delta^\mu_\rho\,\hat{D}Z^\nu + i\hbar\,dT\,(\nabla^\lambda Z^\nu)\,\Gamma^\mu{}_{\rho\lambda}\,. \tag{6.26}$$

The last expression is left-covariant with respect to the index $\rho$ and right-covariant with respect to the indices $\mu$ and $\nu$. The generalized covariant exterior derivative of $Y^\mu Z^\nu$ or, more generally, of a tensor $Y^{\mu\nu}$ can also be written as

$$\hat{D}Y^{\mu\nu} = \hat{d}Y^{\mu\nu} + Y^{\rho\sigma}\,{}_{\rho\sigma}\hat{\Gamma}^{\mu\nu} \tag{6.27}$$

with

$${}_{\rho\sigma}\hat{\Gamma}^{\mu\nu} = {}_\rho\hat{\Gamma}^\mu\,\delta^\nu_\sigma + \delta^\mu_\rho\,{}_\sigma\hat{\Gamma}^\nu + i\hbar\,dT\,g^{\kappa\lambda}\,\Gamma^\mu{}_{\rho\kappa}\,\Gamma^\nu{}_{\sigma\lambda}\,. \tag{6.28}$$

Inserting our expression for ${}_\nu\hat{\Gamma}^\mu$, we find

$$\hat{D}Y^{\mu\nu} = \frac{i\hbar}{2}\,dT\,(\nabla^2 Y^{\mu\nu} + M^\mu{}_\rho\,Y^{\rho\nu} + M^\nu{}_\rho\,Y^{\mu\rho}) + \hat{D}x^\lambda\,\nabla_\lambda Y^{\mu\nu}\,. \tag{6.29}$$

In particular, this yields

$$\hat{D}g^{\mu\nu} = \frac{i\hbar}{2}\,dT\,M^{(\mu\nu)}\,. \tag{6.30}$$

Similarly, we have

$$\hat{D}(\alpha_\mu Z^\nu) = (\hat{D}\alpha_\mu)\,Z^\nu + \alpha_\rho\,{}^\rho(\hat{D}Z^\nu)_\mu \tag{6.31}$$

with

$${}^\rho(\hat{D}Z^\nu)_\mu := \delta^\rho_\mu\,\hat{D}Z^\nu - i\hbar\,dT\,(\nabla^\lambda Z^\nu)\,\Gamma^\rho{}_{\mu\lambda}\,. \tag{6.32}$$

Again, this can be written, more generally for a tensor $Y_\mu{}^\nu$, as

$$\hat{D}Y_\mu{}^\nu = \hat{d}Y_\mu{}^\nu + Y_\rho{}^\sigma\,{}^\rho_\sigma\hat{\Gamma}^\nu_\mu \tag{6.33}$$

with

$${}^\rho_\sigma\hat{\Gamma}^\mu_\nu = {}_\sigma\hat{\Gamma}^\mu\,\delta^\rho_\nu - \delta^\mu_\sigma\,{}^\rho\hat{\Gamma}_\nu - i\hbar\,dT\,g^{\kappa\lambda}\,\Gamma^\mu{}_{\sigma\kappa}\,\Gamma^\rho{}_{\nu\lambda}\,. \tag{6.34}$$

In particular,

$$\hat{D}\delta^\nu_\mu = -\frac{i\hbar}{2}\,dT\,(\tilde{M}^\nu{}_\mu - M^\nu{}_\mu)\,. \tag{6.35}$$

Consistency of the generalized Leibniz rule for $\hat{D}$ now requires

$$\tilde{M}^\nu{}_\mu = M^\nu{}_\mu\,. \tag{6.36}$$



Then

$$\hat{D}(\alpha_\mu Z^\mu) = \hat{d}(\alpha_\mu Z^\mu) \tag{6.37}$$

as it should be since $\alpha_\mu Z^\mu$ is a scalar. The extension of $\hat{D}$ as a right-covariant exterior derivative to higher tensors is now straightforward. In particular, the generalized Leibniz rule for a product of two arbitrary tensors reads

$$\hat{D}(Y^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s}\,Z^{\kappa_1\ldots}_{\lambda_1\ldots}) = (\hat{D}Y^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s})\,Z^{\kappa_1\ldots}_{\lambda_1\ldots} + Y^{\rho_1\ldots\rho_r}_{\sigma_1\ldots\sigma_s}\;{}^{\sigma_1\ldots\sigma_s}_{\rho_1\ldots\rho_r}(\hat{D}Z^{\kappa_1\ldots}_{\lambda_1\ldots})^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s} \tag{6.38}$$

with

$$\begin{aligned}
{}^{\sigma_1\ldots\sigma_s}_{\rho_1\ldots\rho_r}(\hat{D}Z^{\kappa_1\ldots}_{\lambda_1\ldots})^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s} \;=\;& \delta^{\sigma_1}_{\nu_1}\cdots\delta^{\sigma_s}_{\nu_s}\,\delta^{\mu_1}_{\rho_1}\cdots\delta^{\mu_r}_{\rho_r}\,\hat{D}Z^{\kappa_1\ldots}_{\lambda_1\ldots} - i\hbar\,dT\,(\nabla^\gamma Z^{\kappa_1\ldots}_{\lambda_1\ldots})\,[\,(\Gamma^{\sigma_1}{}_{\nu_1\gamma}\,\delta^{\sigma_2}_{\nu_2}\cdots\delta^{\sigma_s}_{\nu_s} \\
&+\ldots+\Gamma^{\sigma_s}{}_{\nu_s\gamma}\,\delta^{\sigma_1}_{\nu_1}\cdots\delta^{\sigma_{s-1}}_{\nu_{s-1}})\,\delta^{\mu_1}_{\rho_1}\cdots\delta^{\mu_r}_{\rho_r} \\
&-(\Gamma^{\mu_1}{}_{\rho_1\gamma}\,\delta^{\mu_2}_{\rho_2}\cdots\delta^{\mu_r}_{\rho_r}+\ldots+\Gamma^{\mu_r}{}_{\rho_r\gamma}\,\delta^{\mu_1}_{\rho_1}\cdots\delta^{\mu_{r-1}}_{\rho_{r-1}})\,\delta^{\sigma_1}_{\nu_1}\cdots\delta^{\sigma_s}_{\nu_s}\,]\,.
\end{aligned} \tag{6.39}$$

*Remark:* In virtue of (6.30) metric compatibility of $\hat{D}$ would require $M^{(\mu\nu)} = 0$, i.e. $M$ had to be antisymmetric. We do not have a good reason, however, to demand this condition. On the other hand there is another notion of metric compatibility in our framework which we consider to be more adequate. Let us define a hermitean scalar product for 1-forms by

$$(\alpha,\beta) := \alpha_\mu\,g^{\mu\nu}\,\beta^*_\nu\,. \tag{6.40}$$

Compatibility of $\hat{D}$ with this metric means

$$\hat{d}(\alpha,\beta) = (\hat{D}\alpha,\beta) + (\alpha,\hat{D}\beta) \tag{6.41}$$

which implies

$$\hat{d}g^{\mu\nu} + {}^\mu\hat{\Gamma}_\lambda\,g^{\lambda\nu} + g^{\mu\lambda}\,({}^\nu\hat{\Gamma}_\lambda)^* = 0\,. \tag{6.42}$$

The last equation leads to $M^{\mu\nu} - (M^{\nu\mu})^* = 0$. If $M$ is real this means that $M$ is symmetric.

## 6.3 Higher order differential forms

We would like to construct higher order differential forms which are right-covariant. In order to do this we have to find a way to build up right-covariant 2-forms from the basic 1-forms $\hat{D}x^\mu$. The product $\hat{D}x^\mu\hat{D}x^\nu$ is neither right-covariant nor antisymmetric since

$$\hat{D}x^{(\mu}\,\hat{D}x^{\nu)} = \hat{d}x^{(\mu}\,\hat{d}x^{\nu)} = -i\hbar\,dT\,dg^{\mu\nu} \tag{6.43}$$

by virtue of (6.6). But with a correction similar to (6.26), i.e.

$$ {}_\rho(\hat{D}x^\nu)^\mu := \delta^\mu_\rho \hat{D}x^\nu + i\hbar\, dT\, g^{\lambda\nu}\, \Gamma^\mu{}_{\rho\lambda} \,, \tag{6.44} $$

we find that

$$ \hat{D}x^\rho\, {}_\rho(\hat{D}x^\nu)^\mu = \hat{D}x^\mu\, \hat{D}x^\nu - i\hbar\, dT\, dx^\rho\, g^{\sigma\nu}\, \Gamma^\mu{}_{\rho\sigma} \tag{6.45} $$

is right-covariant. But

$$ \hat{D}x^\rho\, {}_\rho(\hat{D}x^\nu)^\mu + \hat{D}x^\rho\, {}_\rho(\hat{D}x^\nu)^\mu = -i\hbar\, dT\, dx^\rho\, Q^{(\mu}{}_{\sigma\rho}\, g^{\nu)\sigma} \tag{6.46} $$

shows that we have antisymmetry only if torsion vanishes. Let us therefore define the new wedge product

$$ \hat{D}x^\mu \wedge \hat{D}x^\nu := \hat{D}x^\mu\, \hat{D}x^\nu - i\hbar\, dT\, dx^\rho\, g^{\sigma\nu}\, \Gamma^\mu{}_{\sigma\rho} \,. \tag{6.47} $$

This is right-covariant and antisymmetric.[24] We will use $\hat{D}x^\mu \wedge \hat{D}x^\nu$ as a basis for right-covariant 2-forms. With any tensor which is antisymmetric in two indices we can now associate a (tensor-valued) right-covariant 2-form. For example, we may define a right-covariant torsion 2-form by

$$ \hat{\Theta}^\mu = \frac{1}{2}\, \hat{D}x^\kappa \wedge \hat{D}x^\lambda\, Q^\mu{}_{\kappa\lambda} \,. \tag{6.48} $$

Applying $\hat{D}$ to differential forms we have to take care of signs corresponding to their grades. For example,

$$ \hat{D}(\hat{D}x^\mu) = \hat{d}\hat{D}x^\mu - \hat{D}x^\nu\, {}_\nu\hat{\Gamma}^\mu \,. \tag{6.49} $$

Using (6.7), (6.8), (6.12) and (6.16) with (6.18), we obtain

$$ \hat{D}\hat{D}x^\mu = \hat{\Theta}^\mu - \frac{i\hbar}{2}\, dT\, dx^\nu (R^\mu{}_\nu - M^\mu{}_\nu - \nabla^\rho Q^\mu{}_{\rho\nu} + Q^{\mu\rho\sigma}\, Q_{\rho\sigma\nu}) \tag{6.50} $$

where $R_{\mu\nu}$ is the Ricci tensor of the connection $\Gamma^\mu{}_{\nu\rho}$. Our conventions for the curvature and the Ricci tensor are as follows:

$$ R^\mu{}_{\nu\kappa\lambda} = \partial_\kappa \Gamma^\mu{}_{\nu\lambda} - \partial_\lambda \Gamma^\mu{}_{\nu\kappa} + \Gamma^\mu{}_{\sigma\kappa}\, \Gamma^\sigma{}_{\nu\lambda} - \Gamma^\mu{}_{\sigma\lambda}\, \Gamma^\sigma{}_{\nu\kappa} \quad , \quad R_{\mu\nu} = R^\kappa{}_{\mu\kappa\nu} \,. \tag{6.51} $$

The r.h.s. of (6.50) suggests to relate the tensor $M_{\mu\nu}$ to the energy momentum tensor of matter and in this way "geometrize" the latter. The condition of vanishing torsion and

[24] Replacing $\Gamma^\mu{}_{\rho\lambda}$ by $\Gamma^\mu{}_{\lambda\rho}$ in (6.26) retains (6.25). Then (6.45) simply becomes (6.47). But then it is no longer possible to write (6.25) in the form (6.27).



Einstein's equations in the form $R_{\mu\nu} = M_{\mu\nu}$ can then be expressed as the integrability condition $\hat{D}\hat{D}x^\mu = 0$. We should view the l.h.s. of (6.50) as a generalized torsion.

For a scalar 1-form

$$ A = \frac{i\hbar}{2}\, dT\, f + \hat{D}x^\mu\, \alpha_\mu = \frac{i\hbar}{2}\, dT\, \tilde{f} + \tilde{D}x^\mu\, \alpha_\mu \tag{6.52} $$

where $\tilde{f} := f - Q^\mu\, \alpha_\mu$, the (generalized) exterior derivative can be evaluated as follows

$$ \begin{aligned} \hat{d}A &= -\frac{i\hbar}{2}\, dT\, df + (\hat{D}\hat{D}x^\mu)\, \alpha_\mu - \hat{D}x^\rho\, {}_\rho(\hat{D}\alpha_\mu)^\mu \\ &= \frac{1}{2}\, \hat{D}x^\mu \wedge \hat{D}x^\nu\, \partial_{[\mu}\alpha_{\nu]} + \frac{i\hbar}{2}\, dT\, dx^\nu \Big[\nabla^2\alpha_\nu - \alpha_\mu(R^\mu{}_\nu - \nabla^\rho Q^\mu{}_{\rho\nu} - Q^{\mu\rho\sigma}\, Q_{\rho\sigma\nu}) \\ &\quad + Q^{\mu\rho}{}_\nu \nabla_\rho \alpha_\mu - \partial_\nu f\Big] \,. \end{aligned} \tag{6.53} $$

Using the Levi-Civita connection, this equation reads

$$ \hat{d}A = \frac{i\hbar}{2}\, dT\, (\Box\alpha - d\tilde{f}) + \frac{1}{2}\, \tilde{D}x^\mu \wedge \tilde{D}x^\nu\, \partial_{[\mu}\alpha_{\nu]} \tag{6.54} $$

where $\alpha := \alpha_\mu\, dx^\mu$ and $\Box := d\delta + \delta d$ is the Laplace-Beltrami operator for the metric $g$.

The generalization of (6.44) is

$$ \begin{aligned} {}^{\sigma_1\ldots\sigma_s}_{\rho_1\ldots\rho_r}(\hat{D}x^\nu)^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_s} &= \delta^{\sigma_1}_{\lambda_1}\cdots\delta^{\sigma_s}_{\lambda_s}\,\delta^{\mu_1}_{\rho_1}\cdots\delta^{\mu_r}_{\rho_r}\,\hat{D}x^\nu \\ &\quad - i\hbar\, dT\, g^{\nu\gamma}\big[(\Gamma^{\sigma_1}{}_{\lambda_1\gamma}\,\delta^{\sigma_2}_{\lambda_2}\cdots\delta^{\sigma_s}_{\lambda_s} + \ldots + \Gamma^{\sigma_s}{}_{\lambda_s\gamma}\,\delta^{\sigma_1}_{\lambda_1}\cdots\delta^{\sigma_{s-1}}_{\lambda_{s-1}})\,\delta^{\mu_1}_{\rho_1}\cdots\delta^{\mu_r}_{\rho_r} \\ &\quad - (\Gamma^{\mu_1}{}_{\rho_1\gamma}\,\delta^{\mu_2}_{\rho_2}\cdots\delta^{\mu_r}_{\rho_r} + \ldots + \Gamma^{\mu_r}{}_{\rho_r\gamma}\,\delta^{\mu_1}_{\rho_1}\cdots\delta^{\mu_{r-1}}_{\rho_{r-1}})\,\delta^{\sigma_1}_{\lambda_1}\cdots\delta^{\sigma_s}_{\lambda_s}\big] \,. \end{aligned} \tag{6.55} $$

If torsion vanishes, there is a simple way to construct a right-covariant basis for $r+1$-forms inductively by

$$ \hat{D}x^{\mu_1} \wedge \cdots \wedge \hat{D}x^{\mu_{r+1}} := (\hat{D}x^{\nu_1} \wedge \cdots \wedge \hat{D}x^{\nu_r})\quad {}_{\nu_1\ldots\nu_r}(\hat{D}x^{\mu_{r+1}})^{\mu_1\ldots\mu_r} \,. \tag{6.56} $$

That these right-covariant expressions are indeed antisymmetric (which justifies the wedges) is most easily varified in the gauge $\Gamma = 0$.

## 6.4 Generalized curvature 2-forms

Applying $\hat{d}$ to (6.17), using $\hat{d}^2 = 0$ and (6.17) again, we obtain the curvature 2-form

$$ {}_\nu\hat{\mathcal{R}}^\mu := \hat{d}\, {}_\nu\hat{\Gamma}^\mu - {}_\nu\hat{\Gamma}^\rho\, {}_\rho\hat{\Gamma}^\mu \tag{6.57} $$

which transforms according to

$$ {}_\nu\hat{\mathcal{R}}'^\mu = (\partial'_\nu x^\beta)\, {}_\beta\hat{\mathcal{R}}^\alpha\, (\partial_\alpha x'^\mu) \,. \tag{6.58} $$

This is also evident from the generalized Ricci identity

$$\hat{D}\hat{D}Y^\mu = Y^\nu\,{}_\nu\hat{\mathcal{R}}^\mu\,. \tag{6.59}$$

In order to read off covariant components from the curvature form we need a suitable basis of 2-forms. To simplify the calculations we will assume vanishing torsion in the following. Then

$$\begin{aligned} {}_\rho(\hat{D}x^\mu \wedge \hat{D}x^\nu)^\sigma &:= {}_\rho(\hat{D}x^\kappa)^\lambda\,{}_{\kappa\lambda}(\hat{D}x^\nu)^{\mu\sigma} \\ &= (\hat{D}x^\mu \wedge \hat{D}x^\nu)\,\delta^\sigma_\rho - i\hbar\,dT\,dx^{[\mu}\,g^{\nu]\lambda}\,\Gamma^\sigma{}_{\rho\lambda}\,. \end{aligned} \tag{6.60}$$

has the desired properties. The left index transforms to the left and all other indices transform to the right. Now (6.57) can be written in the form

$${}_\nu\hat{\mathcal{R}}^\mu = {}_\nu\hat{\Omega}^\mu - \frac{i\hbar}{2}\,dT\,dx^\lambda\,(\nabla^\rho R^\mu{}_{\nu\lambda\rho} + \nabla_\lambda\,M^\mu{}_\nu) \tag{6.61}$$

where

$${}_\nu\hat{\Omega}^\mu := \frac{1}{2}\,{}_\nu(\hat{D}x^\rho \wedge \hat{D}x^\sigma)^\lambda\,R^\mu{}_{\lambda\rho\sigma} \tag{6.62}$$

and $R^\mu{}_{\nu\rho\sigma}$ is the ordinary curvature tensor.

Repeating the steps above starting from (6.21), we obtain the curvature 2-form

$${}^\nu\hat{\mathcal{R}}_\mu := \hat{d}^\nu\hat{\Gamma}_\mu + {}^\nu\hat{\Gamma}_\rho\,{}^\rho\hat{\Gamma}_\mu \tag{6.63}$$

which transforms according to

$${}^\nu\hat{\mathcal{R}}'_\mu = (\partial_\beta x'^\nu)^\beta\hat{\mathcal{R}}_\alpha\,(\partial'_\mu x^\alpha) \tag{6.64}$$

and satisfies the Ricci identity

$$\hat{D}\hat{D}\alpha_\mu = -\alpha_\nu\,{}^\nu\hat{\mathcal{R}}_\mu\,. \tag{6.65}$$

Using the basis of 2-forms

$${}^\rho(\hat{D}x^\mu \wedge \hat{D}x^\nu)_\sigma := {}^\rho(\hat{D}x^\kappa)_\lambda\,{}^\lambda_\kappa(\hat{D}x^\nu)^\mu_\sigma = (\hat{D}x^\mu \wedge \hat{D}x^\nu)\,\delta^\rho_\sigma + i\hbar\,dT\,dx^{[\mu}\,g^{\nu]\lambda}\,\Gamma^\rho{}_{\sigma\lambda} \tag{6.66}$$

we have

$${}^\nu\hat{\mathcal{R}}_\mu = {}^\nu\hat{\Omega}_\mu - \frac{i\hbar}{2}\,dT\,dx^\lambda\,(\nabla^\rho R^\nu{}_{\mu\lambda\rho} + \nabla_\lambda\,M^\nu{}_\mu) \tag{6.67}$$

where

$${}^\nu\hat{\Omega}_\mu := \frac{1}{2}\,{}^\nu(\hat{D}x^\rho \wedge \hat{D}x^\sigma)_\lambda\,R^\lambda{}_{\mu\rho\sigma}\,. \tag{6.68}$$



Now we note that

$$Y^\nu\,{}_\nu(\hat{D}x^\rho \wedge \hat{D}x^\sigma)^\mu = (\hat{D}x^\rho \wedge \hat{D}x^\sigma)\,Y^\mu - i\hbar\,dT\,dx^{[\rho}\nabla^{\sigma]}\,Y^\mu \tag{6.69}$$

and

$$\alpha_\nu\,{}^\nu(\hat{D}x^\rho \wedge \hat{D}x^\sigma)_\mu = (\hat{D}x^\rho \wedge \hat{D}x^\sigma)\,\alpha_\mu - i\hbar\,dT\,dx^{[\rho}\nabla^{\sigma]}\,\alpha_\mu\,. \tag{6.70}$$

The first identity allows us to write (6.59) in the form

$$\hat{D}\hat{D}Y^\mu = \hat{\Omega}^\mu{}_\nu Y^\nu - \frac{i\hbar}{2}\,dT\,dx^\lambda\,(\nabla^\rho R^\mu{}_{\nu\lambda\rho}\,Y^\nu + (\nabla_\lambda M^\mu{}_\nu)\,Y^\nu + 2\,R^\mu{}_{\nu\lambda\rho}\nabla^\rho\,Y^\nu) \tag{6.71}$$

with

$$\hat{\Omega}^\mu{}_\nu := \frac{1}{2}\,\hat{D}x^\rho \wedge \hat{D}x^\sigma\,R^\mu{}_{\nu\rho\sigma} \tag{6.72}$$

which is a right-covariant left-form. Similarly, we can use (6.70) to derive a corresponding expression for $\hat{D}\hat{D}\alpha_\mu$.

## 6.5 Covariant exterior derivatives of generalized differential forms

So far we have only considered covariant derivatives of right-covariant tensorial forms. But in the last subsection we also encountered expressions with mixed (left and right) covariance properties, in particular the generalized curvature 2-forms. We therefore have to extend the covariant exterior derivative $\hat{D}$ to such expressions. This extension is completely determined by our rules for the covariant derivative of right-covariant tensorial forms. For example, $Y^\nu\,{}_\nu\hat{\mathcal{R}}^\mu$ with a contravariant 0-form $Y^\nu$ is right-covariant, so that

$$\hat{D}(Y^\nu\,{}_\nu\hat{\mathcal{R}}^\mu) = \hat{d}(Y^\nu\,{}_\nu\hat{\mathcal{R}}^\mu) + Y^\nu\,{}_\nu\hat{\mathcal{R}}^\rho\,{}_\rho\hat{\Gamma}^\mu\,. \tag{6.73}$$

Using the Leibniz rule for $\hat{d}$, this can be written as

$$\hat{D}(Y^\nu\,{}_\nu\hat{\mathcal{R}}^\mu) = (\hat{D}Y^\nu)\,{}_\nu\hat{\mathcal{R}}^\mu + Y^\nu\,[\hat{d}\,{}_\nu\hat{\mathcal{R}}^\mu - {}_\nu\hat{\Gamma}^\rho\,{}_\rho\hat{\mathcal{R}}^\mu + {}_\nu\hat{\mathcal{R}}^\rho\,{}_\rho\hat{\Gamma}^\mu] \tag{6.74}$$

from which we read off

$$\hat{D}({}_\nu\hat{\mathcal{R}}^\mu) = \hat{d}\,{}_\nu\hat{\mathcal{R}}^\mu - {}_\nu\hat{\Gamma}^\rho\,{}_\rho\hat{\mathcal{R}}^\mu + {}_\nu\hat{\mathcal{R}}^\rho\,{}_\rho\hat{\Gamma}^\mu\,. \tag{6.75}$$

Inserting the definition of ${}_\nu\hat{\mathcal{R}}^\mu$ we obtain the second Bianchi identity

$$\hat{D}({}_\nu\hat{\mathcal{R}}^\mu) = 0\,. \tag{6.76}$$

In the same way we find

$$\hat{D}({}^{\nu}\hat{\mathcal{R}}_{\mu}) = \hat{d}\,{}^{\nu}\hat{\mathcal{R}}_{\mu} + {}^{\nu}\hat{\Gamma}_{\rho}\,{}^{\rho}\hat{\mathcal{R}}_{\mu} - {}^{\nu}\hat{\mathcal{R}}_{\rho}\,{}^{\rho}\hat{\Gamma}_{\mu} = 0\,. \tag{6.77}$$

Another example is

$$\hat{D}(Y^{\rho}{}_{\rho}(\hat{D}x^{\mu})^{\sigma}) = (\hat{D}Y^{\rho})\,{}_{\rho}(\hat{D}x^{\mu})^{\sigma} + Y^{\rho}\,[\hat{d}_{\rho}(\hat{D}x^{\mu})^{\sigma} - {}_{\rho}(\hat{D}x^{\kappa})^{\lambda}\,{}_{\kappa\lambda}\hat{\Gamma}^{\mu\sigma} - {}_{\rho}\hat{\Gamma}^{\lambda}\,{}_{\lambda}(\hat{D}x^{\mu})^{\sigma}]\,. \tag{6.78}$$

The l.h.s. and the first term on the r.h.s. are manifestly right-covariant, hence also the last term. Consequently

$$\begin{aligned}\hat{D}_{\rho}(\hat{D}x^{\mu})^{\sigma} &= \hat{d}_{\rho}(\hat{D}x^{\mu})^{\sigma} - {}_{\rho}(\hat{D}x^{\kappa})^{\lambda}\,{}_{\kappa\lambda}\hat{\Gamma}^{\mu\sigma} - {}_{\rho}\hat{\Gamma}^{\lambda}\,{}_{\lambda}(\hat{D}x^{\mu})^{\sigma} \\ &= -\frac{i\hbar}{2}\,dT\,dx^{\nu}\,\delta^{\sigma}_{\rho}\,(R^{\mu}{}_{\nu} - M^{\mu}{}_{\nu}) - i\hbar\,dT\,dx^{\kappa}\,R^{\sigma}{}_{\rho\kappa}{}^{\mu} \\ &= -\frac{i\hbar}{2}\,dT\,\delta^{\sigma}_{\rho}\,N^{\mu} + i\hbar\,dT\,g^{\mu\nu}\partial_{\nu}\lrcorner\Omega^{\sigma}{}_{\rho}\end{aligned} \tag{6.79}$$

has the same transformation properties as ${}_{\rho}(\hat{D}x^{\mu})^{\sigma}$. In the last step we have introduced the abbreviation

$$N^{\mu} := (R^{\mu}{}_{\nu} - M^{\mu}{}_{\nu})\,dx^{\nu} \tag{6.80}$$

and the usual insertion operator $\lrcorner$. A similar calculation leads to

$$\begin{aligned}\hat{D}\,{}^{\rho}(\hat{D}x^{\mu})_{\sigma} &= \hat{d}\,{}^{\rho}(\hat{D}x^{\mu})_{\sigma} - {}^{\rho}(\hat{D}x^{\kappa})_{\lambda}\,{}^{\lambda}_{\kappa}\hat{\Gamma}^{\mu}_{\sigma} + {}^{\rho}\hat{\Gamma}_{\lambda}\,{}^{\lambda}(\hat{D}x^{\mu})_{\sigma} \\ &= -\frac{i\hbar}{2}\,dT\,\delta^{\rho}_{\sigma}\,N^{\mu} - i\hbar\,dT\,g^{\mu\nu}\,\partial_{\nu}\lrcorner\Omega^{\rho}{}_{\sigma}\,.\end{aligned} \tag{6.81}$$

These formulae extend additively to higher tensor degrees. For example,

$$\hat{D}_{\rho_1\rho_2}(\hat{D}x^{\mu})^{\sigma_1\sigma_2} = -\frac{i\hbar}{2}\,dT\,\delta^{\sigma_1}_{\rho_1}\,\delta^{\sigma_2}_{\rho_2}\,N^{\mu} + i\hbar\,dT\,g^{\mu\nu}\,\partial_{\nu}\lrcorner(\Omega^{\sigma_1}{}_{\rho_1}\,\delta^{\sigma_2}_{\rho_2} + \delta^{\sigma_1}_{\rho_1}\,\Omega^{\sigma_2}{}_{\rho_2})\,. \tag{6.82}$$

With these preparations and the first Bianchi identity $R^{\mu}{}_{[\rho\kappa\lambda]} = 0$ we can prove by induction that

$$\begin{aligned}\hat{D}(\hat{D}x^{\mu_1}\wedge\cdots\wedge\hat{D}x^{\mu_r}) &= -\frac{i\hbar}{2}\,dT\,\{\frac{1}{(r-1)!}\,dx^{[\mu_1}\cdots dx^{\mu_{r-1}}\,N^{\mu_r]} \\ &\quad -\frac{1}{(r-2)!}\,dx^{[\mu_1}\cdots dx^{\mu_{r-2}}\,\Omega^{\mu_{r-1}\mu_r]}\}\,.\end{aligned} \tag{6.83}$$

For an ordinary differential $r$-form $\phi := (1/r!)\,dx^{\mu_1}\cdots dx^{\mu_r}\,\phi_{\mu_1\cdots\mu_r}$ let $\hat{\phi}$ denote[25] the right-covariant $r$-form which is obtained from it replacing $dx^{\mu_1}\cdots dx^{\mu_r}$ by $\hat{D}x^{\mu_1}\wedge\ldots\wedge\hat{D}x^{\mu_r}$.

[25] This notation is not entirely consistent with our previous use of the ˆ symbol.



We are now able to calculate

$$\begin{aligned}\hat{d}\hat{\phi} = \hat{D}\hat{\phi} &= \widehat{d\phi} + \frac{i\hbar}{2}\,dT\,dx^{\mu_1}\cdots dx^{\mu_r}\,[\frac{1}{r!}\nabla^2\phi_{\mu_1\cdots\mu_r} - \frac{1}{(r-1)!}\phi_{\mu_1\cdots\mu_{r-1}\nu}\,R^{\nu}{}_{\mu_r} \\ &\quad + \frac{1}{2\,(r-2)!}\,\phi_{\mu_1\cdots\mu_{r-2}\rho\sigma}\,R^{\rho\sigma}{}_{\mu_{r-1}\mu_r}]\end{aligned} \tag{6.84}$$

which is nothing else but

$$\hat{d}\hat{\phi} = \widehat{d\phi} + \frac{i\hbar}{2}\,dT\,\Box\phi \tag{6.85}$$

involving the Laplace-Beltrami operator. For example, in four space-time dimensions the generalized differential form

$$\hat{\epsilon} := \frac{1}{4!}\,\hat{D}x^{\mu}\wedge\hat{D}x^{\nu}\wedge\hat{D}x^{\rho}\wedge\hat{D}x^{\sigma}\,\epsilon_{\mu\nu\rho\sigma} \tag{6.86}$$

associated with the classical volume 4-form satisfies

$$\hat{d}\,\hat{\epsilon} = 0\,. \tag{6.87}$$

The corresponding classical equation $d\epsilon = 0$ is a trivial identity since there are no 5-forms in four dimensions. But here we do have nonvanishing 5-forms and (6.87) is therefore not obvious.

# 7 Harmonic Maps

To understand the following it is useful to take a closer look at the structure of (6.1). In particular, it is important to keep in mind that the product between elements of $\mathcal{A}$ and generalized differentials depends on the metric $g$ as a consequence of this equation. To stress this let us put an index $g$ at the commutator and also express the r.h.s. in a suitable way:

$$[x^{\mu}, \hat{d}x^{\nu}]_g = i\hbar\,dT\,C_g(\hat{d}x^{\mu}, \hat{d}x^{\nu})\,. \tag{7.1}$$

Here $C_g$ is the $\mathcal{A}$-bilinear map $\bigwedge^1(\mathcal{A})\otimes\bigwedge^1(\mathcal{A})\to\mathcal{A}$ given by

$$C_g(\hat{d}f, \hat{d}h) = g^{-1}(\pi\hat{d}f, \pi\hat{d}h) \tag{7.2}$$

where the inverse metric $g^{-1}$ is regarded as a map $\bigwedge^1_{cl}(\mathcal{A})\otimes\bigwedge^1_{cl}(\mathcal{A})\to\mathcal{A}$ and $\pi$ stands for the projection $\bigwedge^1(\mathcal{A})\to\bigwedge^1_{cl}(\mathcal{A})$.

Now let $(\mathcal{N}, \bar{g})$ and $(\mathcal{M}, g)$ be two smooth differential manifolds with metrics $\bar{g}$ and $g$, respectively, such that there is an isometric embedding $\Phi : \mathcal{N}\to\mathcal{M}$ of $\mathcal{N}$ in $\mathcal{M}$. $\Phi$

induces a homomorphism $\Phi^* : \mathcal{A}_\mathcal{M} \to \mathcal{A}_\mathcal{N}$ of the algebras of smooth (complex-valued) functions on the two manifolds, given by $f \mapsto f \circ \Phi$. This map extends to a homomorphism $\Phi^* : \wedge(\mathcal{A}_\mathcal{M}) \to \wedge(\mathcal{A}_\mathcal{N})$ satisfying $\Phi^* \hat{d}_\mathcal{M} = \hat{d}_\mathcal{N} \Phi^*$. [26] Since $\Phi$ is an isometry we have

$$\bar{g} = \Phi^* g \tag{7.3}$$

or in components

$$\bar{g}_{ij} = g_{\mu\nu} (\partial_i x^\mu)(\partial_j x^\nu) \tag{7.4}$$

where $\partial_i = \partial/\partial y^i$ with coordinate functions $y^i$ on $\mathcal{N}$. We have written $\partial_i x^\mu$ as a short notation for $\partial_i(\Phi^* x^\mu) = \partial_i(x^\mu \circ \Phi)$.

Taking the pull back of (7.1) and setting $\Phi^* dT = dS$, we find

$$\begin{aligned} [y^i, \hat{d}y^j]_g (\partial_i x^\mu)(\partial_j x^\nu) &= [\Phi^* x^\mu, \hat{d}\Phi^* x^\nu]_{\Phi^* g} \\ &= \Phi^*([x^\mu, \hat{d}x^\nu]_g) \\ &= i\hbar\, \Phi^*(dT)\, \Phi^* C_g(\hat{d}x^\mu, \hat{d}x^\nu) \\ &= i\hbar\, dS\, C_{\Phi^* g}(\Phi^* \hat{d}x^\mu, \Phi^* \hat{d}x^\nu) \\ &= i\hbar\, dS\, C_g(\hat{d}y^i, \hat{d}y^j)(\partial_i x^\mu)(\partial_j x^\nu) \end{aligned} \tag{7.5}$$

from which we conclude that

$$[y^i, \hat{d}y^j]_g = i\hbar\, dS\, C_g(\hat{d}y^i, \hat{d}y^j)\,. \tag{7.6}$$

In our previous notation this takes the form

$$[y^i, \hat{d}y^j] = i\hbar\, dS\, \bar{g}^{ij}\,. \tag{7.7}$$

A simple calculation yields

$$\Phi^* \hat{D}x^\mu = \frac{i\hbar}{2}\, dS\, \bar{g}^{ij} (\bar{\nabla}_i \partial_j x^\mu + \Gamma^\mu{}_{\nu\rho} (\partial_i x^\nu)(\partial_j x^\rho)) + \hat{D}y^i (\partial_i x^\mu) \tag{7.8}$$

where $\bar{\nabla}_i$ is the covariant derivative associated with the metric $\bar{g}^{ij}$. The equation

$$\bar{g}^{ij} (\bar{\nabla}_i \partial_j x^\mu + \Gamma^\mu{}_{\nu\rho} (\partial_i x^\nu)(\partial_j x^\rho)) = 0 \tag{7.9}$$

characterizes a harmonic map [31]. For example, in the one dimensional case we may choose the coordinate $y$ to be an affine parameter. Then $\bar{\nabla} = d/dy$ and we recover the geodesic equation

$$\frac{d^2 x^\mu}{dy^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{dy} \frac{dx^\rho}{dy} = 0\,. \tag{7.10}$$

In the two dimensional case we obtain the equations of the bosonic string.

[26] We will omit the indices of $\hat{d}$ in the following.



# 8 Conclusions

An analysis of the notion of time in physics led us to the idea of a universal dynamical structure from which the field equations for the various physical fields could be derived. The universality of such a dynamics required it to be encoded at a very basic level which is common to the dynamical description of all fields. This is the differential structure. In this spirit we studied a "dynamical deformation" of the ordinary (de Rham) differential calculus on a manifold, respectively on the corresponding algebra $\mathcal{A}$ (of functions on the manifold). It involves an "evolution parameter" $T$ which turned out to be closely related to the independent time parameter of "proper time formulations" of relativistic theories. Let us summarize to which amount we succeeded to reach our goal.

For a scalar field (uncharged, without gravity) we have

$$\hat{d}\phi = dT\, \frac{i\hbar}{2}\, \Box\phi + \hat{d}x^\mu\, \partial_\mu \phi\,. \tag{8.1}$$

Our stationarity principle demands the coefficient of $dT$ to vanish which leads to the wave equation (massless Klein-Gordon equation) for the field $\phi$.

In case of a (multicomponent) charged field $\psi$, we need to consider instead of $\hat{d}\psi$ a *covariant* differential. This requires the introduction of a generalized connection $\hat{A}$. Then

$$\hat{\mathcal{D}}\psi = dT\, \frac{i\hbar}{2} (\nabla^2 \psi + M\psi) + \hat{\mathcal{D}}x^\mu\, \nabla_\mu \psi\,. \tag{8.2}$$

Here $M$ is an unspecified tensorial part of the generalized connection. A special choice for $M$ would be an ordinary mass term. Our stationarity principle leads to the gauge covariant Klein-Gordon equation

$$\nabla^2 \psi + M\psi = 0\,. \tag{8.3}$$

For the connection $\hat{A}$ itself, we should consider the covariant extension of $\hat{d}\hat{A}$ which is the field strength $\hat{F}$. We obtained

$$\hat{F} = dT\, \frac{i\hbar}{2} (\mathcal{D}^* F - \mathcal{D}M) + \hat{\mathcal{F}} \tag{8.4}$$

and our stationarity condition leads to the Yang-Mills equation

$$\mathcal{D}^* F = \mathcal{D}M \tag{8.5}$$

where $\mathcal{D}M$ may be regarded as a source.

Gravity is a *universal* interaction in the sense that it affects *all* matter fields and cannot be shielded away. Therefore, it already has to appear on the fundamental dynamical level,

i.e. in the differential calculus. Whereas in case of the fields discussed above this just leads to generally covariant equations, we should not expect to obtain the gravity field equations in precisely the same way as for the other fields. Indeed, covariantizing $\hat{d}g^{\mu\nu}$ simply leads to a (generalized) metric compatibility condition. We found that the Einstein equations (together with the condition of vanishing torsion) can be expressed in a very simple way, namely as the vanishing of a generalized torsion 2-form (see (6.50)). If we regard the (right-) covariant differentials $\hat{D}x^\mu$ (which coincide with $\hat{d}x^\mu$ in the harmonic gauge) as fundamental fields, then the same line of arguments applied to the other fields leads to the Einstein equations.

With an ordinary differential $r$-form $\phi$ we associated a (right-) covariant $r$-form $\hat{\phi}$ (see section 6.5) and found

$$\hat{d}\hat{\phi} = dT\,\frac{i\hbar}{2}\,\Box\phi + \widehat{d\phi} \tag{8.6}$$

where $\Box$ is the Laplace-Beltrami operator for the metric $g$. The stationarity principle leads to the equation $\Box\phi = 0$. This generalizes our result for the scalar field.

The common origin of the kinetic terms in all these second order field equations is the "dynamical deformation" of the differential calculus.

So far everything comes out nicely and is in accordance with our general ideas. One problem which remains is how to relate the tensor $M_{\mu\nu}$ – which appears as an unspecified part of the generalized linear connection – to the energy momentum tensor of matter fields in a natural way. Furthermore, in our framework there is no natural place for higher order field equations (which are known to exhibit instabilities) in the sense that such equations can only be obtained by application of the stationarity principle to a quantity which already contains the derivative of a (fundamental) field and is in this sense a derived quantity (like the curvature 2-form (6.61), for example). But we also do not know yet whether it is possible to obtain the (first order) Dirac equation. These are the problems which should be addressed next.

In particular concerning interpretative issues of our formalism the restriction to quantities not explicitly depending on $T$ should be relaxed. The origin of the "mass term" $M$ in the generalized gauge theory and of $M_{\mu\nu}$ (which we would like to relate to the energy-momentum tensor of matter) could then be understood by analogy with the origin of mass in the gauge invariant Stueckelberg equation (2.14) via breaking of $T$-gauge invariance (see section 2). In this extended formalism we then should look for a kind of "universal Higgs mechanism" which will determine the particular "mass terms" and fix the relation to one another.

In applications to physics the use of noncommutative geometry is sometimes advocated



as a possible way to avoid ultraviolet divergences in quantum field theories (cf. [7, 8], for example). The root of these divergencies is attributed to the notion of a point in space-time and the assumption that a particle can be localized with arbitrary precision. If the coordinates $x^\mu$ are replaced by noncommuting operators satisfying appropriate commutation relations, they cannot be simultaneously measured and there is no longer the notion of a coincidence at a single point (see in particular [8] for a discussion of a concrete model). From this point of view the commutative algebra we started with should be replaced by some noncommutative algebra. However, ultraviolet divergencies appear in *integrated* expressions and therefore already the introduction of a weaker kind of noncommutativity, namely a noncommutativity between functions and differentials, (accompanied by a convenient definition of the integral) might do a good job.

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