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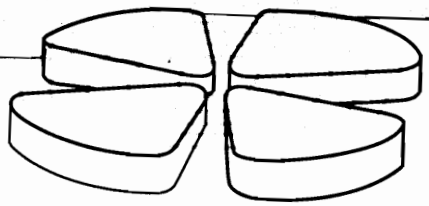

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Symmetries in the structure of nuclei

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Symmetries in the structure of nuclei

P Van Isacker

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Abstract. The use of spectrum generating algebras in the description of the nuclear many-body system is reviewed. General notions of symmetry and dynamical symmetry in quantum mechanics are introduced with the help of simple examples. It is then indicated how techniques based on symmetry considerations can be used to find the analytical solutions for the problem of an aggregate of interacting particles (bosons and/or fermions). Some older ideas due to Wigner, Racah and Elliott are succinctly summarised to put more recent advances in a proper perspective. It is then shown that similar techniques are used in a model of the nucleus in terms of interacting bosons due to Arima and Iachello. Subsequent extensions of this model to odd-mass nuclei lead to the consideration of mixed systems of bosons and fermions and, most notably, to supersymmetry.

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1. Introduction

The attractive feature of the nuclear many-body problem is that it is too hard to be solved exactly; and, even if exact solutions were available, they would be too complex to be of value to the average nuclear physicist. The challenge—and the charm—of nuclear physics is that it requires judicious approximations, good enough to leave intact the essential features of the solution (which may change with the particular application under consideration) yet leading to an amenable problem. The nuclear models considered in this review all share this property: they are simple enough to be analytically solvable yet they retain enough realism to bring forward at least one essential feature of nuclear structure. They also share another property, namely the use group theoretical methods.

The title of this review promises more than can be delivered: since the birth of quantum mechanics the idea of symmetry has acquired such a central rôle in physics that it is clearly beyond the scope of any reasonable review to summarise its impact on a particular subfield such as nuclear structure physics. The scope of this review is rather to reflect on the use of symmetry techniques in the building of nuclear models and, specifically, the use of *dynamical symmetries* and *spectrum generating algebras.* In general terms, the concept of a dynamical symmetry of a many-body system (e.g., the nucleus) is based on the assumption of a 'primary' symmetry with an associated dynamical algebra which has the property that the hamiltonian of the system can be expressed in terms of its generators. The hamiltonian reduces this primary symmetry to a lower, true symmetry [the angular momentum algebra O(3) in the nuclear case] and generates in this way the energy spectrum. For this reason the dynamical algebra is also referred to as the spectrum generating algebra. An important aspect of this process of reduction to the true symmetry (or of spectrum generation) is that it sometimes can be achieved analytically, that is, while preserving the solvable character of the many-body problem.

A comprehensive overview of all applications of dynamical symmetries in nuclear physics still remains a gargantuan task and no attempt at that is made here either. This review, however, aims to be comprehensive in time: early applications of symmetries that go back to the very beginnings of nuclear physics are summarised and presented from a modern and coherent perspective. These early ideas due to Wigner, Racah and Elliott can be considered as precursors to the more modern models of spectrum

generating algebras such as the interacting boson model of Arima and Iachello. Of the early developments, only the most important are summarised. Throughout the text many examples are given which show that, even if these ideas are old, they still continue to inspire experiments at the forefront of today's research in nuclear physics.

2. Symmetry in quantum mechanics

2.1. Symmetry

A hamiltonian $\hat{H}$, invariant under a set of infinitesimal transformations $\{\hat{g}_i\}$ which together form a Lie algebra G, that is

$$[\hat{H}, \hat{g}_i] = 0, \qquad \text{for } \hat{g}_i \in G, \tag{2.1}$$

is said to have a *symmetry* G or, alternatively, to be *invariant under* G. An ingredient borrowed from group theory concerns the construction of operators like $\hat{H}$ in (2.1) that commute with all elements of G. Such operators are called *Casimir operators* [1] and are denoted here as $\hat{C}_n[G]$, the index n referring to the order of the operator in $\hat{g}_i$.

2.2. Degeneracy and state labelling

A well-known consequence of a symmetry is the occurrence of degeneracies in the eigenspectrum of $\hat{H}$. Given an eigenstate $|\alpha\rangle$ of $\hat{H}$ with energy E, the condition (2.1) implies that the states $\hat{g}_i|\alpha\rangle$ have the same energy. An arbitrary eigenstate of $\hat{H}$ can thus be written as $|\Gamma\gamma\rangle$, where the first quantum number Γ is different for states with different energies and the second quantum number γ is needed to label degenerate eigenstates. The eigenvalues of a hamiltonian that satisfies (2.1) depend on Γ only,

$$\hat{H}|\Gamma\gamma\rangle = E(\Gamma)|\Gamma\gamma\rangle, \tag{2.2}$$

and, furthermore, the transformations $\hat{g}_i$ do not admix states with different Γ:

$$\hat{g}_i|\Gamma\gamma\rangle = \sum_{\gamma'} a^{\Gamma}_{\gamma'\gamma}(i)|\Gamma\gamma'\rangle. \tag{2.3}$$

2.3. Dynamical symmetry breaking

The concept of a *dynamical symmetry* can now be introduced, for which (at least) two algebras G_1 and G_2 with $G_1 \supset G_2$ are needed. The eigenstates of a hamiltonian $\hat{H}$ with symmetry G_1 are labelled as $|\Gamma_1\gamma_1\rangle$. But, since $G_1 \supset G_2$, a hamiltonian with G_1 symmetry necessarily must also have a symmetry G_2 and, consequently, its eigenstates can also be labelled as $|\Gamma_2\gamma_2\rangle$. Combination of the two properties leads to

the eigenequation †

$$\hat{H}|\Gamma_1\Gamma_2\gamma_2\rangle = E(\Gamma_1)|\Gamma_1\Gamma_2\gamma_2\rangle, \tag{2.4}$$

where the rôle of γ_1 is played by $\Gamma_2\gamma_2$. Hence the eigenvalues depend only on Γ_1. If the condition of G_1 symmetry is too strong, a *possible* breaking of the G_1 symmetry can be imposed via the hamiltonian

$$\hat{H}' = \kappa_1\hat{C}_{n_1}[G_1] + \kappa_2\hat{C}_{n_2}[G_2], \tag{2.5}$$

which consists of a combination of Casimir operators of G_1 *and* G_2. The symmetry properties of the hamiltonian $\hat{H}'$ are now as follows. Since $[\hat{H}', \hat{g}_i] = 0$ for $\hat{g}_i \in G_2$, $\hat{H}'$ is invariant under G_2. The hamiltonian $\hat{H}'$ generally does not, however, commute with all elements of G_1 and for this reason the G_1 symmetry is broken, the extent of the symmetry breaking depending on the ratio κ_2/κ_1. Furthermore, since $\hat{H}'$ is a combination of Casimir operators of G_1 and G_2, its eigenvalues can be obtained in closed form:

$$\Big(\kappa_1\hat{C}_{n_1}[G_1]+\kappa_2\hat{C}_{n_2}[G_2]\Big)|\Gamma_1\Gamma_2\gamma_2\rangle = \Big(\kappa_1 E_{n_1}(\Gamma_1)+\kappa_2 E_{n_2}(\Gamma_2)\Big)|\Gamma_1\Gamma_2\gamma_2\rangle. \tag{2.6}$$

The conclusion is thus that, although $\hat{H}'$ is not invariant under G_1, its eigenstates are the same as those of $\hat{H}$ in (2.4). The hamiltonian $\hat{H}'$ is said to have G_1 as a dynamical symmetry. The essential feature is that, although the eigenvalues of $\hat{H}'$ depend on Γ_1 *and* Γ_2 (and hence G_1 is not a symmetry), the eigenstates do not change during the breaking of the G_1 symmetry: the dynamical symmetry breaking splits but does not admix the eigenstates.

Example: The isobaric multiplet mass equation

Many concrete examples exist in physics of the abstract idea of dynamical symmetry. Perhaps the best known in nuclear physics concerns isospin symmetry and its breaking by the Coulomb interaction.

After the discovery of the neutron by Chadwick, Heisenberg realised that the mathematical apparatus of the Pauli spin matrices could be applied to the labelling of the two nucleonic charge states, the proton and the neutron [2]. Although he thus lead the foundation of an important development in physics—the use of symmetry transformations in abstract spaces—Heisenberg did not realise at that time the connection between his formalism and the requirement of charge independence of nuclear forces. This point was emphasised by Wigner some years later [3] who defined

† In (2.4) the representation $[\Gamma_2]$ is assumed to occur only once in $[\Gamma_1]$, otherwise an additional quantum number α would have been needed to uniquely label the states as $|\Gamma_1\alpha\Gamma_2\gamma_2\rangle$. For the purpose of illustrating the concept of dynamical symmetry, however, this technical complication can be ignored.

isospin for complex nuclei as we know it today and who also coined the name of 'isotopic spin'.

If, in first approximation, the Coulomb interaction between the protons is neglected and, furthermore, if it is assumed that the strong interaction does not distinguish between protons and neutrons, the resulting nuclear hamiltonian $\hat{H}$ is isospin invariant. Explicitly, invariance under the isospin algebra $\mathrm{SU}(2) \equiv \{\hat{T}_z, \hat{T}_\pm\}$ follows from

$$[\hat{H}, \hat{T}_z] = [\hat{H}, \hat{T}_\pm] = 0. \tag{2.7}$$

The $\hat{T}_\mu$ operators are sums over single-particle operators, $\hat{T}_\mu = \sum_k \hat{t}_\mu(k)$, where $\hat{t}_z(k)$ gives $-\frac{1}{2}$ for a proton and $+\frac{1}{2}$ for a neutron while $\hat{t}_\pm(k)$ transforms a proton into a neutron or *vice versa*. The conditions (2.7) give rise to many-particle eigenstates of $\hat{H}$ with good isospin symmetry classified as $|\alpha T M_T\rangle$ where T is the total isospin of the nucleus obtained from the coupling of the individual isospins $\frac{1}{2}$ of all nucleons, M_T is its projection on the z axis in isospin space, $M_T = \frac{1}{2}(N - Z)$, and α denotes any additional quantum number. If isospin were a true symmetry, all states $|\alpha T M_T\rangle$ with $M_T = -T, -T+1, \ldots, +T$ (and with identical other quantum numbers α) would be degenerate in energy; for example, proton and neutron would have the same mass.

The Coulomb interaction between the protons destroys the equivalence between the nucleons and hence breaks isospin symmetry. It can be shown (see chapter 10 of [4]) that the bulk effect of the Coulomb interaction can be represented as

$$\hat{H}' = \kappa_0 + \kappa_1 \hat{T}_z + \kappa_2 \hat{T}_z^2, \tag{2.8}$$

for some particular coefficients κ_0, κ_1 and κ_2. In the notation $G_1 \supset G_2$ introduced in subsection (2.3), this can be viewed as a dynamical symmetry breaking of the type $\mathrm{SU}(2) \supset \mathrm{O}(2) \equiv \{\hat{T}_z\}$. The hamiltonian $\hat{H}'$ splits but does not admix the eigenstates $|\alpha T M_T\rangle$ with $M_T = -T, -T+1, \ldots, +T$, and has the eigenspectrum

$$\hat{H}'|\alpha T M_T\rangle = (\kappa_0 + \kappa_1 M_T + \kappa_2 M_T^2)|\alpha T M_T\rangle. \tag{2.9}$$

The expression (2.8) is but an approximation to the true Coulomb interaction; it represents the diagonal part of it, with the T-mixing isovector and isotensor parts being neglected. In that approximation isospin remains a good quantum number. The excitation spectra of the different nuclei belonging to the same isospin multiplet (with the same T but different M_T) are identical but their ground states do not have the same binding energy.

The energy formula (2.9) was derived by Wigner [5] who introduced the name of *isobaric multiplet mass equation* (IMME). Early applications of IMME were considered by Wilkinson [6]. Countless experimental examples of nuclear isospin multiplets are known at present [7]. A recently measured one where the idea has been tested to high angular momentum, is shown in figure 1 [8]. The ground-state energies of the two nuclei of the $T = \frac{1}{2}$ isospin doublet ($^{49}_{24}\mathrm{Cr}_{25}$ with $M_T = +\frac{1}{2}$ and $^{49}_{25}\mathrm{Mn}_{24}$ with $M_T = -\frac{1}{2}$) are

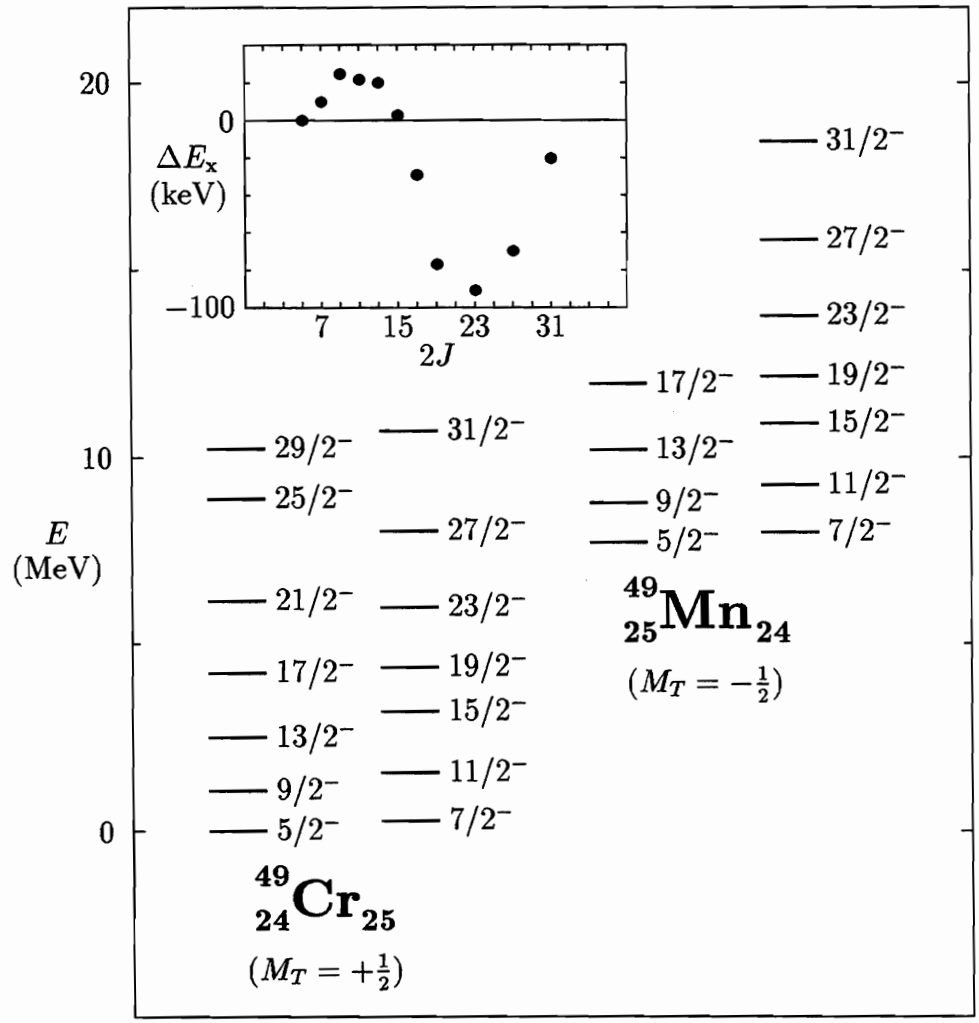

Figure 1. Level schemes of the mirror nuclei ^{49}Cr and ^{49}Mn relative to the ground state of the first nucleus. Levels are labelled by their angular momentum and parity J^π. The inset shows the difference in excitation energy $\Delta E_x \equiv E_x(^{49}\mathrm{Cr}; J) - E_x(^{49}\mathrm{Mn}; J)$ as a function of $2J$.

shifted with respect to each other but the excitation energies of yrast states are indeed very similar as required by isospin (dynamical) symmetry. (The non-observation of the $J^\pi = 21/2^-$, $25/2^-$ and $29/2^-$ states in ^{49}Mn has experimental reasons.) Nevertheless, the spectra are not identical as is clear from the inset in figure 1 where the difference in excitation energy is plotted as a function of the angular momentum J. The deviations from zero signal a break down of the dynamical symmetry approximation (2.8) to the Coulomb interaction and, specifically, reveal subtle differences in alignment properties of the protons and neutrons in the two mirror nuclei [9].

2.4. Selection rules

The most important consequence of a symmetry, which remains valid under the process of a dynamical symmetry breaking, is the existence of conserved quantum numbers. Frequently, these quantum numbers give rise to selection rules in electromagnetic transition or particle transfer processes. The measurement of transition or transfer probabilities is thus a method to establish the goodness of quantum numbers and this in turn indicates to what extent a given (dynamical) symmetry is valid.

The link between symmetries and selection rules can be given a precise quantitative formulation through a generalisation of the Wigner–Eckart theorem. This theorem is well known for the case SU(2) $\supset$ O(2) with associated labels of angular momentum J and its projection M_J. The generalisation involves an arbitrary labelling of the type

$$\begin{array}{ccc} G_1 & \supset & G_2 \\ \downarrow & & \downarrow \\ \Gamma & & \gamma \end{array}, \tag{2.10}$$

where below each algebra the associated label is given. Suppose the calculation is required of a transition or transfer matrix element between an initial state $|\Gamma_i\gamma_i\rangle$ and a final state $|\Gamma_f\gamma_f\rangle$. To find the corresponding matrix element, it is first necessary to determine the tensor character of the operator associated with the transition or transfer which generally is achieved by writing the operator as $\sum_{\Gamma\gamma} a_\gamma^\Gamma \hat{T}_\gamma^{(\Gamma)}$. Each piece $\hat{T}_\gamma^{(\Gamma)}$ can now be dealt with separately by the *generalised Wigner–Eckart theorem* [1] which states that †

$$\langle\Gamma_f\gamma_f|\hat{T}_\gamma^{(\Gamma)}|\Gamma_i\gamma_i\rangle = \langle\Gamma_i\gamma_i\ \Gamma\gamma|\Gamma_f\gamma_f\rangle\langle\Gamma_f\|\hat{T}^{(\Gamma)}\|\Gamma_i\rangle. \tag{2.11}$$

The matrix element can be written as the product of a *generalised coupling coefficient* (denoted as $\langle\cdot\cdot\ \cdot\cdot|\cdot\cdot\rangle$) and a reduced matrix element (written as $\langle\cdot\|\cdot\|\cdot\rangle$). The essential point is that all dependence on the quantum numbers associated with the subalgebra G_2 is contained in the generalised coupling coefficient. The calculation of the latter is a purely algebraic problem that boils down to a matrix diagonalisation. In addition, selection rules follow from (2.11): if $[\Gamma_f]$ is not contained in the product $[\Gamma_i] \times [\Gamma]$, the generalised coupling coefficient is zero and the matrix element vanishes.

Example: E1 transitions in self-conjugate nuclei

A well-known example of the idea of selection rules concerns electric dipole transitions in self-conjugate nuclei [10, 11], that is, nuclei with an equal number of protons and

† Technical complications due to multiplicities are ignored here for simplicity. These can be twofold in this case: i) $[\Gamma_f]$ may occur more than once in the product $[\Gamma_i] \times [\Gamma]$; ii) $[\gamma]$ may be contained more than once in $[\Gamma]$.

neutrons ($Z = N$). The E1 operator is, in lowest order of the long-wave approximation, given by

$$\hat{T}_\mu(\mathrm{E1}) = \sum_{k=1}^{A} e_k r_\mu(k), \tag{2.12}$$

where the sum runs over all nucleons in the nucleus. Since the charge e_k of the kth nucleon is e for a proton and zero for a neutron, the E1 operator (2.12) can be rewritten as

$$\hat{T}_\mu(\mathrm{E1}) = \frac{e}{2}\sum_{k=1}^{A}(1-\tau_z(k))r_\mu(k) = \frac{e}{2}\left[R_\mu - \sum_{k=1}^{A}\tau_z(k)r_\mu(k)\right], \tag{2.13}$$

where $\tau_z(k) \equiv 2t_z(k)$ gives -1 for a proton and $+1$ for a neutron. The first term R_μ in (2.13) is the centre-of-mass coordinate of the total nucleus and does not contribute to an internal E1 transition. (It is responsible for Thomson scattering off a nucleus.) The conclusion is that the electric dipole operator is, in lowest order of the long-wave approximation, of pure isovector character $\hat{T}^{(T=1)}_{M_T=0}$. Direct application of the Wigner–Eckart theorem (2.11) in isospin space gives

$$\langle \alpha_\mathrm{f} T_\mathrm{f} M_{T_\mathrm{f}} | \hat{T}^{(1)}_0 | \alpha_\mathrm{i} T_\mathrm{i} M_{T_\mathrm{i}} \rangle = \langle T_\mathrm{i} M_{T_\mathrm{i}}\ 10 | T_\mathrm{f} M_{T_\mathrm{f}} \rangle \langle \alpha_\mathrm{f} T_\mathrm{f} \| \hat{T}^{(1)} \| \alpha_\mathrm{i} T_\mathrm{i} \rangle. \tag{2.14}$$

The coupling coefficient here is the usual Clebsch–Gordan coefficient associated with SU(2) $\supset$ O(2). Self-conjugate nuclei have $M_{T_\mathrm{i}} = M_{T_\mathrm{f}} = 0$ and, as a consequence of (2.14), exhibit a simple selection rule: E1 transitions are forbidden between levels with the same isospin $T_\mathrm{i} = T_\mathrm{f} = T$ because of the vanishing Clebsch–Gordan coefficient, $\langle T0\ 10 | T0 \rangle = 0$.

This selection rule has been verified to hold approximately in light self-conjugate nuclei [12] (see also chapter 1 of [13]). Deviations occur because of higher-order terms in the E1 operator but also, and more importantly, because isospin is not an exactly conserved quantum number. Isospin mixing can be estimated in a variety of nuclear models. They all show that the mixing (i.e., the non-dynamical breaking of isospin symmetry) is maximal in $Z = N$ nuclei and rapidly decreases with proton or neutron excess; in addition, the mixing increases with nuclear mass. This is illustrated in figure 2 where a simple estimate [14] of the admixture $P(T = M_T + 1)$ in the ground state of even–even nuclei is plotted as a function of nuclear charge Z for neutron numbers $N = Z$, $Z + 4$ and $Z + 8$. This quantity gives the square of the component with isospin $T = M_T + 1$ in the ground state of a nucleus with isospin projection M_T. Isospin mixing effects, caused mainly by Coulomb, should thus be looked for in heavy $Z = N$ nuclei where they are largest. The spectrum of the heaviest $Z = N$ nucleus studied so far in this respect, ${}^{64}_{32}\mathrm{Ge}_{32}$, is shown in figure 3. The crucial transition is the E1 from 5^- to 4^+ (indicated by the down arrow) which should be strictly forbidden if the isospin (dynamical) symmetry were exact. A small $B(\mathrm{E1}; 5^- \to 4^+)$ value is measured

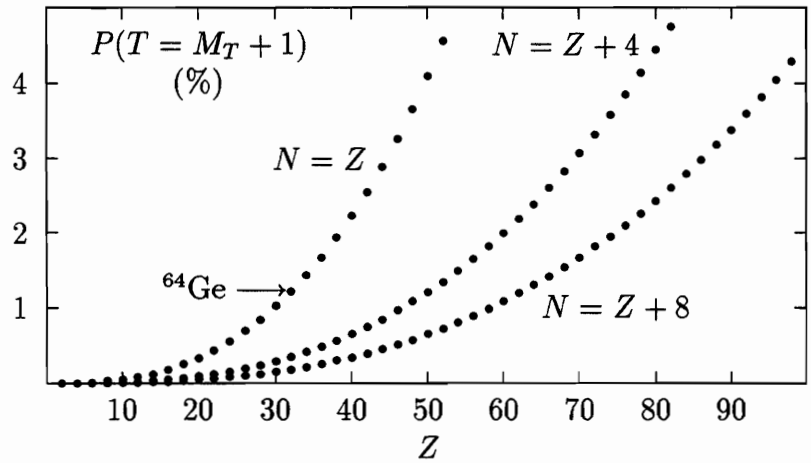

Figure 2. Calculated isospin mixing $P(T = M_T + 1)$ in the ground state of even–even nuclei as a function of nuclear charge Z for neutron numbers $N = Z$, $Z + 4$ and $Z + 8$.

nevertheless [15] and this is explained through the mixing with higher-lying 5^- and 4^+ levels in ^{64}Ge with $T = 1$. Although an estimate of the isospin mixing can be made ($P \approx 1.2\%$ [16], in agreement with figure 2) the analysis is hampered by experimental uncertainties: a competing E3 transition from 5^- to 4^+, the fact that the $T = 1$ states in ^{64}Ge have not been measured but rather inferred from their isospin analogue states in ^{64}Ga, etc. A reliable measurement of isospin admixtures as a function of $Z = N$ is still very much a declared goal of the current experimental efforts with radioactive beams.

3. Symmetries of interacting fermion models

The nuclear many-problem consists in finding the solution of the A-body Schrödinger equation

$$\hat{H}\Psi(1, 2, \ldots, A) = E\Psi(1, 2, \ldots, A). \tag{3.1}$$

The hamiltonian $\hat{H}$ can be written as

$$\hat{H} = \sum_{k=1}^{A} \left(\hat{T}(k) + \hat{U}(k)\right) + \sum_{k<l=1}^{A} \hat{V}(k, l), \tag{3.2}$$

where $\hat{T}(k)$ is the kinetic energy of nucleon k, $\hat{U}(k)$ is the mean field felt by the kth nucleon and $\hat{V}(k, l)$ is the residual interaction between nucleons k and l.

A simple but reasonably adequate form of the mean-field potential consists of a three-dimensional harmonic oscillator corrected with a spin–orbit and an orbit–orbit term,

$$\hat{U}(k) = \tfrac{1}{2}m\omega^2 r_k^2 + v_{ls}\bar{l}_k \cdot \bar{s}_k + v_{ll}\bar{l}_k \cdot \bar{l}_k, \tag{3.3}$$

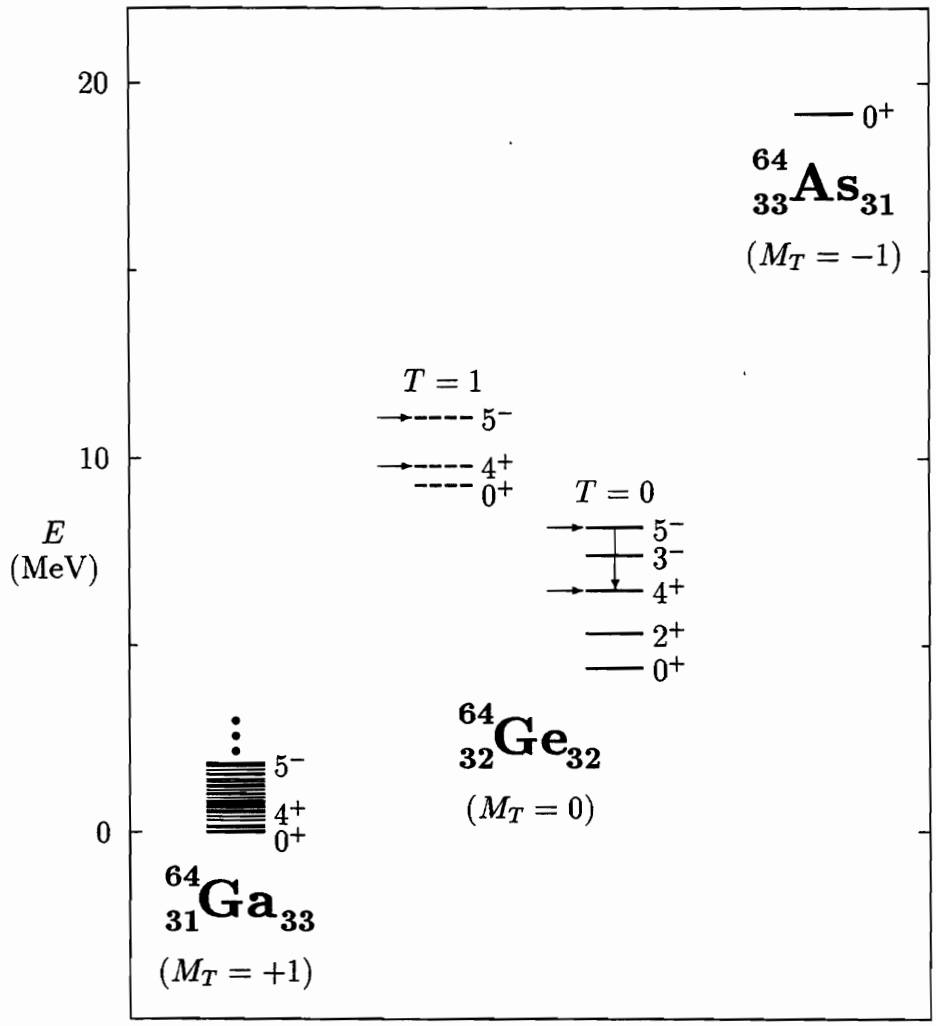

Figure 3. Level schemes of the nuclei in the $A = 64$ isospin triplet ^{64}Ga, ^{64}Ge and ^{64}As relative to the ground state of the first nucleus. Levels are labelled by their angular momentum and parity J^{π}. The observed $5^- \rightarrow 4^+$ E1 transition between $T = 0$ states in ^{64}Ge is explained through mixing with the $T = 1$ states, indicated by the arrows. The levels in broken lines are inferred from the isospin analogue levels in ^{64}Ga.

where m is the nucleon mass and ω the angular frequency, assumed to be identical for protons and neutrons. An attractive spin–orbit term ($v_{ls} < 0$) is required to achieve shell closures at the observed magic numbers. An orbit–orbit term with $v_{ll} < 0$ pushes down single-particle states with high orbital angular momentum l which is one of the main effects if the schematic harmonic oscillator mean field is substituted by the more realistic Woods–Saxon potential. The residual interaction, on the other hand, can approximately

be written as pairing-plus-quadrupole:

$$\hat{V}(k,l) = \hat{V}_{\text{pairing}}(k,l) + \hat{V}_{\text{quadrupole}}(k,l). \tag{3.4}$$

For particular values or limits of the parameters in the mean field and the residual interaction, the eigenproblem (3.1) can be solved analytically. Three situations can be envisaged:

(i) *No residual interaction.* If $\hat{V}(k,l) = 0$, the solution of (3.2) reduces to a Slater determinant built from harmonic oscillator eigenstates.

(ii) *Pairing interaction in jj coupling.* In case of a strong spin–orbit coupling (v_{ls} large) a jj-coupling scheme is obtained. If, in addition, the residual interaction has a pairing character ($\langle\hat{V}_{\text{pairing}}\rangle \gg \langle\hat{V}_{\text{quadrupole}}\rangle$) Racah's SU(2) model of pairing results.

(iii) *Quadrupole interaction in LS coupling.* In case of a weak spin–orbit coupling (v_{ls} small) an LS or Russel–Saunders coupling is obtained. If, in addition, the residual interaction has a quadrupole character ($\langle\hat{V}_{\text{quadrupole}}\rangle \gg \langle\hat{V}_{\text{pairing}}\rangle$) Elliott's SU(3) model of rotation results.

The situation can be represented schematically as in figure 4: the vertices correspond

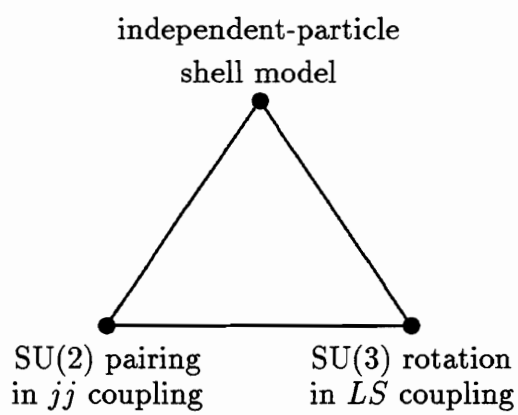

Figure 4. Schematic representation of the shell-model parameter space with its three analytically solvable vertices.

to the three limits that can be solved analytically while an arbitrary point of the triangle corresponds to a general hamiltonian of the schematic form (3.2)–(3.4).

A hamiltonian of the top vertex yields uncorrelated Hartree–Fock type wavefunctions. This limit is reached if the single-particle energies (ϵ_j in jj coupling or ϵ_l in LS coupling) are large in comparison with a 'typical' matrix element K of the residual interaction. The transition between jj and LS coupling is controlled by the ratio v_{ls}/K, the strength of the spin–orbit coupling over the strength of the residual interaction. The

search for the correct coupling scheme was one of the central questions in the early days of the nuclear shell and soon it transpired that v_{ls}/K is the crucial parameter in this respect [17, 18]. The transition from SU(2) pairing to SU(3) rotation not only involves the ratio v_{ls}/K but in addition requires a change of the residual interaction from pairing to quadrupole. The character of the residual interaction depends on the numbers of protons and neutrons in the valence shell: it is, essentially, a pairing interaction between like nucleons while the quadrupole component becomes dominant if sufficient numbers of both protons and neutrons are active (i.e., are valence nucleons).

The uncorrelated limit is not further discussed here. A brief *exposé* of the other basic symmetries of the shell model now follows, preceded by a discussion of Wigner's supermultiplet model which is at the basis of the nuclear LS coupling scheme.

3.1. Wigner's spin–isopin or SU(4) symmetry

Some five years after Heisenberg's suggestion [2] of isospin symmetry in nuclei, Wigner [3] (and, independently of him, Hund [19]) proposed a beautiful extension of this idea by assuming nuclear forces to be invariant under rotations in *spin* as well as *isospin* space. This invariance is expressed by the following commutation relations:

$$[\hat{H}, \sum_{k=1}^{A} s_\mu(k)] = [\hat{H}, \sum_{k=1}^{A} t_\mu(k)] = [\hat{H}, \sum_{k=1}^{A} s_\mu(k) t_\nu(k)] = 0, \tag{3.5}$$

where $s_\mu(k)$ and $t_\mu(k)$ are the spin and isospin components, respectively, of nucleon k. The 15 operators $\sum_k s_\mu(k)$, $\sum_k t_\mu(k)$ and $\sum_k s_\mu(k) t_\nu(k)$ generate the Lie algebra SU(4). According to the discussion in section 2, any hamiltonian satisfying the conditions (3.5) has SU(4) symmetry. Because of the first two conditions in (3.5), the eigenstates of any such hamiltonian must also have total spin S and total isospin T as good quantum numbers. Furthermore, the additional requirement of rotational invariance,

$$[\hat{H}, \sum_{k=1}^{A} j_\mu(k)] \equiv [\hat{H}, \sum_{k=1}^{A} (l_\mu(k) + s_\mu(k))] = 0, \tag{3.6}$$

then also implies the conservation of total orbital angular momentum L since combination of (3.5) and (3.6) yields

$$[\hat{H}, \sum_{k=1}^{A} l_\mu(k)] = 0. \tag{3.7}$$

As a consequence of the invariances (3.5), the A-particle eigenstates of $\hat{H}$ can be labelled as follows:

$$\Psi(1, 2, \ldots, A) = \Psi_{\alpha L M_L}(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_A) \Psi_{(\lambda\mu\nu) S M_S T M_T}(1, 2, \ldots, A), \tag{3.8}$$

where the coupling of L and S to total angular momentum J and projection M_J is not shown. The first part on the right-hand side of (3.8) depends on the spatial coordinates

of the A nucleons while the second part involves their spin–isospin coordinates. Besides the labels L, S and T associated with $O_L(3)$, $SU_S(2)$ and $SU_T(2)$†, and the magnetic sublabels associated with their respective O(2) subalgebras, there appear in (3.8) also the labels $(\lambda\mu\nu)$ connected with SU(4). They can be defined mathematically in terms of U(4) Young tableaux [20, 21] with lengths f_1, f_2, f_3 and f_4 which are related to $(\lambda\mu\nu)$ through $\lambda = f_1 - f_2$, $\mu = f_2 - f_3$ and $\nu = f_3 - f_4$. The complete wavefunction Ψ is totally antisymmetric; the separate wavefunctions $\Psi_{\alpha LM_L}$ and $\Psi_{(\lambda\mu\nu)SM_STM_T}$, however, are not but must have conjugate symmetry in the sense that rows and columns are interchanged in their respective Young tableaux.

To further clarify the meaning the SU(4) labels, it is instructive to analyse the two-particle case which is summarised in table 1 for the *sd* shell. Total antisymmetry of the

Table 1. Classification of one and two particle(s) in the *sd* shell.

particle number	spatial symmetry	L	spin–isospin symmetry	$(\lambda\mu\nu)$	(S,T)
1	□	0, 2	□	(100)	$(\frac{1}{2},\frac{1}{2})$
2	□□ (S)	$0^2, 2^2, 4$	□ □ (A)	(010)	(0,1) (1,0)
	□ □ (A)	1, 2, 3	□□ (S)	(200)	(0,0) (1,1)

Note: S stands for symmetric, A for antisymmetric.

wavefunction ensures in this case that the spatial part is symmetric and the spin–isospin part antisymmetric or that the spatial part is antisymmetric and the spin–isospin part symmetric, with the corresponding values of L, S and T as given in the table. This can be generalised to an arbitrary number of particles and the result emerges that the SU(4) quantum numbers $(\lambda\mu\nu)$ specify the way in which the overall antisymmetry is distributed over the spatial and spin–isospin parts of the wavefunction.

The relevance of the classification (3.8) is its connection with the short-range attractive nature of the residual interaction as a result of which states with spatial symmetry are favoured energetically. To see this point, one may return once more to the example of two particles and consider an extreme form of a short-range interaction, namely a delta interaction. It can be shown that (see chapter 11 of [22])

$$\langle \alpha LM_L; (200)SM_STM_T|\delta(\bar{r}_1 - \bar{r}_2)|\alpha LM_L; (200)SM_STM_T\rangle = 0, \qquad (3.9)$$

that is, the interaction matrix element vanishes identically for $(\lambda\mu\nu) = (200)$. Intuitively this must be so since a spatially antisymmetric two-particle state has zero probability of

† Since L is always integer while S and T can be both integer and half-integer, the associated algebras are denoted by O(3) in the former and by SU(2) in the latter case.

having $\bar{r}_1 = \bar{r}_2$. In contrast, the matrix element is attractive in the spatially symmetric case $(\lambda\mu\nu) = (010)$. Again, this result can be generalised to many particles, leading to the conclusion that the energy of a state depends strongly on $(\lambda\mu\nu)$. This statement can be quantified (see chapter 29 of [22]) by constructing a Majorana space exchange operator [related to the Casimir operator of SU(4)] that 'measures' the symmetry of the spatial part of the wavefunction.

Wigner's supermultiplet model is a nuclear LS coupling scheme. With the advent of the nuclear shell model the importance of the spin–orbit coupling has become clear and, as a result, the SU(4) model is now largely abandoned. In spite of its limited applicability, Wigner's idea remains important because it demonstrates the connection between the short-range character of the residual interaction and the spatial symmetry of the many-body wavefunction. The break down of SU(4) symmetry is a consequence of the $\bar{l}_k \cdot \bar{s}_k$ term in the nuclear mean field (3.3) which does not satisfy the second and third commutator in (3.5). The spin–orbit term breaks SU(4) symmetry [in a non-dynamical manner in the sense that SU(4) representations are admixed by it] and does so increasingly in heavier nuclei since the energy splitting of the spin doublets $l - \frac{1}{2}$ and $l + \frac{1}{2}$ increases with nuclear mass number A. In addition, SU(4) symmetry is also broken by the Coulomb interaction [which has a non-vanishing first and third commutator in (3.5)]—an effect that also increases with nuclear mass—and it is broken by spin-dependent residual interactions.

At the time of Wigner's original suggestion insufficient data were available to test his SU(4) symmetry hypothesis. Many years later, in 1963, Franzini and Radicati [23] suggested the use of a combination of nuclear masses as a test of the symmetry and showed that the observed values agree rather well the SU(4) predictions for nuclei up to mass $A \approx 110$. It was demonstrated subsequently [24], however, that the particular combination employed in [23] is not very sensitive to SU(4) symmetry breaking and hence does not provide a critical test of the symmetry. More recent analyses regarding Gamow–Teller β decay [25] and nuclear masses [26] clearly indicate the break down of SU(4) symmetry with increasing nuclear mass. This is also confirmed by realistic shell-model calculations [27]. The following example discusses the evidence from Gamow–Teller β decay.

Example: Gamow–Teller β decay along the $Z = N$ line

In lowest-order approximation the operator for Gamow–Teller β decay is given by

$$\hat{T}^{\pm}_{\mu}(\mathrm{GT}) = 2g_{\mathrm{A}} \sum_{k=1}^{A} s_{\mu}(k) t_{\pm 1}(k) = \sqrt{\tfrac{1}{2}}\, g_{\mathrm{A}} \sum_{k=1}^{A} \sigma_{\mu}(k) \tau_{\pm 1}(k), \tag{3.10}$$

where $\pm$ applies to $\beta^{\pm}$ decay, g_{A} is the axial-vector coupling constant ($g_{\mathrm{A}} = 1.267$) and σ_{μ} and τ_{μ} are the Pauli matrices in spin and isospin space. Since $\hat{T}^{\pm}_{\mu}(\mathrm{GT})$ is a generator

of SU(4) it follows that Gamow–Teller β decay cannot take place between two different SU(4) representations. [A generator cannot change representation, see (2.3).] This gives rise to a very characteristic prediction for the Gamow–Teller β^+ decay into $Z = N$ nuclei as a function of nuclear mass number A [25]. It can be shown (see chapter 29 of [22]) that the ground state of an even–even nucleus has SU(4) labels $(\lambda\mu\nu) = (0T0)$ and that those for an odd–odd nucleus are $(\lambda\mu\nu) = (1, T-1, 1)$ except for an odd–odd $Z = N$ nucleus for which they are $(\lambda\mu\nu) = (010)$. With reference to figure 5, the Gamow–Teller

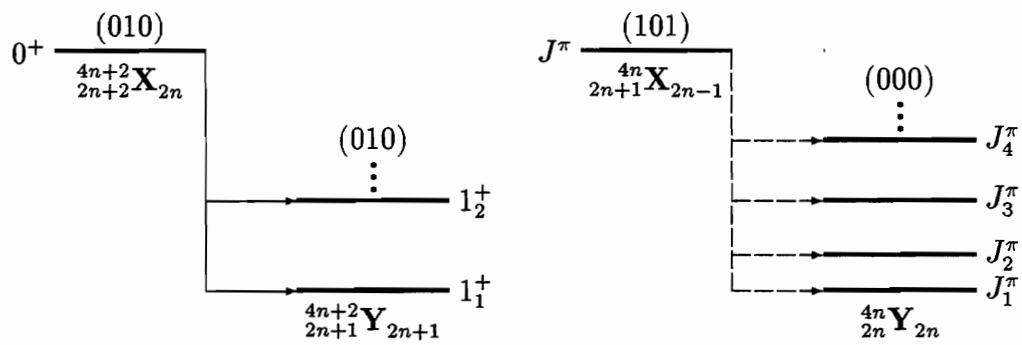

Figure 5. The β^+ decay into a $Z = N$ nucleus for an even–even (left) and an odd–odd (right) parent. Nuclei are denoted by $^A_Z\mathrm{X}_N$. Levels are labelled by their angular momentum and parity J^π, and by the supermultiplet labels $(\lambda\mu\nu)$.

β^+ decay from an even–even nucleus with $Z = N + 2$ into an odd–odd $Z = N$ nucleus takes place between SU(4) representations (010) and should be strong. The decay from an odd–odd nucleus with $Z = N + 2$ into an even–even $Z = N$ nucleus, on the other hand, goes from (101) to (000) and should be forbidden. The resulting zig-zag pattern for the Gamow–Teller strength $B(\mathrm{GT})$ [in units $(4\pi)^{-1}g_A^2$] is compared with the data [28] in figure 6 up to mass $A = 40$. (The decay into the doubly magic nuclei $^{16}_8\mathrm{O}_8$ and $^{40}_{20}\mathrm{Ca}_{20}$ is excluded from the present considerations because the odd–odd parent nucleus involves a proton particle and a neutron hole in different oscillator shells and, consequently, in these cases the β decay is first forbidden with a change in l or occurs through cross-shell correlations.) The analysis of the $0^+ \to 1^+$ decay into an odd–odd nucleus is fairly straightforward: significant decay proceeds to very few 1^+ states at low energy in the daughter nucleus (never more than two) and the summed strength is shown in figure 6. The decay into an even–even nucleus is more delicate to analyse because the strength is considerably more fragmented to higher energies. The figure shows the sum of all observed strength. This can be an underestimate if some of the fragmented strength is not observed; it can also be an overestimate if some strength is included to states that do not belong to the favoured SU(4) multiplet (000). In spite of the shortcomings of this analysis, the message from figure 6 is clear: a zig-zag is observed in the summed Gamow–Teller strength but it fades away with increasing mass especially towards the

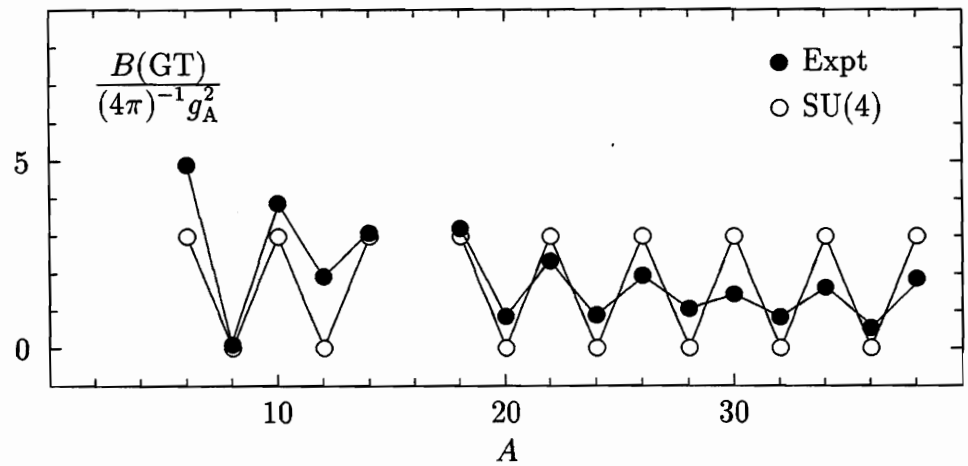

Figure 6. Observed Gamow–Teller β-decay strength for $(T = 1) \rightarrow (T = 0)$ transitions as a function of nuclear mass number A compared with the SU(4) prediction.

end of the *sd* shell. This is an experimental proof of the breaking of SU(4) symmetry as a result mainly of the strong spin–orbit coupling in heavier nuclei. A similar conclusion is reached from nuclear binding energies [26].

3.2. Racah's SU(2) pairing model

If the coefficient v_{ls} of the spin–orbit term in the mean-field potential (3.3) is sufficiently large, a jj-coupling scheme arises. Suppose initially that the splitting between the different j states is large compared to the residual interaction strength, so that only nucleons in the last j shell must be considered as valence particles. Suppose, furthermore, that the residual interaction among those valence nucleons has a pairing character which is attractive for two particles coupled to angular momentum $J = 0$ and is zero otherwise,

$$\langle j^2 J M_J | \hat{V}_{\mathrm{pairing}} | j^2 J M_J \rangle = \begin{cases} -\frac{1}{2}(2j+1)G, & \text{if } J = 0 \\ 0, & \text{if } J \neq 0 \end{cases} . \tag{3.11}$$

This is a schematic albeit reasonable approximation to the residual interaction between *identical* nucleons and hence can only be appropriate in semi-magic nuclei. Under the above assumptions the schematic shell-model hamiltonian (3.2) reduces to

$$\hat{H} = E_0 + \hat{V}_{\mathrm{pairing}}, \tag{3.12}$$

where E_0 represents a constant energy contribution from all nucleons in shells below the valence j shell.

The hamiltonian (3.12) can be diagonalised analytically [29] in a space of n identical fermions in a j shell by noting the second-quantised form of the pairing interaction,

$\hat{V}_{\text{pairing}} = -G\hat{S}_+\hat{S}_-$, with

$$\hat{S}_+ = \tfrac{1}{2}\sqrt{2j+1}\,(a^\dagger_{l\frac{1}{2},j} \times a^\dagger_{l\frac{1}{2},j})^{(0)}_0, \qquad \hat{S}_- = \left(\hat{S}_+\right)^\dagger, \tag{3.13}$$

where $a^\dagger_{l\frac{1}{2},jm}$ creates a particle in the $(l\frac{1}{2})j$ shell with projection m and the bracket denotes the coupling of the angular momenta j. If, in addition to $\hat{S}_\pm$, also the operator

$$\hat{S}_z = \tfrac{1}{4}(2\hat{n} - 2j - 1) = \frac{1}{4}\left(\sum_{m=-j}^{j} 2a^\dagger_{l\frac{1}{2},jm}a_{l\frac{1}{2},jm} - 2j - 1\right) \tag{3.14}$$

is considered, then the following commutation relations hold:

$$[\hat{S}_z, \hat{S}_\pm] = \pm\hat{S}_\pm, \qquad [\hat{S}_+, \hat{S}_-] = 2\hat{S}_z. \tag{3.15}$$

This shows that the set of operators $\{\hat{S}_z, \hat{S}_\pm\}$ forms an SU(2) algebra, which is referred to as the *quasi-spin* algebra. Because of this relation with the quasi-spin SU(2) algebra, the pairing hamiltonian can be solved. From the commutation relations (3.15) it follows that $\hat{S}_+\hat{S}_- = \hat{S}^2 - \hat{S}_z^2 + \hat{S}_z$, which shows that the pairing hamiltonian can be written as a combination of Casimir operators belonging to

$$\begin{array}{ccc} \mathrm{SU}(2) & \supset & \mathrm{O}(2) \\ \downarrow & & \downarrow \\ s & & m_s \end{array}. \tag{3.16}$$

The associated eigenvalue problem can be solved instantly. The quantum numbers s and m_s can be put in relation to the more usual ones of seniority v [30] and (valence) particle number n,

$$s = \tfrac{1}{4}(2j - 2v + 1), \qquad m_s = \tfrac{1}{4}(2n - 2j - 1). \tag{3.17}$$

The interpretation of the seniority quantum number v (found by repeated action of $\hat{S}_+$ on a state with $n = v$) is that it gives the number of nucleons not in pairs coupled to angular momentum zero.

The above concepts of seniority and quasi spin have found repeated application in nuclear physics and have been the subject of fruitful generalisations. An important extension is due to Flowers [31] and concerns the seniority classification of protons and neutrons in jj coupling; in doing so the concept of *reduced isospin* t is established, which is the isospin of the nucleons not in pairs coupled to angular momentum zero. This brings about the generalisation of the quasi-spin algebra from SU(2) to O(5) with properties thoroughly investigated by Hecht [32, 33].

Another generalisation concerns that towards several orbits. In case of degenerate orbits this can be achieved by making the substitution $\hat{S}_+ \to \sum_j \hat{S}_+(j)$ which leaves all previous results such as the algebraic structure (3.15) unchanged. The ensuing formalism can then be applied to semi-magic nuclei but since it requires the assumption

of a pairing interaction with degenerate orbits, its applicability is limited. A much more generally valid model is obtained if one imposes the following condition on the shell-model hamiltonian:

$$[[\hat{H}, \hat{S}_+], \hat{S}_+] = \Delta \left(\hat{S}_+\right)^2, \tag{3.18}$$

where Δ is a constant interpreted as the pairing gap. This condition of *generalised seniority* [34] is much weaker than the assumption of a pairing interaction and, in particular, it does not require that the commutator $[\hat{S}_+, \hat{S}_-]$ yields (up to a constant) the number operator which is central to the quasi-spin formalism. In spite of the absence of a closed algebraic structure, it is still possible to compute the exact ground-state eigenvalue of hamiltonians satisfying (3.18).

A third extension towards isoscalar pairing which is of current topical interest is discussed in the example subsection.

Example: The O(8) model of $T = 0$ and $T = 1$ pairing

The pairing interaction considered so far acts between identical nucleons in a j shell coupled to angular momentum $J = 0$ and isospin $T = 1$. If protons and neutrons are considered, the isospin of a pair of nucleons is not necessarily $T = 1$ but can also be $T = 0$, and the pairing interaction can be generalised correspondingly. One way of doing so is to consider the LS coupling limit of the shell model with a set of degenerate l shells and a pairing interaction between pairs of nucleons coupled to total *orbital* angular momentum $L = 0$. According to the discussion of subsection 3.1, overall antisymmetry implies for $L = 0$ that $S = 0, T = 1$ or $S = 1, T = 0$. Thus the pairing interaction can be of spin-scalar, isovector or of spin-vector, isoscalar type. This leads to a generalised pairing interaction of the form

$$\hat{V}_{\text{pairing}} = -\tfrac{1}{2}(1-x)G\sum_{\mu} \hat{S}^{10}_{+,\mu}\hat{S}^{10}_{-,\mu} - \tfrac{1}{2}(1+x)G\sum_{\mu} \hat{S}^{01}_{+,\mu}\hat{S}^{01}_{-,\mu}, \tag{3.19}$$

with

$$\begin{aligned} \hat{S}^{10}_{+,\mu} &= \sqrt{\tfrac{1}{2}}\sum_l \sqrt{2l+1}\,(a^\dagger_{l\frac{1}{2}\frac{1}{2}} \times a^\dagger_{l\frac{1}{2}\frac{1}{2}})^{(010)}_{0\mu 0}, \qquad \hat{S}^{10}_{-,\mu} = \left(\hat{S}^{10}_{+,\mu}\right)^\dagger, \\ \hat{S}^{01}_{+,\mu} &= \sqrt{\tfrac{1}{2}}\sum_l \sqrt{2l+1}\,(a^\dagger_{l\frac{1}{2}\frac{1}{2}} \times a^\dagger_{l\frac{1}{2}\frac{1}{2}})^{(001)}_{00\mu}, \qquad \hat{S}^{01}_{-,\mu} = \left(\hat{S}^{01}_{+,\mu}\right)^\dagger, \end{aligned} \tag{3.20}$$

where $a^\dagger_{lm_l\frac{1}{2}m_s\frac{1}{2}m_t}$ creates a partcile in the l shell with quantum numbers as indicated and the bracket denotes coupling in orbital angular momentum, spin and isospin. The pairing interaction contains two parameters: an overall strength G and a parameter x which determines the relative strengths of isovector and isoscalar pairing. For $x = +1$ ($x = -1$) pure isovector (isoscalar) pairing is obtained while for $x = 0$ both pairings are of equal strength.

To recognise the algebraic structure of this model, one considers, in addition to the pair creation and annihilation operators (3.20), particle-number conserving operators [35, 36]. This is similar to the construction of the SU(2) quasi-spin algebra where the pair creation and annihilation operators $\hat{S}_\pm$ are supplemented with $\hat{S}_z$ to obtain a closed algebraic structure, only in this case, to ensure closure, one is compelled to add several operators of the form

$$\begin{aligned}
\hat{n} &= 2\sum_l \sqrt{2l+1}\,(a^\dagger_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}_{l\frac{1}{2}\frac{1}{2}})^{(000)}_{000}, \\
\hat{S}_\mu &= \sum_l \sqrt{2l+1}\,(a^\dagger_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}_{l\frac{1}{2}\frac{1}{2}})^{(010)}_{0\mu 0}, \\
\hat{T}_\mu &= \sum_l \sqrt{2l+1}\,(a^\dagger_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}_{l\frac{1}{2}\frac{1}{2}})^{(001)}_{00\mu}, \\
\hat{Y}_{\mu\nu} &= \sum_l \sqrt{2l+1}\,(a^\dagger_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}_{l\frac{1}{2}\frac{1}{2}})^{(011)}_{0\mu\nu},
\end{aligned} \tag{3.21}$$

where the $\tilde{a}_{lm_l\frac{1}{2}m_s\frac{1}{2}m_t}$ are annihilation operators with correct transformation properties under rotations in orbital, spin and isospin spaces,

$$\tilde{a}_{lm_l\frac{1}{2}m_s\frac{1}{2}m_t} \equiv (-)^{l+m_l+\frac{1}{2}+m_s+\frac{1}{2}+m_t} a_{l-m_l\frac{1}{2}-m_s\frac{1}{2}-m_t}. \tag{3.22}$$

The set of 28 operators $\{\hat{S}^{10}_{\pm,\mu}, \hat{S}^{01}_{\pm,\mu}, \hat{n}, \hat{S}_\mu, \hat{T}_\mu, \hat{Y}_{\mu\nu}\}$ forms the Lie algebra O(8) [35]; besides the pair creation and annihilation operators $\hat{S}^{10}_{\pm,\mu}$ and $\hat{S}^{01}_{\pm,\mu}$ it contains the number operator $\hat{n}$, the spin and isospin operators $\hat{S}_\mu$ and $\hat{T}_\mu$, and the operators $\hat{Y}_{\mu\nu}$ that are vectors in spin and isospin. Physically relevant classifications in the O(8) model conserve spin S and isospin T and hence one is interested in algebraic reductions of O(8) that contain $\mathrm{SU}_S(2) \otimes \mathrm{SU}_T(2)$. Three such limits (or dynamical symmetries, in the terminology of section 2) exist and they are specified by the lattice of algebras

$$\mathrm{O}(8) \supset \left\{ \begin{array}{c} \mathrm{O}_S(5) \otimes \mathrm{SU}_T(2) \\ \mathrm{SU}(4) \\ \mathrm{O}_T(5) \otimes \mathrm{SU}_S(2) \end{array} \right\} \supset \mathrm{SU}_S(2) \otimes \mathrm{SU}_T(2), \tag{3.23}$$

where $\mathrm{O}_S(5) \equiv \{\hat{S}^{10}_{\pm,\mu}, \hat{n}, \hat{S}_\mu\}$, $\mathrm{O}_T(5) \equiv \{\hat{S}^{01}_{\pm,\mu}, \hat{n}, \hat{T}_\mu\}$ and $\mathrm{SU}(4) \equiv \{\hat{S}_\mu, \hat{T}_\mu, \hat{Y}_{\mu\nu}\}$. Mathematical details of these classifications are given in [35, 36] and are not of concern here. Their physical significance is rather easily understood from the hamiltonian (3.19). The upper and lower limits in (3.23) are obtained in the case of pure isoscalar ($x = +1$) and isovector ($x = -1$) pairing, respectively. The middle limit is obtained with equal pairing strengths ($x = 0$) in which case it turns out that the hamiltonian (3.19) can be written in terms of the generators of Wigner's SU(4) algebra. The entire range of pairing strengths can thus be simulated by varying x and the extreme situations are analytically solvable. These properties make that O(8) is a schematic model that can be used to study the question of $T = 0$ versus $T = 1$ pairing and therefore it receives currently

renewed attention in relation to $Z \approx N$ nuclei (see, e.g., [37]). It can, however, only have limited applicability since it assumes an LS classification and no spin–orbit coupling. Once a spin–orbit term is added to the hamiltonian (3.19), the subspace constructed out of $L = 0$ pairs no longer is decoupled and one again is forced to a solution of the eigenproblem in the full shell-model space.

3.3. Elliott's SU(3) model of rotation

In Wigner's supermultiplet model no statement is made about the nature of the classification of the spatial part of the wavefunction (3.8) except that the total orbital angular momentum L is assumed to be a good quantum number. Additional labels are left unspecified as indicated by α. The main feature of Elliott's model [38] is that it provides an orbital classification scheme which incorporates rotational characteristics. Elliott's SU(3) model of rotation presupposes Wigner's SU(4) classification and assumes in addition that the residual interaction has a *quadrupole* character. The latter is a reasonable hypothesis if both protons and neutrons are filling the valence shell. In terms of the schematic hamiltonian (3.2) one requires that it reduces to

$$\hat{H} = \sum_{k=1}^{A} \left(\frac{p_k^2}{2m} + \tfrac{1}{2} m\omega^2 r_k^2 \right) + \hat{V}_{\text{quadrupole}}, \tag{3.24}$$

where $\hat{V}_{\text{quadrupole}} = \kappa \hat{Q} \cdot \hat{Q}$ contains the quadrupole operator given by

$$\hat{Q}_\mu = \sqrt{\frac{3}{2}} \left(\sum_{k=1}^{A} (\bar{r}_k \times \bar{r}_k)_\mu^{(2)} / b^2 + b^2 \sum_{k=1}^{A} (\bar{p}_k \times \bar{p}_k)_\mu^{(2)} / \hbar^2 \right), \tag{3.25}$$

in terms of coordinates $\bar{r}_k$ and momenta $\bar{p}_k$, and where b is the oscillator length parameter, $b = \sqrt{\hbar / m\omega}$. Note that $\hat{Q} \cdot \hat{Q}$ contains one-body ($k = l$) as well as two-body ($k \neq l$) terms.

To recognise that the shell-model hamiltonian (3.24) is analytically solvable, it is best to write it in second-quantised form. Because of its symmetric structure in $\bar{r}$ and $\bar{p}$, the quadrupole operator $\hat{Q}_\mu$ does not couple to states outside a given valence shell and particle operators $a^\dagger_{lm_l \frac{1}{2} m_s \frac{1}{2} m_t}$ can be assigned l quantum numbers of that shell (e.g., s, p, sd,...), together with the spin and isospin labels. The quadrupole operator (3.25) can then be rewritten as

$$\hat{Q}_\mu = \sum_l \sqrt{8(2l+1)}\, (a^\dagger_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}_{l\frac{1}{2}\frac{1}{2}})^{(200)}_{\mu 00}, \tag{3.26}$$

since it must be scalar in spin and isospin. Likewise, the orbital angular momentum operator

$$\hat{L}_\mu = \sum_{k=1}^{A} (\bar{r}_k \wedge \bar{p}_k)_\mu / \hbar, \tag{3.27}$$

becomes in second quantisation

$$\hat{L}_\mu = \sum_l \sqrt{4l(l+1)(2l+1)/3}\,(a^\dagger_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}_{l\frac{1}{2}\frac{1}{2}})^{(100)}_{\mu 00}. \tag{3.28}$$

The hamiltonian (3.24) can be rewritten as

$$\hat{H} = \hbar\omega\left(\hat{n} + \tfrac{3}{2}\right) + \kappa\hat{Q}\cdot\hat{Q}, \tag{3.29}$$

where $\hat{n}$ is the number operator. For a given number of nucleons in the valence shell the first term in (3.29) reduces to a constant; the second term, however, generates a spectrum as can be seen as follows.

The hamiltonian (3.24) satisfies the commutation relations (3.5) and hence has SU(4) symmetry. Its additional symmetry character depends somewhat on the orbital space available to the valence particles. Denoting the orbital degeneracy by Ω (i.e., $\Omega = 1, 3, 6, \ldots$ for the s, p, sd,... shells), the orbital–spin-isospin decomposition of the wavefunction (3.8) is equivalent to the algebraic reduction

$$\begin{array}{ccccc} \mathrm{U}(4\Omega) & \supset & \mathrm{U}(\Omega) & \otimes & \mathrm{U}(4) \\ \downarrow & & \downarrow & & \downarrow \\ [1^M] & & [\bar{f}_1\bar{f}_2\bar{f}_3\bar{f}_4] & & [f_1f_2f_3f_4] \end{array}. \tag{3.30}$$

As pointed out in subsection 3.1, the overall antisymmetry $[1^M]$ of the wavefunction requires conjugate symmetries for U(Ω) and U(4). The operators $\hat{L}_\mu$ and $\hat{Q}_\mu$ are scalar in spin and isospin and hence are generators of U(Ω). Furthermore, from their explicit expressions, equations (3.26) and (3.28), one derives the commutation relations

$$\begin{aligned} [\hat{L}_\mu, \hat{L}_\nu] &= -\sqrt{2}\,\langle 1\mu\ 1\nu|1\mu+\nu\rangle \hat{L}_{\mu+\nu}, \\ [\hat{L}_\mu, \hat{Q}_\nu] &= -\sqrt{6}\,\langle 1\mu\ 2\nu|2\mu+\nu\rangle \hat{Q}_{\mu+\nu}, \\ [\hat{Q}_\mu, \hat{Q}_\nu] &= 3\sqrt{10}\,\langle 2\mu\ 2\nu|1\mu+\nu\rangle \hat{L}_{\mu+\nu}, \end{aligned} \tag{3.31}$$

which show that they generate an SU(3) Lie algebra that must then be a subalgebra of U(Ω). From the commutation relations (3.31) it can also be shown that the quadratic combination $\frac{1}{4}\hat{Q}\cdot\hat{Q} + \frac{3}{4}\hat{L}\cdot\hat{L}$ commutes with all generators of SU(3). It then follows that the hamiltonian (3.24) implies the orbital reduction

$$\begin{array}{ccccc} \mathrm{U}(\Omega) & \supset & \mathrm{SU}(3) & \supset & \mathrm{O}(3) \\ \downarrow & & \downarrow & & \downarrow \\ [\bar{f}_1\bar{f}_2\bar{f}_3\bar{f}_4] & & (\bar{\lambda}\bar{\mu}) & & KL \end{array}, \tag{3.32}$$

and represents an example of dynamical symmetry breaking: the degeneracy within a given Wigner supermultiplet $(\lambda\mu\nu)$ is lifted (dynamically) by the quadrupole interaction.

The importance of Elliott's idea is that it gives rise to a rotational classification of states in the context of the shell model through mixing of spherical orbits. The quantum number K emerging from his analysis is associated with the projection of total orbital

angular momentum on the axis of symmetry. As such, the SU(3) model established a link between the nuclear shell model of Mayer [39] and of Jensen *et al.* [40], and the droplet model of Bohr and Mottelson [41] which up to that time (1958) existed as two separate views of the nucleus. It is telling that this connection was made through an analytically solvable, algebraic model rather than via some cumbersome numerical diagonalisation.

Given that Elliott's SU(3) model uses Wigner's supermultiplet classification as a starting point, it likewise breaks down as a result of the spin–orbit term in the nuclear mean field and it cannot be applied to heavy nuclei. Since in the s and p shells the SU(3) model reduces to Wigner's supermultiplet theory, the first real test case where orbital mixing and associated deformation may occur, is for nuclei in the sd shell. Elliott's SU(3) model has thus found its main application in sd-shell nuclei [42, 43].

Example: Quasi SU(3) symmetry in heavy nuclei

Although Elliott's model shows how deformed, non-spherical shapes may arise out of the spherical shell model, the argument is not *a priori* applicable in heavy nuclei where the spin–orbit coupling causes a considerable rearrangement of the single-particle levels. Over the years several schemes have been proposed with the aim of transposing the SU(3) scheme to those modified situations. One such modification (others are discussed in subsections 3.4 and 3.5) has been suggested recently by Zuker *et al.* [44] under the name of quasi SU(3). These authors start from the observation that in jj coupling the single-particle matrix elements of the quadrupole operator are largest for $\Delta j = 0, \pm 2$, those with $\Delta j = \pm 1$ being significantly smaller (see table 2). As a result, the spaces

Table 2. Matrix elements of the quadrupole operator $\hat{Q}_{20}$ in LS and jj coupling.

		LS coupling		$l \gg \lvert m \rvert$
$\langle lm \lvert \hat{Q}_{20} \rvert lm \rangle$	$=$	$\dfrac{l(l+1)-3m^2}{(2l-1)(2l+3)}$	$\rightarrow$	$\dfrac{1}{4}$
$\langle lm \lvert \hat{Q}_{20} \rvert l+2\ m \rangle$	$=$	$\dfrac{3[(l+1)^2-m^2]^{1/2}[(l+2)^2-m^2]^{1/2}}{2(2l+1)^{1/2}(2l+3)(2l+5)^{1/2}}$	$\rightarrow$	$\dfrac{3}{8}$
		jj coupling		$j \gg \lvert m \rvert$
$\langle jm \lvert \hat{Q}_{20} \rvert jm \rangle$	$=$	$\dfrac{j(j+1)-3m^2}{4j(j+1)}$	$\rightarrow$	$\dfrac{1}{4}$
$\langle jm \lvert \hat{Q}_{20} \rvert j+1\ m \rangle$	$=$	$\dfrac{3m[(j+1)^2-m^2]^{1/2}}{4j(j+1)(j+2)}$	$\rightarrow$	$\dfrac{3m}{4j} \sim 0$
$\langle jm \lvert \hat{Q}_{20} \rvert j+2\ m \rangle$	$=$	$\dfrac{3[(j+1)^2-m^2]^{1/2}[(j+2)^2-m^2]^{1/2}}{8(j+1)(j+2)}$	$\rightarrow$	$\dfrac{3}{8}$

built from $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ are effectively decoupled. Furthermore, as table 2

shows, the numerical values of the quadrupole matrix elements do not differ too much in LS and jj coupling. With reference to figure 7, the diagonalisation of $\hat{Q}_{20}$ in the spaces $j = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots$ (or $j = \frac{3}{2}, \frac{7}{2}, \ldots$) and $l = 0, 2, 4, \ldots$ (or $l = 1, 3, \ldots$) yields similar results. This argument has been invoked by Zuker *et al.* [44] as an explanation of the rotational

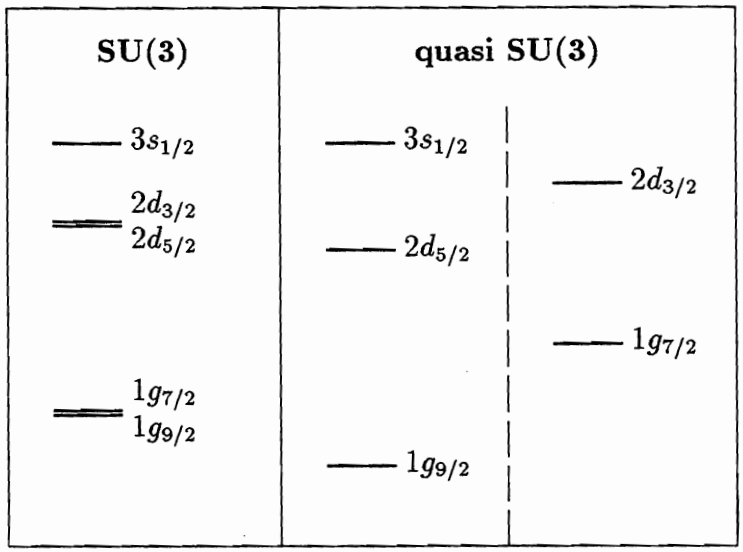

Figure 7. Single-particle spaces of SU(3) and quasi SU(3) for the example of the *sdg* shell. In SU(3) $v_{ls} = 0$ while in quasi SU(3) $v_{ls} = 2v_{ll}$ is taken. The dashed line in quasi SU(3) separates the two spaces that are approximately decoupled.

behaviour even in the presence of a large spin–orbit splitting. It is also instrumental as a guiding principle in the choice of an appropriate shell-model space.

3.4. Pseudo spin symmetry

A very successful way of extending applications of the SU(3) model to heavy nuclei (and to high angular momentum) is based upon the concept of pseudo spin symmetry. The starting point for the explanation of this symmetry is the single-particle part of the hamiltonian (3.2),

$$\hat{h} = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 r^2 + v_{ls}\bar{l}\cdot\bar{s} + v_{ll}\bar{l}\cdot\bar{l}, \tag{3.33}$$

where for notational simplicity the particle index k is omitted. For $v_{ls} = v_{ll} = 0$ a three-dimensional isotropic harmonic oscillator is obtained which exhibits degeneracies associated with U(3) symmetry; for arbitrary non-zero values of v_{ls} and v_{ll} this symmetry is broken as argued in previous subsections. However, for the particular combination $v_{ls} = 4v_{ll}$ some degree of degeneracy, associated with a so-called *pseudo spin* symmetry,

is restored in the spectrum of $\hat{h}$. To understand the nature of pseudo spin symmetry, consider the unitary transformation

$$\hat{u} = 2i\frac{\bar{s}\cdot\bar{r}}{r}, \tag{3.34}$$

and apply this transformation to the single-particle hamiltonian (3.33). One finds [45]

$$\hat{u}^{-1}\hat{h}\hat{u} = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 r^2 + (4v_{ll} - v_{ls})\bar{l}\cdot\bar{s} + v_{ll}\bar{l}\cdot\bar{l} + \hbar\omega - v_{ls} + 2v_{ll}. \tag{3.35}$$

It can be concluded that the hamiltonian

$$\hat{h}' = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 r^2 + (4v_{ll} - v_{ls})\bar{l}\cdot\bar{s} + v_{ll}\bar{l}\cdot\bar{l} \tag{3.36}$$

has the same excitation spectrum as the original one $\hat{h}$ since they are related, up to the constant $\hbar\omega - v_{ls} + 2v_{ll}$, by a unitary transformation. In particular, it shows that for $v_{ls} = 4v_{ll}$ the spectrum of $\hat{h}$ is identical (up to a constant) to that with only an orbit–orbit and no spin–orbit term. This results from the single-particle levels with $j = l + \frac{1}{2}$ and $j = (l+2) - \frac{1}{2}$ being degenerate for all values of l. These single-particle levels can be considered as originating from a pseudo orbital angular momentum $\tilde{l} = l + 1$, in the presence of zero pseudo spin–orbit splitting $\tilde{l}\cdot\tilde{s}$. This is illustrated in figure 8 for the *sdg* shell: with the appropriate choice of $v_{ls} = 4v_{ll}$ the levels $(2d_{5/2}, 1g_{7/2})$ and $(3s_{1/2}, 2d_{3/2})$ occur in doublets and can be considered as originating from $\tilde{l} = 3$ and $\tilde{l} = 1$, respectively. Note, however, that there is no partner for $1g_{9/2}$ in this scheme.

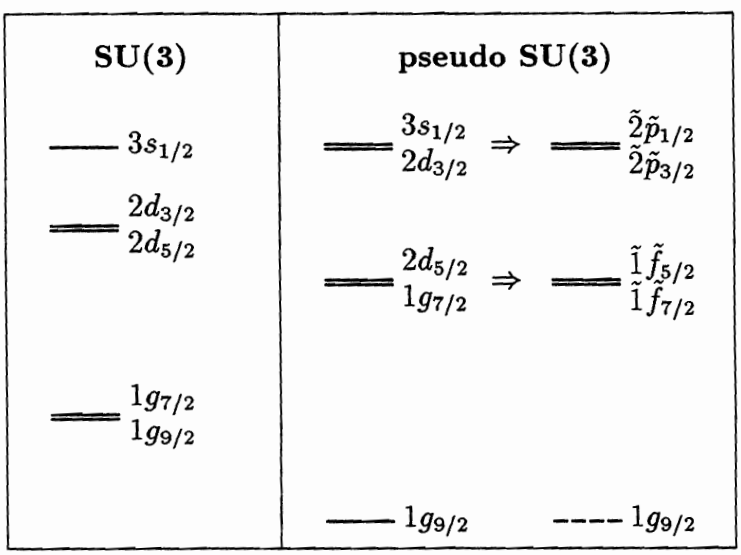

Figure 8. Single-particle spaces of SU(3) and pseudo SU(3) for the example of the *sdg* shell. In SU(3) $v_{ls} = 0$ while pseudo SU(3) requires $v_{ls} = 4v_{ll}$ in which case the level degeneracies can be interpreted in terms of a pseudo spin symmetry.

Pseudo spin symmetry has a long history in nuclear physics. The existence of nearly degenerate pseudo spin doublets in the nuclear mean-field potential was noted already thirty years ago by Hecht and Adler [46] and, independently, by Arima *et al.* [47]. These authors also realised that, because of the small pseudo spin–orbit splitting, pseudo LS (or $\tilde{L}\tilde{S}$) coupling should be a reasonable starting point in medium-mass and heavy nuclei where LS coupling becomes unacceptable. With $\tilde{L}\tilde{S}$ coupling as a premiss, an SU(3) model can be constructed in much the same way as Elliott's SU(3) model can be defined in LS coupling. The ensuing pseudo SU(3) model was investigated seriously for the first time in [48] with many applications following afterwards (for a review, see [49]). Although the pseudo SU(3) model is probably the most important emanation of pseudo spin symmetry, it must be emphasised that the latter is a broader concept than just pseudo SU(3), as is illustrated in the example of this subsection. The formal definition of the pseudo spin transformation (3.34) in terms of a helicity operator was given by Bohr *et al.* [45] and later generalised by Castaños *et al.* [50], to include transformations that not only act on the spin–angular part of the wavefunction—as does (3.34)—but also on its orbital part. An important development was the application of pseudo spin symmetry to superdeformed nuclei [51] and its possible relevance in the explanation of identical bands [52, 53]. Finally, it is only recently that an explanation of pseudo spin symmetry was suggested in terms of the relativistic mean-field model of the nucleus. First, the condition $v_{ls} = 4v_{ll}$ was found to be approximately valid in numerical calculations [54] and later the pseudo spin symmetry was proven to be a symmetry of the Dirac equation which occurs if the scalar and vector potentials are equal in size but opposite in sign [55, 56].

A vanishing pseudo spin–orbit splitting does not necessarily imply the validity of $\tilde{L}\tilde{S}$ coupling which can be broken by pseudo spin dependent terms in the residual interaction. The assumption of $\tilde{L}\tilde{S}$ coupling can be verified by analysing the wavefunctions of a shell-model calculation with realistic interactions. An example of such an analysis is given in [58] and leads to the conclusion that $\tilde{L}\tilde{S}$ coupling is a reasonable *ansatz* for nuclei in the mass $A \approx 60$ region. Intuitively, this result is a combined effect of the short-range nature of the residual interaction (cfr. the discussion in subsection 3.1) and the fact that nucleons interact predominantly at the surface of the nucleus. Because of the latter property, matrix elements of the residual interaction are not very sensitive to the radial structure of the wavefunction at the interior of the nucleus. As a result, the problem of n nucleons in a pseudo harmonic oscillator shell is approximately equivalent to that of n nucleons in a normal harmonic oscillator shell. In fact, this equivalence is exact for the surface delta interaction [59, 60] which is known to be a reasonable approximation to the true effective interaction in nuclei.

Example: Pseudo SU(4) symmetry in $A \approx 60$ nuclei

Just as the SU(2) symmetries of spin and isospin can be combined to yield the larger SU(4) symmetry, one can equally well combine the SU(2) symmetries of pseudo spin and isospin to give what can be called a pseudo SU(4) [or $\widetilde{\mathrm{SU}}(4)$] symmetry [57, 58]. A hamiltonian with pseudo SU(4) symmetry satisfies the following commutations relations:

$$[\hat{H}, \sum_{k=1}^{A} \tilde{s}_\mu(k)] = [\hat{H}, \sum_{k=1}^{A} t_\mu(k)] = [\hat{H}, \sum_{k=1}^{A} \tilde{s}_\mu(k) t_\nu(k)] = 0, \tag{3.37}$$

where $\tilde{s}_\mu$ is the transformed spin operator, $\tilde{s}_\mu = \hat{u}^{-1} s_\mu \hat{u}$. For example, the schematic shell-model hamiltonian (3.2) satisfies the commutation relations (3.37) if $v_{ls} = 4v_{ll}$. The hamiltonian in (3.37) has eigenstates labelled by $|(\widetilde{\lambda\mu\nu})\tilde{L}\tilde{S}JT\rangle$ and conserves the total pseudo orbital angular momentum $\tilde{L}$ and the total pseudo spin $\tilde{S}$, which result from the separate coupling of all individual pseudo orbital angular momenta $\tilde{l}_k$ and pseudo spins $\tilde{s}_k$. The labels $(\widetilde{\lambda\mu\nu})$ are associated with pseudo SU(4) in direct analogy with Wigner's supermultiplet scheme.

The existence of a pseudo SU(4) symmetry requires minimally that the valence shell coincides with a pseudo oscillator shell. A region where this is possibly the case concerns $Z \approx N$ nuclei at the beginning of the 28–50 shell where the dominant orbits are $2p_{1/2}$, $2p_{3/2}$ and $1f_{5/2}$ which can be considered as a pseudo *sd* shell.

It is instructive to compare the predictions of pseudo SU(4) symmetry concerning Gamow–Teller β decay with those of SU(4). A typical example of the latter, the β^+ decay of ${}^{18}_{8}\mathrm{Ne}_{10}$, is shown in figure 9: the Gamow–Teller decay proceeds to two 1^+ states in ${}^{18}_{9}\mathrm{F}_9$ with $\log(ft)$ values of 3.1 and 4.5, respectively. In SU(4) these correspond to transitions within the supermultiplet (010); the first occurs without change of orbital structure (${}^1S_0 \rightarrow {}^3S_1$) and is fast, while the second (${}^1S_0 \rightarrow {}^3D_1$) is forbidden. The SU(4) classification thus provides a qualitative understanding of Gamow–Teller β decay in ^{18}Ne.

The analysis of the analogous problem in pseudo SU(4) is more complicated because the Gamow–Teller operator is not a generator of pseudo SU(4). The situation is illustrated in figure 9. The transition without change in orbital structure (${}^1\tilde{S}_0 \rightarrow {}^3\tilde{S}_1$) is about one order of magnitude weaker than the corresponding one in SU(4) (4.0 versus 3.0). Furthermore, the second transition (${}^1\tilde{S}_0 \rightarrow {}^3\tilde{D}_1$) no longer is forbidden. Its matrix element depends upon the amount and character of orbital mixing; the numbers shown in the figure ($3.3 \sim 3.7$) represent a typical range between prolate and oblate deformation. It is also seen that the observed ft values in the decay of ${}^{58}_{30}\mathrm{Zn}_{28}$ [61] strongly differ from those for ^{18}Ne, and are more akin to the pseudo SU(4) prediction.

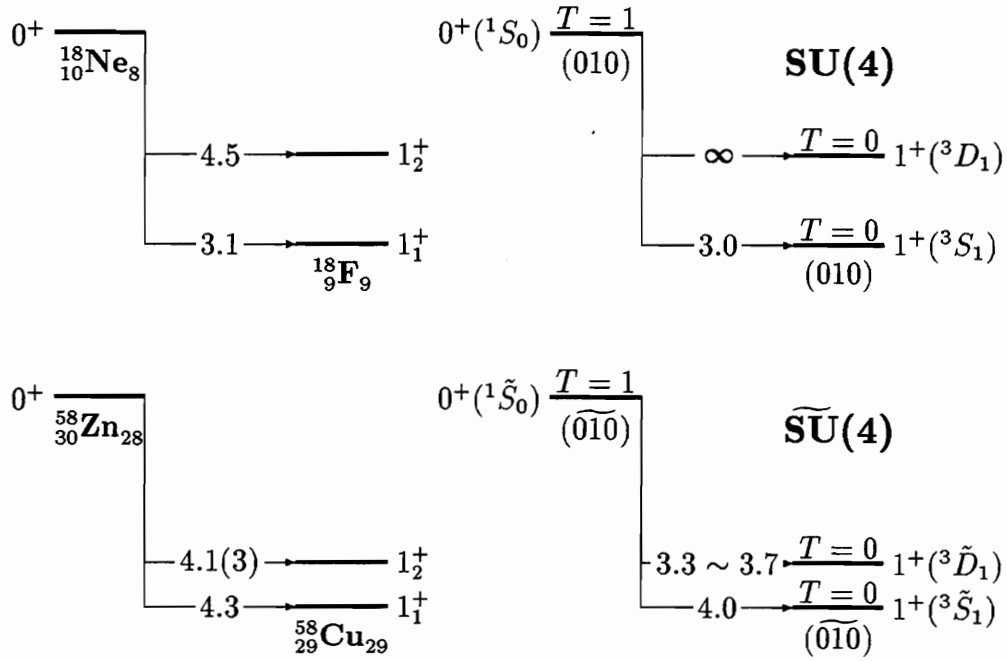

Figure 9. The log(ft) values for Gamow–Teller matrix elements in the β^+ decay of ^{18}Ne and ^{58}Zn as contrasted with the corresponding values in standard and pseudo SU(4).

3.5. The fermion dynamical symmetry model

An important property of Racah's pairing hamiltonian is that it does not couple states constructed out of S pairs (3.13) to the rest of the shell-model space. Thus, for example, the state with seniority $v = 0$, entirely built from S pairs, is an eigenstate of the pairing hamiltonian $\hat{S}_+\hat{S}_-$, and hence decoupled from all other states. The question now arises whether this property of decoupling can be generalised to more realistic scenarios. In particular, given the important rôle of the quadrupole degree of freedom in nuclei, one would like an extension of this idea towards a decoupled space in terms of monopole S and quadrupole D pairs. A systematic procedure for constructing such hamiltonians was devised by Ginocchio [62], drawing on earlier ideas by Hecht *et al.* [63] and using a method which resembles that of pseudo spin [46, 47]. The theory was later developed under the name of *fermion dynamical symmetry model* or FDSM by Wu and others [64, 65].

The starting point of the method is the separation of the nucleon angular momentum $\bar{\jmath}$ into a pseudo orbital part $\bar{k}$ and a pseudo spin part $\bar{\imath}$: $\bar{\jmath} = \bar{k} + \bar{\imath}$. This is similar to the pseudo spin scheme of subsection 3.4. By convention, k is taken integer and i half-

integer but no other condition is imposed *a priori*. This separation carries with it the definition of the particle creation operators†

$$a^{\dagger}_{ki,jm_j} = \sum_{m_k m_i} \langle km_k\, im_i | jm_j \rangle a^{\dagger}_{km_k im_i}, \tag{3.38}$$

where $a^{\dagger}_{km_k im_i}$ is related as follows to the usual $(l\frac{1}{2})j$-coupled creation operators:

$$a^{\dagger}_{km_k im_i} = \sum_{jm_j} \langle km_k\, im_i | jm_j \rangle a^{\dagger}_{l\frac{1}{2},jm_j}. \tag{3.39}$$

Pairs of particles can now be defined in a KI-coupled instead of a jj-coupled basis, the two being related through

$$\begin{aligned} \hat{B}_+(KI, JM_J) &\equiv (a^{\dagger}_{ki} \times a^{\dagger}_{k'i'})^{(KI,J)}_{M_J} \\ &= \sum_{jj'} \begin{bmatrix} k & k' & K \\ i & i' & I \\ j & j' & J \end{bmatrix} (a^{\dagger}_{l\frac{1}{2},j} \times a^{\dagger}_{l'\frac{1}{2},j'})^{(J)}_{M_J}, \end{aligned} \tag{3.40}$$

where the coefficient between square brackets represents the transformation from the coupling $|kk'(K)ii'(I); J\rangle$ to $|ki(j)k'i'(j'); J\rangle$ and is related to a nine-j symbol [22]. Any two-body nucleon interaction can be written equivalently in terms of either jj-coupled or KI-coupled pair creation and annihilation operators since these are connected by the unitary transformation (3.40). In addition to the two-particle creation operators (3.40) and their hermitian conjugates, a closed algebraic structure requires multipole operators of the form

$$\hat{P}(KI, JM_J) \equiv (a^{\dagger}_{ki} \times \tilde{a}_{k'i'})^{(KI,J)}_{M_J}, \tag{3.41}$$

where the annihilation operators $\tilde{a}_{km_k im_i}$ have the correct transformation properties in pseudo orbital and pseudo spin space,

$$\tilde{a}_{km_k im_i} \equiv (-)^{k+m_k+i+m_i} a_{k-m_k i-m_i}. \tag{3.42}$$

It is now a matter of straightforward algebra to work out the commutation relations between the operators $\hat{B}_+(KI, JMJ)$, their hermitian conjugates $\hat{B}_-(KI, JMJ)$ and $\hat{P}(KI, JM_J)$. These will depend on the choice of k and i, and their coupled values K and I, and a judicious choice will define a set of pairs that decouple from all others.

The search is now on for sets of pair operators that include a monopole S and a quadrupole D pair and close under commutation to an algebra which is smaller than the one that spans the entire shell-model space. Two such algebras can be defined [62].

(i) *The O(8) algebra.* This case is obtained from K-scalar pairs with $i = \frac{3}{2}$. The choice of k is free (different values are allowed) but they should couple to $K = 0$

† As only one kind of nucleon is considered in this subsection, isospin indices are omitted.

in all pairs. Overall antisymmetry then requires even I and leads to either $J = I = 0$ (S) or $J = I = 2$ (D). The set of pairs that decouples from all others consists of the pair operators $\hat{S}_+ \equiv \hat{B}_+(K = 0\, I = 0, J = 0\, M_J = 0)$, $\hat{D}_+ \equiv \hat{B}_+(K = 0\, I = 2, J = 2\, M_J)$, their hermitian conjugates $\hat{S}_-$ and $\hat{D}_-$, and the multipole operators $\hat{P}_+(K = 0\, I, J = I\, M_J), I = 0, 1, 2, 3$. These 28 operators generate the Lie algebra O(8).

(ii) *The Sp(6) algebra.* This case is obtained from I-scalar pairs with $k = 1$. Due to overall antisymmetry the coupling to $I = 0$ requires either $J = K = 0$ (S) or $J = K = 2$ (D). The set of pairs that decouples from all others consists in this case of $\hat{S}_+ \equiv \hat{B}_+(K = 0\, I = 0, J = 0\, M_J = 0)$, $\hat{D}_+ \equiv \hat{B}_+(K = 2\, I = 0, J = 2\, M_J)$, their hermitian conjugates $\hat{S}_-$ and $\hat{D}_-$, and the multipole operators $\hat{P}_+(K\, I = 0, J = K\, M_J), K = 0, 1, 2$. These 21 operators generate the Lie algebra Sp(6).

What is required for the above algebraic realisations and the associated decoupling from the rest of the shell-model space to occur? First, it needs the appropriate single-particle orbits which should be consistent with the representation in terms of k and i. Cases of relevance to the nuclear shell model are summarised in table 3. All combinations

Table 3. Possible combinations of single-particle orbits in the FDSM.

		O(8) $(i = \frac{3}{2})$		
k =	1	2	0 + 3	1 + 5
	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$	$\frac{1}{2}, \frac{3}{2}, \ldots, \frac{9}{2}$	$\frac{1}{2}, \frac{3}{2}, \ldots, \frac{13}{2}$
		Sp(6) $(k = 1)$		
i =	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2} + \frac{7}{2}$	$\frac{3}{2} + \frac{9}{2}$
	$\frac{1}{2}, \frac{3}{2}$	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$	$\frac{1}{2}, \frac{3}{2}, \ldots, \frac{9}{2}$	$\frac{1}{2}, \frac{3}{2}, \ldots, \frac{11}{2}$

$j = \frac{1}{2}, \frac{3}{2}, \ldots, j_{\max}$ are possible, sometimes in two different ways, but note that this sequence does not include the unnatural-parity orbit which should thus be dealt with separately. A far more restrictive condition concerns the decoupling of part of the shell-model space. This requires that the hamiltonian be written in terms of generators of a single algebra, either O(8) or Sp(6). This condition, in fact, may lead to unrealistic hamiltonians.

Once the algebraic structures, O(8) and Sp(6), are identified, analytically solvable limits can be studied following the generic procedure outlined in section 2. This amounts to finding algebraic embeddings that reduce to the angular momentum algebra O(3);

they can be specified by the lattices

$$\mathrm{O}(8) \supset \left\{ \begin{array}{c} \mathrm{O}(5) \otimes \mathrm{SU}(2) \\ \mathrm{O}(6) \\ \mathrm{O}(7) \end{array} \right\} \supset \mathrm{O}(5) \supset \mathrm{O}(3), \tag{3.43}$$

and

$$\mathrm{Sp}(6) \supset \left\{ \begin{array}{c} \mathrm{SU}(3) \\ \mathrm{SU}(2) \otimes \mathrm{O}(3) \end{array} \right\} \supset \mathrm{O}(3). \tag{3.44}$$

The condition that the hamiltonian be expressed in terms of Casimir operators of a single reduction chain places further restrictions on its form which depend on the precise realisation of the various algebras in terms of pair and multipole operators. The analysis of the different limits, the classification of states, the wavefunction analysis, etc. is rather involved and is not pursued here. More details can be found in [49] and references therein.

A valuable aspect of the FDSM—which it shares with all other models discussed in this section—is that the Pauli principle is correctly treated without any approximation. A vivid illustration of this feature can be obtained in the SU(3) limit of Sp(6) by studying the allowed SU(3) representations as a function of particle number in a given set of shells. The example of the $j = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{9}{2}$ shells is shown in table 4 [66]. For

Table 4. Classification of n identical particles in shells with $j = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{9}{2}$.

n	$(\lambda\mu)$							
0	(00)							
2	(20)							
4	(40)	(02)						
6	(60)	(22)	(00)					
8	(80)	(42)	(04)	(20)				
10	(10,0)	(62)	(24)	(40)	(02)			
12	—	(82)	(44)	(06)	(60)	(22)	(00)	
14	—	—	(64)	(26)	(80)	(42)	(04)	(20)
16	—	—	(46)	(62)	(08)	(24)	(40)	(02)
18	—	(28)	(44)	(60)	(06)	(22)	(00)	
20	(0,10)	(26)	(42)	(04)	(20)			
22	(08)	(24)	(40)	(02)				
24	(06)	(22)	(00)					
26	(04)	(20)						
28	(02)							
30	(00)							

particle numbers n smaller than one third of the shell size, $n \leq \frac{1}{3}\Omega$, the leading SU(3)

representation is stretched with $\lambda = n$ and $\mu = 0$. Beyond $\frac{1}{3}\Omega$ this representation is Pauli forbidden and the leading one has $\mu \neq 0$. This situation should be contrasted with the SU(3) limit of the IBM (see subsection 4.1) where the leading representation is *always* ($\lambda 0$). An effective way of dealing with this problem in the IBM, is to convert to a hole representation of the bosons once the number of particles exceeds more than half the shell size. This amounts to interchanging λ and μ and corrects the neglect of the Pauli principle at the top of the shell. Nevertheless, as is clear from table 4, for $\frac{1}{3}\Omega < n < \frac{2}{3}\Omega$ the leading SU(3) representation has $\lambda \neq 0$ and $\mu \neq 0$, a result with which the IBM is at variance.

On the negative side in the evaluation of the FDSM is that the structure of the pairs is algebraically imposed. This is clear from the expression (3.40) for $\hat{B}_+(KI, JM_J)$ which gives a well-defined expansion in terms of the usual jj-coupled pairs. In exceptional cases the algebraic pairs might correspond to those favoured by the nuclear interaction; more often they will not. In such situations the FDSM can only be viewed as a useful shell-model truncation scheme.

3.6. Non-compact extensions

The models discussed so far all share the property of being confined to a single shell [a j shell in the case of the pairing model, an oscillator shell in SU(4), a pseudo oscillator shell in pseudo SU(4), etc.] It is well known, however, that a full description of nuclear collective motion requires correlations that involve configurations outside a single shell. The proper framework for such correlations invokes the concept of a non-compact algebra.

The idea is perhaps best illustrated [67, 68] starting from Elliott's SU(3) model. The *algebraic* quadrupole operator (3.25) of SU(3) is chosen such that it reproduces exactly the matrix elements of the *real* collective quadrupole operator

$$\hat{Q}^{c}_{\mu} = \sqrt{6}\sum_{k=1}^{A}(\bar{r}_k \times \bar{r}_k)^{(2)}_{\mu}/b^2. \tag{3.45}$$

This is true as long as the matrix element involves states of the same oscillator shell such as those shown in table 2. By construction, the matrix elements of $\hat{Q}_\mu$ between different oscillator shells vanish, and it is precisely this property which gives rise to the algebraic SU(3) structure. The real quadrupole operator $\hat{Q}^{c}_{\mu}$, however, has non-zero $\Delta n = \pm 2$ matrix elements and those are the ones responsible for the increased quadrupole collectivity. For the sake of constructing a closed algebra these correlations are thus lost from the SU(3) model.

Can this situation be restored by extending SU(3) to a larger algebra that is capable of accommodating cross-shell effects? The answer is yes, by embedding SU(3) into the symplectic algebra Sp(3,R). The essence of this idea is contained in the following relation

between the algebraic and collective quadrupole operators [49]:

$$\hat{Q}^{c}_{\mu} = \hat{Q}_{\mu} + \sqrt{\tfrac{3}{2}}\,(\hat{B}^{2}_{+,\mu} + \hat{B}^{2}_{-,\mu}), \tag{3.46}$$

where $\hat{B}^{2}_{\pm,\mu}$ are quadrupole $2\hbar\omega$ raising and lowering operators. The Sp(3,R) algebra is comprised of $\{\hat{n}, \hat{L}_{\mu}, \hat{Q}_{\mu}, \hat{B}^{0}_{\pm,0}, \hat{B}^{2}_{\pm,\mu}\}$ where $\hat{n}$ is the number operator, $\hat{L}_{\mu}$ and $\hat{Q}_{\mu}$ are the angular momentum and quadrupole operators forming SU(3) and $\hat{B}^{l}_{\pm,\mu}$ are the monopole ($l = 0$) and quadrupole ($l = 2$) $2\hbar\omega$ raising and lowering operators. These operators close under commutation and the collective quadrupole operator (3.45) can be constructed in terms of them. The resulting algebra is called non-compact since the (unitary) representation space is infinite-dimensional (because particles can be raised *ad infinitum*).

The symplectic extensions of the SU(3) and pseudo SU(3) models have been applied to *sd*-shell, and rare-earth and actinide nuclei, generally accounting for the observed correlation effects.

4. Symmetries of interacting boson models

Although the symmetry techniques employed in section 3 are of a general nature, the discussion of that section is entirely focussed on the *nuclear* many-body system: the choice of mean field is appropriate for nuclei, the interactions are chosen as to mimic the residual nucleon–nucleon interaction in the nuclear medium, etc. It is clear, however, that the symmetry methods explained there are quite general and can be applied to other fields of physics. For this reason the work of Wigner, Racah and Elliott without any doubt remains of undisputed importance. Unfortunately, only a limited number of nuclei correspond to the analytically solvable vertices of figure 4 (or other analytically solvable situations discussed in section 3). The vast majority of nuclei are somewhere in between ('transitional'), requiring the numerical solution of an eigenvalue problem which generally is of a formidable complexity. Hence, the direct impact of fermionic symmetry methods, measured by the number of nuclei that can be described with them, has remained limited.

The interacting boson model (IBM) of Arima and Iachello [69] proved to be much more useful in this respect. In this model, proposed in 1975, the nucleus is described in terms of interacting s and d bosons. Again, as in the shell model, for particular choices of boson energies and boson–boson interactions, the ensuing eigenvalue problem can be solved analytically. For an IBM hamiltonian with up to two-body interactions between the bosons, three different classes of analytical solutions or 'limits' exist: the vibrational U(5) limit [70], the rotational SU(3) limit [71] and the γ-unstable O(6) limit [72]. This is illustrated in figure 10: the vertices correspond to analytically solvable limits while an arbitrary point of the triangle corresponds to a general IBM hamiltonian. Moreover,

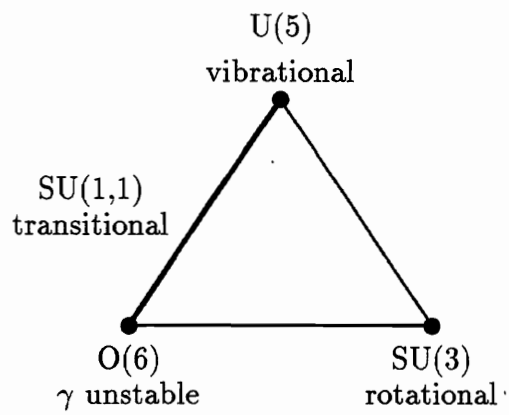

Figure 10. Schematic representation of the boson-model parameter space with its three analytically solvable vertices and its analytically solvable edge.

as shown by Pan and Draayer [73], the U(5) and O(6) vertices are connected by an analytically solvable path in terms of an SU(1,1) algebra.

There is an important practical difference between the triangles in figures 4 and 10: while a numerical solution of the shell-model eigenvalue problem in general rapidly becomes intractable, the corresponding bosonic problem remains tractable at all times, requiring the diagonalisation of a matrix with dimension of no more than $\sim 10^2$. One of the main reasons of the success of the IBM is that it provides a workable, albeit approximate scheme which allows an easy description of transitional nuclei. Its descriptive power of a wide range of nuclear structure data is indeed impressive, as is argued, for example, in the review of Casten and Warner [74].

Another important aspect of the IBM is its geometrical interpretation [75, 76]. Geometry is derived from the algebraic description by taking the limit of infinite boson number; this leads to a Bohr–Mottelson type hamiltonian with kinetic and potential energy terms in the variables β and γ that parametrise quadrupole vibrations of the nuclear surface around a static equilibrium shape [77]. An analysis of this type shows that the three limits of the IBM have simple geometric counterparts that are frequently encountered in nuclei. First, the geometric interpretation of the U(5) limit is the anharmonic vibrator model of Brink *et al.* [78]; in fact, in this case the equivalence is exact even for a finite number of bosons. Second, the SU(3) limit for infinite boson number becomes equivalent to a statically deformed droplet that exhibits quadrupole oscillations around an axially symmetric shape which is a well-established description of the nucleus since the work of Bohr and Mottelson [41]. Third, the O(6) limit yields a γ-unstable rotor known as the Wilets–Jean model [79]. Finally, the entire SU(1,1) limit [or U(5)–O(6) transition] has a geometric counterpart in the γ-unstable model of Elliott *et al.* [80].

Many of the group theoretical aspects of the IBM go back to those developed for

the collective model of the nucleus introduced by Bohr and Mottelson [41]. In fact, an explicit and complete determination of states characterised by U(5) ⊃ O(5) ⊃ O(3) is given by Moshinsky and collaborators in [81] with the aim of providing the group theoretical background of the collective model. As shown in [82] these results are immediately extendable to the U(5) limit of the IBM while a full analysis of the other two limits, SU(3) and O(6), requires the derivation of additional transformation brackets, also given in [82]. The conclusion of the work by Moshinsky's group is thus that an analysis of the symmetry properties of the IBM can be considered as including those of the collective model.

The O(6) symmetry still stands out as an excellent example of the value and power of symmetry methods. It emerged from the formalism as the third limit of the IBM; its origins were purely algebraic, but its structure was later found to resemble that of a γ-unstable rotor. Its predictions were found to correspond closely to the empirical structure of some Pt nuclei [83] and indeed, the transition between SU(3) and O(6) was found to provide a natural explanation for the entire region of Os–Pt nuclei [84], which had always been regarded as one of the difficult challenges in nuclear structure. Since that early work, it has become increasingly evident that the O(6) symmetry in fact represents the third commonly occurring class of nuclei, which have been identified in several regions, most notably around $A = 130$ [85].

The following subsections do not aim to give a comprehensive overview of all ramifications of the IBM—with well over a thousand papers published to date concerning the IBM that would be a hard task indeed! Instead, the most important developments are sketched and illustrated with recent examples. The important extension to odd-mass nuclei is, for reasons of logical consistency, deferred to section 5.

4.1. The interacting boson model

In the original version of the IBM, applicable to even–even nuclei, the basic building blocks are s and d bosons [86]. Unitary transformations among the six components occurring in the model ($s^\dagger$ and $\mathrm{d}^\dagger_m, m = 0, \pm1, \pm2$) generate the Lie algebra U(6). A low-lying collective state of an even–even nucleus with $2N$ valence nucleons is approximated as an N-boson state. Although the separate boson numbers n_s and n_d are not necessarily conserved, their sum $n_s + n_d = N$ is; this implies a total-boson-number conserving hamiltonian of the generic form

$$\hat{H} = \hat{H}_0 + \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \cdots, \tag{4.1}$$

where the index refers to the order of the interaction in the generators of U(6). The first three terms are the ones routinely included:

- a constant

$$\hat{H}_0 = E_0\hat{1}; \tag{4.2}$$

- the one-body part

$$\hat{H}_1 = \epsilon_s \hat{n}_s + \epsilon_d \hat{n}_d, \tag{4.3}$$

where ϵ_s and ϵ_d are the single-boson energies of the s and d boson, respectively;

- the two-body interactions†

$$\hat{H}_2 = \sum_{l_1 l_2 l_1' l_2' L} \frac{v^L_{l_1 l_2 l_1' l_2'} \sqrt{2L+1}}{\sqrt{(1+\delta_{l_1 l_2})(1+\delta_{l_1' l_2'})}} \left((b^\dagger_{l_1} \times b^\dagger_{l_2})^{(L)} \times (\tilde{b}_{l_1'} \times \tilde{b}_{l_2'})^{(L)} \right)^{(0)}_0, \tag{4.4}$$

where the v coefficients are the interaction matrix elements between normalised two-boson states,

$$v^L_{l_1 l_2 l_1' l_2'} \equiv \langle l_1 l_2; LM_L | \hat{H}_2 | l_1' l_2'; LM_L \rangle. \tag{4.5}$$

Since the bosons are necessarily symmetrically coupled, allowed two-boson states are s^2 ($L = 0$), sd ($L = 2$) and d^2 ($L = 0, 2, 4$). Thus, seven independent two-body interactions v can be found: three for $L = 0$, three for $L = 2$ and one for $L = 4$. Note in particular the possibility of off-diagonal interactions, v^0_{0022} and v^2_{0222}, which are crucial for the SU(3) and O(6) limits.

This analysis can be carried further to higher-order interactions. Specifically, one may consider

- the three-body interactions

$$v^L_{l_1 l_2 l_3 l_1' l_2' l_3'} \equiv \langle l_1 l_2 l_3; LM_L | \hat{H}_3 | l_1' l_2' l_3'; LM_L \rangle. \tag{4.6}$$

The allowed three-boson states are s^3 ($L = 0$), $s^2 d$ ($L = 2$), sd^2 ($L = 0, 2, 4$) and d^3 ($L = 0, 2, 3, 4, 6$), leading to $6+6+1+3+1 = 17$ independent three-body interactions for $L = 0, 2, 3, 4, 6$, respectively. Note that any three-boson state $s^i d^{3-i}$ is fully characterised by its angular momentum L; this no longer is the case for higher boson numbers.

The number of possible interactions at each order n is summarised in table 5 for up to $n = 3$. Some of these interactions exclusively contribute to the binding energy and do not affect the excitation spectrum of a single nucleus. To determine the number of such interactions, one notes that the hamiltonian $\hat{N}\hat{H}_{n-1}$ for constant N (i.e., a single nucleus) essentially reduces to the $(n-1)$-body hamiltonian $\hat{H}_{n-1}$. Consequently, of the $\mathcal{N}_n$ independent interactions of order n contained in $\hat{H}_n$, $\mathcal{N}_{n-1}$ terms of the type $\hat{N}\hat{H}_{n-1}$ must be discarded if one wishes to retain only those that influence the excitation energies. This argument leads to the numbers quoted in table 5.

Any IBM hamiltonian (4.1) can be solved numerically with relative ease involving the diagonalisation of matrices of modest size. The procedure for doing so has

† The customary notation of l_i for the boson angular momenta and L for their coupled value is followed here. Note, however, that L corresponds to the *total* angular momentum usually denoted by J and *not* to the orbital angular momentum. In fact, in IBM-1,2,3 the character of the boson angular momentum (i.e., orbital or spin) is left unspecified; this connection with underlying fermion quantum numbers is made more explicit in IBM-4 (see subsection 4.3).

Table 5. Enumeration of interactions in the IBM.

order of interaction	number of interactions total	BE[a]	E_x^b
$n=0$	1	1	0
$n=1$	2	1	1
$n=2$	7	2	5
$n=3$	17	7	10

[a]Interactions that do not influence excitation spectra.
[b]Interactions that do influence excitation spectra.

been implemented for the most general hamiltonian up to and including three-body interactions [87] and can, if needed, be extended to higher orders.

The characteristics of the most general hamiltonian which includes up to two-body interactions are by now well established and, specifically, its symmetry properties are understood. Three different types of dynamical symmetries occur in IBM, associated with the algebraic reductions

$$\mathrm{U}(6) \supset \left\{ \begin{array}{c} \mathrm{U}(5) \supset \mathrm{O}(5) \\ \mathrm{SU}(3) \\ \mathrm{O}(6) \supset \mathrm{O}(5) \end{array} \right\} \supset \mathrm{O}(3). \tag{4.7}$$

The algebras appearing in (4.7) are subalgebras of U(6) generated by operators of the type $b^\dagger_{lm} b_{l'm'}$, the explicit form of which is listed, for example, in [86]. With the subalgebras U(5), SU(3), O(6), O(5) and O(3) there are associated one linear [of U(5)] and five quadratic Casimir operators. This matches the number of one- and two-body interactions quoted in the last column of table 5. The total of all one- and two-body interactions can be represented by including in addition the operators $\hat{C}_1[\mathrm{U}(6)]$, $\hat{C}_2[\mathrm{U}(6)]$ and $\hat{C}_1[\mathrm{U}(6)]\hat{C}_1[\mathrm{U}(5)]$. The most general IBM hamiltonian with up to two-body interactions can thus be written equivalently in terms of Casimir operators. Specifically, the hamiltonian reads

$$\begin{aligned} \hat{H}_{1+2} &= \kappa_1 \hat{C}_1[\mathrm{U}(5)] + \kappa_2 \hat{C}_2[\mathrm{U}(5)] + \kappa_3 \hat{C}_2[\mathrm{SU}(3)] + \\ &\quad \kappa_4 \hat{C}_2[\mathrm{O}(6)] + \kappa_5 \hat{C}_2[\mathrm{O}(5)] + \kappa_6 \hat{C}_2[\mathrm{O}(3)] \\ &\equiv \hat{H}_{1+2}\{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6\}, \end{aligned} \tag{4.8}$$

which is just an alternative way of writing $\hat{H}_1 + \hat{H}_2$ of (4.3) and (4.4) when omitting interactions that contribute to the binding energy only.

The representation (4.8) is much more telling when it comes to the symmetry properties of the IBM hamiltonian. If some of the coefficients κ_i vanish such that $\hat{H}_{1+2}$ contains Casimir operators of subalgebras belonging to a *single* reduction in (4.7),

then, according to the discussion of section 2, the eigenvalue problem can be solved analytically. Three classes of spectrum generating hamiltonians can be constructed of the form

$$\begin{aligned}
\hat{H}_{1+2}[\mathrm{U}(5)] &= \kappa_1\hat{C}_1[\mathrm{U}(5)] + \kappa_2\hat{C}_2[\mathrm{U}(5)] + \kappa_5\hat{C}_2[\mathrm{O}(5)] + \kappa_6\hat{C}_2[\mathrm{O}(3)], \\
\hat{H}_{1+2}[\mathrm{SU}(3)] &= \kappa_3\hat{C}_2[\mathrm{SU}(3)] + \kappa_6\hat{C}_2[\mathrm{O}(3)], \\
\hat{H}_{1+2}[\mathrm{O}(6)] &= \kappa_4\hat{C}_2[\mathrm{O}(6)] + \kappa_5\hat{C}_2[\mathrm{O}(5)] + \kappa_6\hat{C}_2[\mathrm{O}(3)].
\end{aligned} \tag{4.9}$$

The dynamical symmetries of the IBM thus arise if combinations of certain coefficients κ_i vanish. The converse, however, cannot be said: even if all parameters κ_i are nonzero, the hamiltonian $\hat{H}_{1+2}$ still can exhibit a dynamical symmetry and be analytically solvable. This is a consequence of the existence of unitary transformations $\hat{u}$ which preserve the eigenspectrum of the hamiltonian $\hat{H}_{1+2}$ (and hence its analyticity properties) and which can be represented as transformations in the parameter space $\{\kappa_i\}$. It can be shown [88] that two such transformations exist for the IBM:

(i) The transformation associated with $\hat{u}_1^{-1}s^\dagger\hat{u}_1 = s^\dagger$ and $\hat{u}_1^{-1}d^\dagger\hat{u}_1 = -d^\dagger$. The hamiltonian (4.8) is transformed into

$$\begin{aligned}
&\hat{u}_1^{-1}\hat{H}_{1+2}\{\kappa_1,\kappa_2,\kappa_3,\kappa_4,\kappa_5,\kappa_6\}\hat{u}_1 \\
&= \hat{H}_{1+2}\{\kappa_1+2\kappa_3, \kappa_2+2\kappa_3, -\kappa_3, \kappa_4+4\kappa_3, \kappa_5-6\kappa_3, \kappa_6+2\kappa_3\}.
\end{aligned} \tag{4.10}$$

(ii) The transformation associated with $\hat{u}_2^{-1}s^\dagger\hat{u}_2 = s^\dagger$ and $\hat{u}_2^{-1}d^\dagger\hat{u}_2 = i\,d^\dagger$ which can be defined if $\kappa_3 = 0$. Up to a constant contribution, the hamiltonian (4.8) is transformed into

$$\begin{aligned}
&\hat{u}_2^{-1}\hat{H}_{1+2}\{\kappa_1,\kappa_2,0,\kappa_4,\kappa_5,\kappa_6\}\hat{u}_2 \\
&= \hat{H}_{1+2}\{\kappa_1+2(N+2)\kappa_4, \kappa_2-4\kappa_4, 0, -\kappa_4, \kappa_5+2\kappa_4, \kappa_6\}.
\end{aligned} \tag{4.11}$$

The transformations $\hat{u}_1$ and $\hat{u}_2$ that leave invariant the eigenspectrum have been known for some time [89, 90] in the specific case of the IBM. The results of Shirokov *et al.* [88] constitute a *systematic* procedure for finding *all* such transformations (or *parameter symmetries*) which can, in fact, be applied to any hamiltonian describing a system of interacting bosons and/or fermions. The enumeration of all symmetries of a hamiltonian system [which includes symmetries obvious from reductions such as (4.7) but also *hidden symmetries* revealed through parameter transformations] is important for a proper understanding of its chaoticity character [91, 92].

As implied by the reduction (4.7), the only subalgebras of U(6) that contain the angular momentum O(3) algebra are U(5), SU(3), O(6) and O(5). Other algebras that are no subalgebras of U(6) can, however, be represented not as linear combinations of the generators $b^\dagger_{lm}b_{l'm'}$ of U(6) but in terms of the creation and annihilation operators $b^\dagger_{lm}$ and $b_{l'm'}$ themselves. This idea can be used to find the image of various fermion

symmetries (such as those discussed in subsection 3.5) in terms of the s and d bosons of the IBM [93]. The boson hamiltonian found in this way generally is not hermitian but can be made so through a similarity transformation. The end result is that a hermitian IBM hamiltonian is found with eigenvalues that exactly correspond to those of the original fermion hamiltonian. This is a somewhat artificial result in the sense that, although the eigenenergies of a particular fermion symmetry are reproduced exactly, the wavefunction must necessarily correspond to one of the three IBM limits. For example, the eigenenergy expression of the O(7) limit of the Ginocchio's O(8) model [62] can be reproduced with an IBM hamiltonian with a U(5) dynamical symmetry through a complicated N dependence of the coefficients κ_i. This correspondence can be extended to other observables by considering higher-order terms in the relevant operators in IBM [93].

It is argued in section 2 that the concept of a dynamical symmetry can be viewed as a generalisation and refinement of that of a symmetry. Yet a further enlargement can be proposed which leads to the idea of a *partial dynamical symmetry.* As the name suggests it concerns hamiltonians of which only a subset of eigenstates is analytically solvable. The procedure for finding all partial dynamical symmetries can be stated as a general algorithm [94]: given a dynamical symmetry, its associated partial dynamical symmetries can be constructed through a tensor decomposition of the interaction. The existence or non-existence of partial dynamical symmetries is crucially related to the order of the interactions. The example of the partial SU(3) dynamical symmetry is discussed below but it must be emphasised that this is only a specific illustration of the generic procedure outlined in [94].

While many if not all of the properties of two-body boson interactions are understood, geometrically as well as algebraically, those of third order are much less known. In the limit of infinite boson number it can be shown [95, 96] that static triaxial shapes (i.e., $\gamma \neq 0°$ and $\gamma \neq 60°$) may occur with cubic interactions, which is not the case for lower orders. Furthermore, a few phenomenological studies are available where a specific (and to some extent arbitrary) cubic interaction in the d bosons is used to improve energy fits [97, 98]. A systematic understanding of the parameter space of the three-body interactions is still very much an open problem. This might become increasingly relevant since with recent advances in experimental detection techniques it is possible to probe ever higher multi-phonon structures in vibrational [99] as well as rotational [100] nuclei, which are particularly sensitive to higher-order interactions. Besides their relevance to phenomenological IBM studies, higher-order interactions may yield some surprises when it comes to symmetry properties of the IBM as example 2 illustrates.

Example 1: Partial SU(3) dynamical symmetry

The starting point in this example is the SU(3) hamiltonian in (4.9), the eigenstates of which can be classified according to

$$\begin{array}{ccccccc} \mathrm{U}(6) & \supset & \mathrm{SU}(3) & \supset & \mathrm{O}(3) & \supset & \mathrm{O}(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [N] & & (\lambda\mu) & & KL & & M_L \end{array}, \tag{4.12}$$

or $|[N](\lambda\mu)KLM_L\rangle$ in short. The structure of the spectrum with SU(3) dynamical symmetry is illustrated in the left-hand panel of figure 11 where the lowest (ground, β and γ) rotational bands are shown. This spectrum is obtained with two Casimir

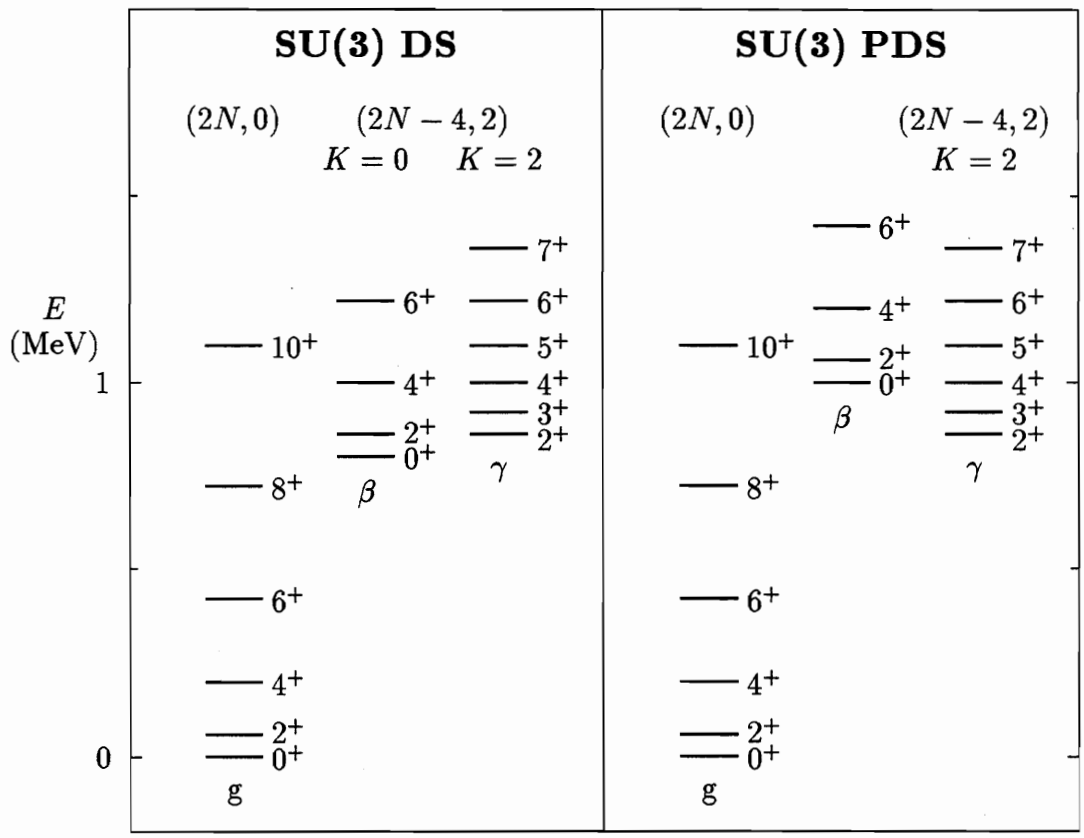

Figure 11. Partial eigenspectrum of a hamiltonian with SU(3) dynamical symmetry (left) and with partial SU(3) dynamical symmetry (right).

operators, $\hat{C}_2[\mathrm{SU}(3)]$ and $\hat{C}_2[\mathrm{O}(3)]$; any additional one- or two-body term leads to a non-solvable hamiltonian. It is, however, possible to find an interaction that preserves the analyticity of *some* of the states.

To construct this interaction, it is first necessary to perform an SU(3) tensor decomposition of the two-body interaction. The two-boson operators $b^\dagger_{lm}b^\dagger_{l'm'}$ can be

coupled to definite SU(3) quantum numbers,

$$B^{\dagger}_{(\lambda\mu)LM_L} \propto \sum_{ll'} \langle (20)l\ (20)l' | (\lambda\mu)L \rangle (b^{\dagger}_{l} \times b^{\dagger}_{l'})^{(L)}_{M_L}, \tag{4.13}$$

where $\langle \cdot\cdot\ \cdot\cdot | \cdot\cdot \rangle$ is a generalised coupling coefficient associated with SU(3) $\supset$ O(3) [see (2.11)]. Expressions for such coefficients are given in [101] whence the normalised two-boson SU(3) tensors shown in table 6 are derived. Any interaction in $\hat{H}_2$ between

Table 6. Normalised two-boson SU(3) tensors $B^{\dagger}_{(\lambda\mu)LM_L}$.

$B^{\dagger}_{(40)00}$	=	$\sqrt{\frac{5}{18}}(s^{\dagger} \times s^{\dagger})^{(0)}_{0} + \sqrt{\frac{2}{9}}(d^{\dagger} \times d^{\dagger})^{(0)}_{0}$
$B^{\dagger}_{(40)2\mu}$	=	$\sqrt{\frac{7}{9}}(s^{\dagger} \times d^{\dagger})^{(2)}_{\mu} - \sqrt{\frac{1}{9}}(d^{\dagger} \times d^{\dagger})^{(2)}_{\mu}$
$B^{\dagger}_{(40)4\mu}$	=	$\sqrt{\frac{1}{2}}(d^{\dagger} \times d^{\dagger})^{(4)}_{\mu}$
$B^{\dagger}_{(02)00}$	=	$\sqrt{\frac{2}{9}}(s^{\dagger} \times s^{\dagger})^{(0)}_{0} - \sqrt{\frac{5}{18}}(d^{\dagger} \times d^{\dagger})^{(0)}_{0}$
$B^{\dagger}_{(02)2\mu}$	=	$\sqrt{\frac{2}{9}}(s^{\dagger} \times d^{\dagger})^{(2)}_{\mu} + \sqrt{\frac{7}{18}}(d^{\dagger} \times d^{\dagger})^{(2)}_{\mu}$

two-boson states $(b^{\dagger}_{l_1} \times b^{\dagger}_{l_2})^{(L)}_{M_L}$ can now be written equivalently as an interaction between the $B^{\dagger}_{(\lambda\mu)LM_L}$; the two-body hamiltonian $\hat{H}_2$ in (4.4) can thus be rewritten as

$$\hat{H}_2 = \sum_{(\lambda\mu)(\lambda'\mu')L} v^{L}_{(\lambda\mu)(\lambda'\mu')} \sqrt{2L+1} \left(B^{\dagger}_{(\lambda\mu)L} \times \tilde{B}_{(\lambda'\mu')L} \right)^{(0)}_{0}, \tag{4.14}$$

in terms of v coefficients that are the interaction matrix elements between normalised two-boson states with good SU(3) labels,

$$v^{L}_{(\lambda\mu)(\lambda'\mu')} \equiv \langle [2](\lambda\mu)LM_L | \hat{H}_2 | [2](\lambda'\mu')LM_L \rangle. \tag{4.15}$$

A hamiltonian with partial SU(3) dynamical symmetry can be constructed by considering a subset of SU(3) interactions with $(\lambda\mu) = (\lambda'\mu') = (02)$, involving the two matrix elements $v^{0}_{(02)(02)}$ and $v^{2}_{(02)(02)}$. The resulting hamiltonian, denoted here as $\hat{H}_2$[pSU(3)],

$$\hat{H}_2[\mathrm{pSU}(3)] = v^{0}_{(02)(02)} B^{\dagger}_{(02)0} \cdot \tilde{B}_{(02)0} + v^{2}_{(02)(02)} B^{\dagger}_{(02)2} \cdot \tilde{B}_{(02)2}, \tag{4.16}$$

has the property that it gives zero acting on specific SU(3) eigenstates such as the ground-band states $|[N](2N,0)LM_L\rangle$,

$$\hat{H}_2[\mathrm{pSU}(3)] | [N](2N,0)LM_L \rangle = 0. \tag{4.17}$$

This is so because the action of $\tilde{B}_{(02)LM_L}$ leads to an $(N-2)$-boson state and none of the representations $(\lambda\mu)$ contained in $[N-2]$ can be coupled with (02) to $(2N,0)$:

$$\{(\lambda\mu) \| (\lambda\mu) \in [N-2] \wedge (\lambda\mu) \times (02) \ni (2N,0)\} = \emptyset. \tag{4.18}$$

The hamiltonian (4.16) can now be rewritten as

$$\hat{H}_2[\mathrm{pSU}(3)] = \nu_1 \left(2\hat{N}(2\hat{N}+3) - \hat{C}_2[\mathrm{SU}(3)]\right) + \nu_2\left(B^\dagger_{(02)0}\cdot\tilde{B}_{(02)0} - B^\dagger_{(02)2}\cdot\tilde{B}_{(02)2}\right), \tag{4.19}$$

where

$$\nu_1 = \tfrac{1}{36}\left(v^0_{(02)(02)} + v^2_{(02)(02)}\right), \qquad \nu_2 = \tfrac{1}{2}\left(v^0_{(02)(02)} - v^2_{(02)(02)}\right), \tag{4.20}$$

and use is made of the identity

$$B^\dagger_{(02)0}\cdot\tilde{B}_{(02)0} + B^\dagger_{(02)2}\cdot\tilde{B}_{(02)2} = \tfrac{1}{18}\left(2\hat{N}(2\hat{N}+3) - \hat{C}_2[\mathrm{SU}(3)]\right). \tag{4.21}$$

The conclusion of this lengthy derivation is that the hamiltonian

$$\hat{H}_{1+2}[\mathrm{pSU}(3)] = \kappa_3\hat{C}_2[\mathrm{SU}(3)] + \kappa_6\hat{C}_2[\mathrm{O}(3)] + \kappa_3'\left(B^\dagger_{(02)0}\cdot\tilde{B}_{(02)0} - B^\dagger_{(02)2}\cdot\tilde{B}_{(02)2}\right), \tag{4.22}$$

has $|[N](2N,0)LM_L\rangle$ as eigenstates. In fact, it can be shown [94] that all states of the type $|[N](2N-2k,k)K=k\ LM_L\rangle, k=0,2,4,\ldots$ are eigenstates of $\hat{H}_{1+2}[\mathrm{pSU}(3)]$; other SU(3) states, however, are not. For the lowest bands of an SU(3) spectrum it means that the ground and the γ bands are analytically solvable but that the β band is not (see right-hand panel of figure 11). The hamiltonian $\hat{H}_{1+2}[\mathrm{pSU}(3)]$ clearly is more general than $\hat{H}_{1+2}[\mathrm{SU}(3)]$ (it contains one more free parameter) but a subset of the former's eigenstates is analytically solvable and coincides with those of the latter.

Example 2: A rotational O(6) limit

A simple but extremely useful parametrisation of the IBM hamiltonian is of the form

$$\hat{H}_{1+2} = \kappa_2\hat{Q}^\chi\cdot\hat{Q}^\chi + \kappa_2'\hat{L}\cdot\hat{L}, \tag{4.23}$$

where $\hat{L}_\mu$ is the boson angular momentum operator and $\hat{Q}^\chi_\mu$ the boson quadrupole operator depending on a parameter χ,

$$\hat{Q}^\chi_\mu = (s^\dagger\times\tilde{d} + d^\dagger\times\tilde{s})^{(2)}_\mu + \chi(d^\dagger\times\tilde{d})^{(2)}_\mu. \tag{4.24}$$

To calculate nuclear E2 transitions, the same $\hat{Q}^\chi_\mu$ can be taken as E2 operator whence the name consistent-Q formalism or CQF [102]. The beauty of CQF is its economy: with just three parameters (κ_2, κ_2' and χ) a wide variety of nuclear structures can be described. In fact, scale-independent energy properties (such as ratios of energies) depend on just two (κ_2/κ_2' and χ), and wavefunctions and E2 transition rates on just one (χ). In addition, two of the three limits of the IBM are recovered naturally in CQF, namely SU(3) for $\chi = \pm\frac{1}{2}\sqrt{7}$ and O(6) for $\chi = 0$.

One may now ask the question how higher-order interactions can be accommodated in CQF and how those will modify the symmetry properties of $\hat{H}_{1+2}$ in (4.23). The most obvious extension is of the form

$$\hat{H}_{1+2+3} = \kappa_2 \hat{Q}^\chi \cdot \hat{Q}^\chi + \kappa_3 (\hat{Q}^\chi \times \hat{Q}^\chi \times \hat{Q}^\chi)^{(0)} + \kappa_2' \hat{L} \cdot \hat{L}, \tag{4.25}$$

which introduces an additional parameter κ_3. For $\chi = \pm\frac{1}{2}\sqrt{7}$, the hamiltonian $\hat{H}_{1+2+3}$ has SU(3) symmetry; the wavefunctions of the states remain unchanged with respect to the SU(3) limit of $\hat{H}_{1+2}$ but the spectrum can be made anharmonic in the β and γ vibrations with an appropriate choice of κ_2 and κ_3. For $\chi = 0$, O(6) is still a symmetry of $\hat{H}_{1+2+3}$ but in this case the structure of the wavefunctions depends on the ratio κ_3/κ_2. The reason is that, while $\hat{Q}^0 \cdot \hat{Q}^0$ preserves O(5) symmetry, $(\hat{Q}^0 \times \hat{Q}^0 \times \hat{Q}^0)^{(0)}$ does not. As a result, the geometric analogue of the cubic term is *not* γ unstable but rather axially symmetric.

The spectral properties of the hamiltonian (4.23) are illustrated in figure 12 for $\kappa_2 = 0$, which thus corresponds to the extreme form of the rotational O(6) limit. Only the lowest bands of the O(6) multiplet with $\sigma = N$ are shown; they compare rather well with the observed spectrum of ${}^{162}_{66}\mathrm{Dy}_{96}$ [28]. This agreement extends to E2 transitions from the $K^\pi = 2^+$ γ band to the $K^\pi = 0^+$ ground band, which are shown in figure 13 by way of a so-called Michailov plot [103]. The B(E2) value for a transition $K_\mathrm{i}J_\mathrm{i} \to K_\mathrm{f}J_\mathrm{f}$ is divided by its geometric estimate $\langle J_\mathrm{i}K_\mathrm{i}\ 2K_\mathrm{f} - K_\mathrm{i} | J_\mathrm{f}K_\mathrm{f}\rangle^2$ and the square root of this ratio is plotted as a function of $J_\mathrm{i}(J_\mathrm{i}+1) - J_\mathrm{f}(J_\mathrm{f}+1)$. The slope of the line thus obtained is indicative of K mixing: a horizontal line corresponds to no mixing and requires rigid axial symmetry, a steep slope indicates large K mixing. For finite boson number no IBM hamiltonian ever corresponds to a pure rigid rotor because of finite-N fluctuations. Figure 13 provides an illustration of this mechanism, and also shows that the mixing thus obtained is of the order of the one observed [104] in the specific case of ^{162}Dy. Although the hamiltonian (4.25) gives rise to a minimum at $\gamma = 0°$, indicating static, axially symmetric deformation, a non-zero slope is obtained in the Michailov plot for the $\gamma \to$ g transitions. In fact, for $\kappa_2 = 0$ and $\chi = 0$ the slope is independent of the other two parameters κ_3 and κ_2' in the hamiltonian (4.25) and is uniquely determined by N. Since for $N \to \infty$ the plot becomes flat, any K mixing in this example must be entirely due to finite-N fluctuations.

4.2. Protons, neutrons and F spin

The bosons of the IBM are interpreted as correlated pairs of nucleons, that is, pairs of nucleons with many components distributed over single-particle orbits coupled to $J = 0$ (s) and $J = 2$ (d). Since these components are mainly confined to the valence orbits, the number of bosons is taken as half the number of valence particles. Given this microscopic interpretation of the bosons, a natural extension of the simplest version of the IBM is

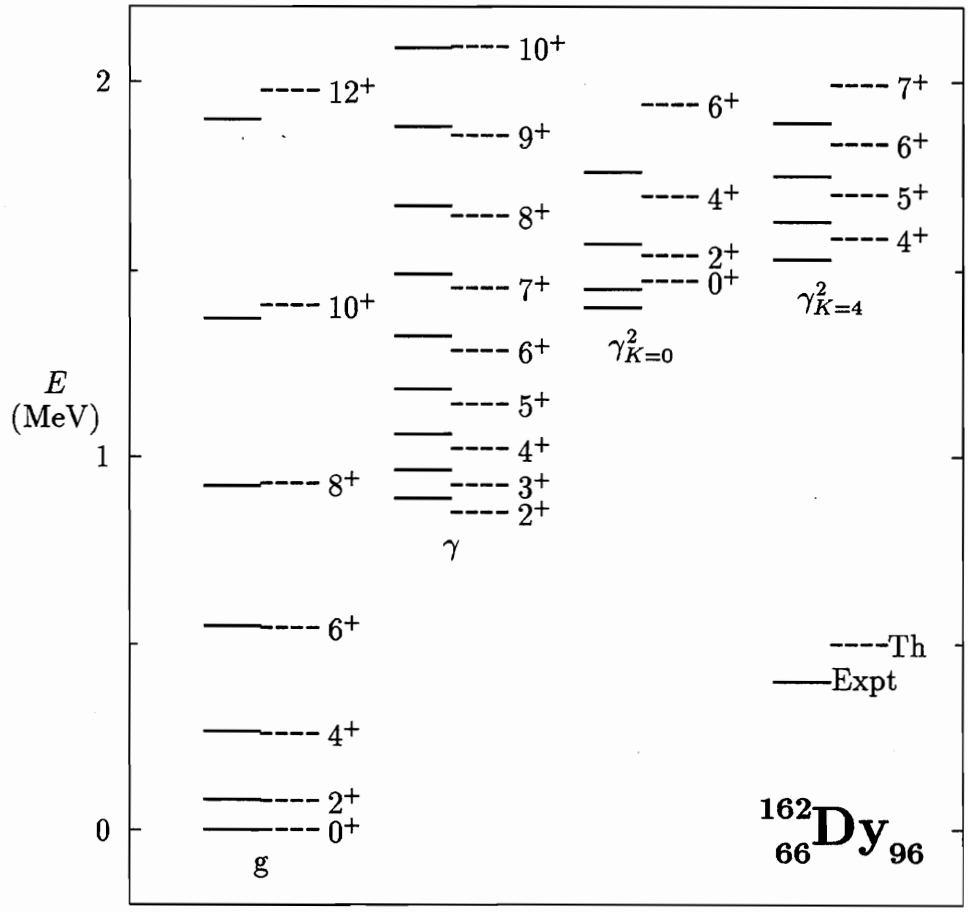

Figure 12. Partial eigenspectrum of the hamiltonian (4.25) with $\kappa_2 = 0$, $\kappa_3 = 2.9$ keV, $\kappa_2' = 8$ keV and $\chi = 0$ for $N = 15$ bosons and its comparison with observed levels of ^{162}Dy. Levels are labelled by their angular momentum and parity J^π, and all belong to the O(6) representation $\sigma = N$.

to assume two different types of bosons, proton and neutron bosons, giving rise to the proton–neutron interacting boson model or IBM-2 [105, 106]. The total number of bosons N is the sum of the proton and neutron boson numbers, N_π and N_ν, which are conserved separately. The appropriate algebraic structure of IBM-2 is a product of U(6) algebras, $\mathrm{U}_\pi(6)\otimes\mathrm{U}_\nu(6)$, consisting of proton $b^\dagger_{\pi lm}b_{\pi l'm'}$ and neutron $b^\dagger_{\nu lm}b_{\nu l'm'}$ generators, respectively. Likewise, the model space of the IBM-2 is the product of symmetric representations $[N_\pi] \times [N_\nu]$ of $\mathrm{U}_\pi(6) \otimes \mathrm{U}_\nu(6)$; in this model space the most general, N_π, N_ν-conserving, rotationally invariant IBM-2 hamiltonian must be diagonalised.

The IBM-2 gives a successful phenomenological description of low-energy collective properties of virtually all medium-mass and heavy nuclei. A comprehensive review of the model and its implications for nuclear structure can be found in [107], while in [108, 109] its microscopic foundation (in terms of the nuclear shell model) is discussed.

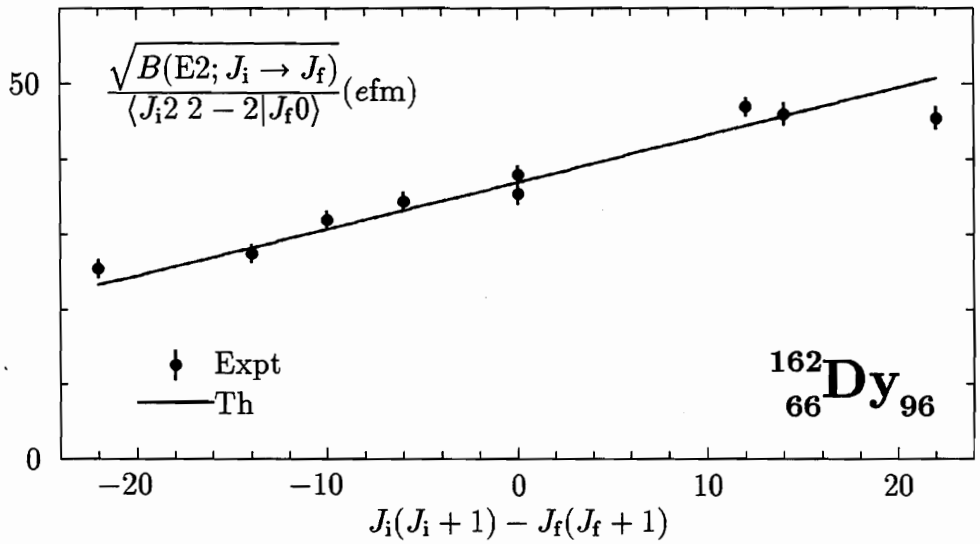

Figure 13. Michailov plot for $\gamma \to$ g E2 transitions in ^{162}Dy. The line is obtained from calculated B(E2) values using the quadrupole operator $\hat{Q}^\chi$ with $\chi = 0$ in the hamiltonian $(\hat{Q}^\chi \times \hat{Q}^\chi \times \hat{Q}^\chi)^{(0)}$ and in the E2 operator. The boson number equals $N = 15$.

The classification and analysis of its symmetry limits is considerably more complex than the corresponding problem in IBM-1 (see, for example, chapter 9 of [21] for an enumeration of all limits). Nevertheless, properties of the most important limits which are of relevance in the analysis of nuclei, are well known from [110]. These points are not repeated here but rather the two main features of IBM-2 are highlighted.

The first important aspect is that the existence of two kinds of bosons offers the possibility to assign an F-spin quantum number to them, $F = \frac{1}{2}$, the boson being in two possible charge states with $F_z = +\frac{1}{2}$ for protons and $F_z = -\frac{1}{2}$ for neutrons [106]. Group theoretically F spin is defined by the reduction

$$\begin{array}{ccccc} \mathrm{U}(12) & \supset & \mathrm{U}(6) & \otimes & \mathrm{U}(2) \\ \downarrow & & \downarrow & & \downarrow \\ [N] & & [N-f, f] & & [N-f, f] \end{array}, \tag{4.26}$$

with $F = \frac{1}{2}[(N-f) - f] = \frac{1}{2}N - f$. The algebra U(12) consists of the generators $b^\dagger_{\rho lm} b_{\rho' l' m'}$, with $\rho, \rho' = \pi$ or ν, which also includes operators that change a proton into neutron or *vice versa* ($\rho \neq \rho'$). Under this algebra U(12) bosons behave symmetrically; as a result the representations of U(6) and U(2) are identical. This should be contrasted with the corresponding situation for fermions where, because of overall *anti*symmetry, representations are conjugate [see, for example, equation (3.29)].

The mathematical structure of F spin is entirely similar to that of isospin. An F-spin SU(2) algebra [which is a subalgebra of U(2) in (4.26)] can be defined which

consists of the diagonal operator $\hat{F}_z = \frac{1}{2}(N_\pi - N_\nu)$ and the ladder operators $\hat{F}_\pm$ that transform a proton boson into a neutron boson or *vice versa*. These are the direct analogues of the isopin generators $\hat{T}_z$ and $\hat{T}_\pm$. The physical meaning of F spin and isospin is different, however: the mapping onto the IBM-2 of a shell-model hamiltonian with isospin symmetry does not necessarily yield an F-spin conserving hamiltonian. Conversely, an F-spin conserving IBM-2 hamiltonian may or may not have eigenstates with good isospin. In fact, if the protons and neutrons occupy different shells, so that the bosons are defined in different shells, then *any* IBM-2 hamiltonian has eigenstates that correspond to shell-model states with good isospin, irrespective of its F-spin symmetry character. If, on the other hand, protons and neutrons occupy the same shell, a general IBM-2 hamiltonian does *not* lead to states with good isospin. The isospin symmetry violation is particularly significant in nuclei with approximately equal numbers of protons and neutrons ($Z \approx N$) and requires the consideration of more elaborate boson models discussed in subsection 4.3. As the difference between the numbers of protons and neutrons in the same shell increases, an approximate equivalence of F spin and isospin is recovered [111] and the need for such elaborate boson models disappears.

Just as isobaric multiplets of nuclei are defined through the connection implied by the raising and lowering operators $\hat{T}_\pm$, *F-spin multiplets* can be defined through the action of $\hat{F}_\pm$ [112]. The states connected are in nuclei with $N_\pi + N_\nu$ constant; these can be isobaric (constant nuclear mass number A) or may differ by multiples of α particles, depending on whether the proton and neutron bosons are of the same or of a different type (which refers to their particle- or hole-like character).

The relation between levels belonging to an F-spin multiplet depends on the F-spin symmetry character of the hamiltonian which comes down to a question of IBM-2 phenomenology. Specifically, a hamiltonian with a *true* F-spin symmetry requires

$$[\hat{H}, \hat{F}_z] = [\hat{H}, \hat{F}_\pm] = 0, \tag{4.27}$$

and implies that all nuclei in the F-spin multiplet have equal binding energies. This is clearly inappropriate and the condition should be relaxed to one of a *dynamical* F-spin symmetry,

$$[\hat{H}, \hat{F}^2] = [\hat{H}, \hat{F}_z] = 0, \tag{4.28}$$

which still implies a classification in terms of F and F_z but now with non-degenerate ground-state energies given by

$$E_{\rm gs}(N_\pi, N_\nu) = \kappa_0 + \kappa_1 F_z + \kappa_2 F_z^2. \tag{4.29}$$

The excitation spectra (i.e., the energies relative to the ground state) of the $F = F_{\rm max}$ states of nuclei belonging to an F-spin multiplet are identical for a hamiltonian satisfying

(4.28). Empirical evidence for F-spin multiplets has thus two aspects: (i) To what extent can ground-state binding energies be described by the *F-spin multiplet mass equation* or FMME (4.29)? (ii) Are low-energy excitation spectra of F-spin multiplet nuclei identical?

The first aspect is illustrated in table 7 where the experimental [113] binding energies of nuclei belonging to an F-spin multiplet with $N_\pi + N_\nu = 12$ are compared to results obtained with (4.29). The FMME is seen to agree with a precision of about 50 keV or

Table 7. Binding energies in an F-spin multiplet with $N_\pi + N_\nu = 12$.

				binding energy (MeV)		
nucleus	N_π	N_ν	F_z	Expt[a]	Error	FMME[b]
$^{156}_{66}Dy_{90}$	8	4	2	1278.025	0.007	1278.025
$^{160}_{68}Er_{92}$	7	5	1	1304.274	0.050	1304.290
$^{164}_{70}Yb_{94}$	6	6	0	#1329.930	0.100	1329.926
$^{168}_{72}Hf_{96}$	5	7	−1	#1354.960	0.100	1354.932
$^{172}_{74}W_{98}$	4	8	−2	#1379.350	0.270	1379.309
$^{176}_{76}Os_{100}$	3	9	−3	#1403.040	0.200	1403.055
$^{180}_{78}Pt_{102}$	2	10	−4	#1426.090	0.200	1426.173
$^{184}_{80}Hg_{104}$	1	11	−5	#1448.710	0.200	1448.660

[a]Energies with # are derived from systematics.

[b]With parameters $\kappa_0 = 1329.926$, $\kappa_1 = -25.321$, $\kappa_2 = -0.315$, in MeV.

better.

The second aspect is illustrated in figure 14 which shows the observed excitation spectra [28] of the nuclei in the same $N_\pi + N_\nu = 12$ F-spin multiplet. With some exceptions levels are seen to be remarkably constant in energy as implied by a dynamical F-spin symmetry. The exceptions concern the first-excited 0^+ level, indicating that this state possibly is of (quasi)particle nature while the IBM can only account for collective excitations involving many nucleons. Similarly, it is seen that the spectrum of $^{184}_{80}Hg_{102}$ strongly differs from that of all other isotopes in the F-spin multiplet; this is a result of the propinquity of the $Z = 82$ shell closure which makes the excitations of that particular isotope less collective.

The phenomenology of F-spin multiplets is similar to that of isobaric multiplets [7] but for one important difference. The nucleon–nucleon interaction favours spatially symmetric configurations and consequently nuclear excitations at low energy generally have $T = T_{\rm min} = |T_z|$; boson–boson interactions also favour spatial symmetry but that leads to low-lying levels with $F = F_{\rm max} = \frac{1}{2}(N_\pi + N_\nu)$. As a result, in the case of an F-spin multiplet a relation is implied between the low-lying spectra of the nuclei in the multiplet while an isobaric multiplet (with $T \geq 1$) involves states at higher excitation

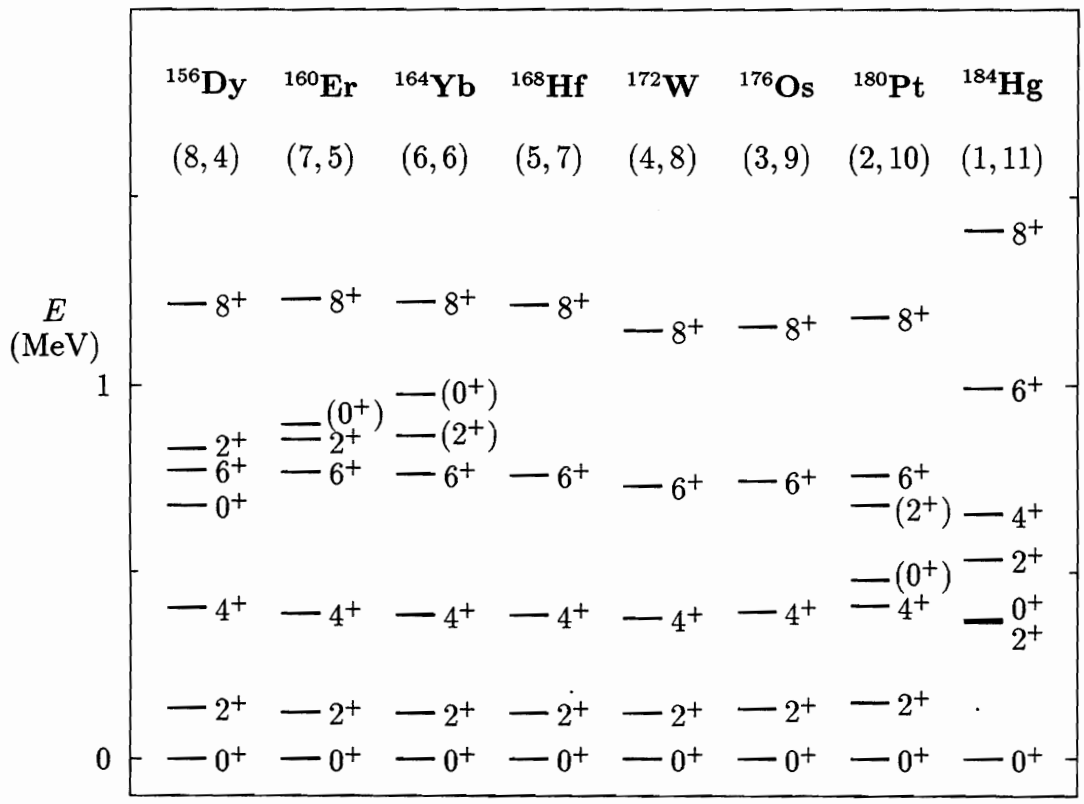

Figure 14. Level schemes of nuclei belonging to an F-spin multiplet with $N_\pi + N_\nu = 12$. Levels are labelled by their angular momentum and parity J^π. Underneath each isotope are given the proton and neutron boson numbers (N_π, N_ν).

energies in some nuclei.

The second important aspect of IBM-2 is that it predicts states which are additional to those found in IBM-1 [114]. Their structure can be understood in terms of the F-spin classification (4.26). States with maximal F spin, $F = \frac{1}{2}N$, are symmetric in U(6) and are the exact analogues of IBM-1 states. The next class of states has $F = \frac{1}{2}N - 1$ and is no longer symmetric in U(6) but corresponds to the representation $[N-1, 1]$. Such states are observed experimentally [115] and seem to be a persistent feature of nuclei [116].

In even–even nuclei the existence of non-symmetric 1^+ states excited in (e, e') or (γ, γ') is by now well established. Their characteristic excitation is of magnetic dipole type. The IBM-2 prediction for the M1 strength towards the 1^+ state is [110]

$$B(\mathrm{M1}; 0_1^+ \to 1_\mathrm{s}^+) = \frac{3}{4\pi}(g_\pi - g_\nu)^2 f(N) N_\pi N_\nu, \tag{4.30}$$

where g_π and g_ν are the boson g factors. The subscript 's' refers to the 'scissors' character of the 1^+ state which is the pictorial image one has of it in the case of deformed nuclei.

The function $f(N)$ is known analytically in the three principal limits of the IBM-2,

$$f(N) = \begin{cases} 0 & \text{U(5)} \\ \dfrac{8}{2N-1} & \text{SU(3)} \\ \dfrac{3}{N+1} & \text{O(6)} \end{cases} . \tag{4.31}$$

Equation (4.30) gives a simple and reasonably accurate estimate of the total M1 strength of orbital nature to 1^+ states in nuclei. As schematically illustrated in figure 15, this M1 strength is observed around an excitation energy of 3 MeV but typically is fragmented into several levels.

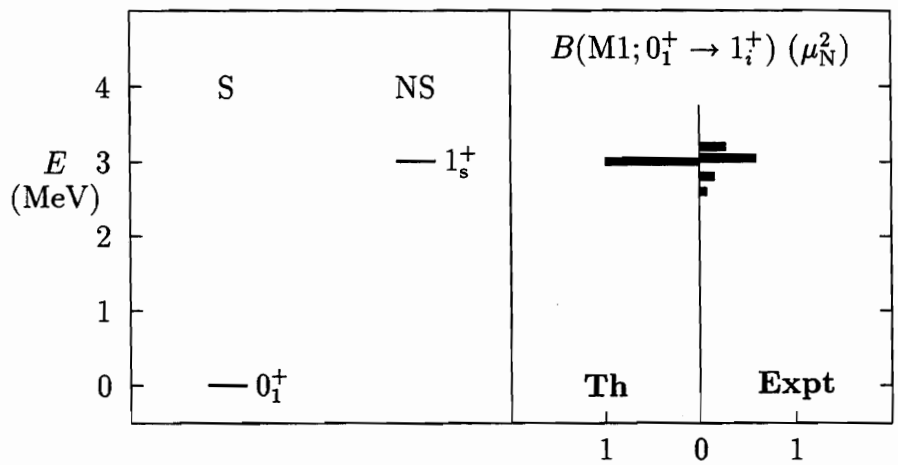

Figure 15. Schematic representation of observed and calculated scissors M1 strength in a typical deformed nucleus.

The geometric interpretation of non-symmetric states can be found by taking the limit of large boson number [117]. From this analysis emerges that they correspond to linear or angular displacement oscillations in which the protons and neutrons are out of phase, in contrast to the symmetric IBM-2 states for which such oscillations are in phase. The occurrence of such states was first predicted in the context of geometric two-fluid models in vibrational [118] and deformed [119] nuclei in which they appear as proton–neutron counter oscillations. The IBM-2 thus confirms these geometric descriptions but at the same time generalises them to *all* nuclei, not only spherical and deformed, but γ unstable and transitional as well.

The IBM-2 description of non-symmetric states has been seen to agree with the data, sometimes to a remarkable extent. Figure 16 shows the example of ${}^{94}_{42}\mathrm{Mo}_{52}$ and compares recently measured E2 and M1 transition rates in this nucleus [120] with a simple IBM-2 calculation in the O(6) limit with (dynamical) F-spin symmetry. As the structure of the wavefunctions is fixed by the dynamical symmetry, the calculated

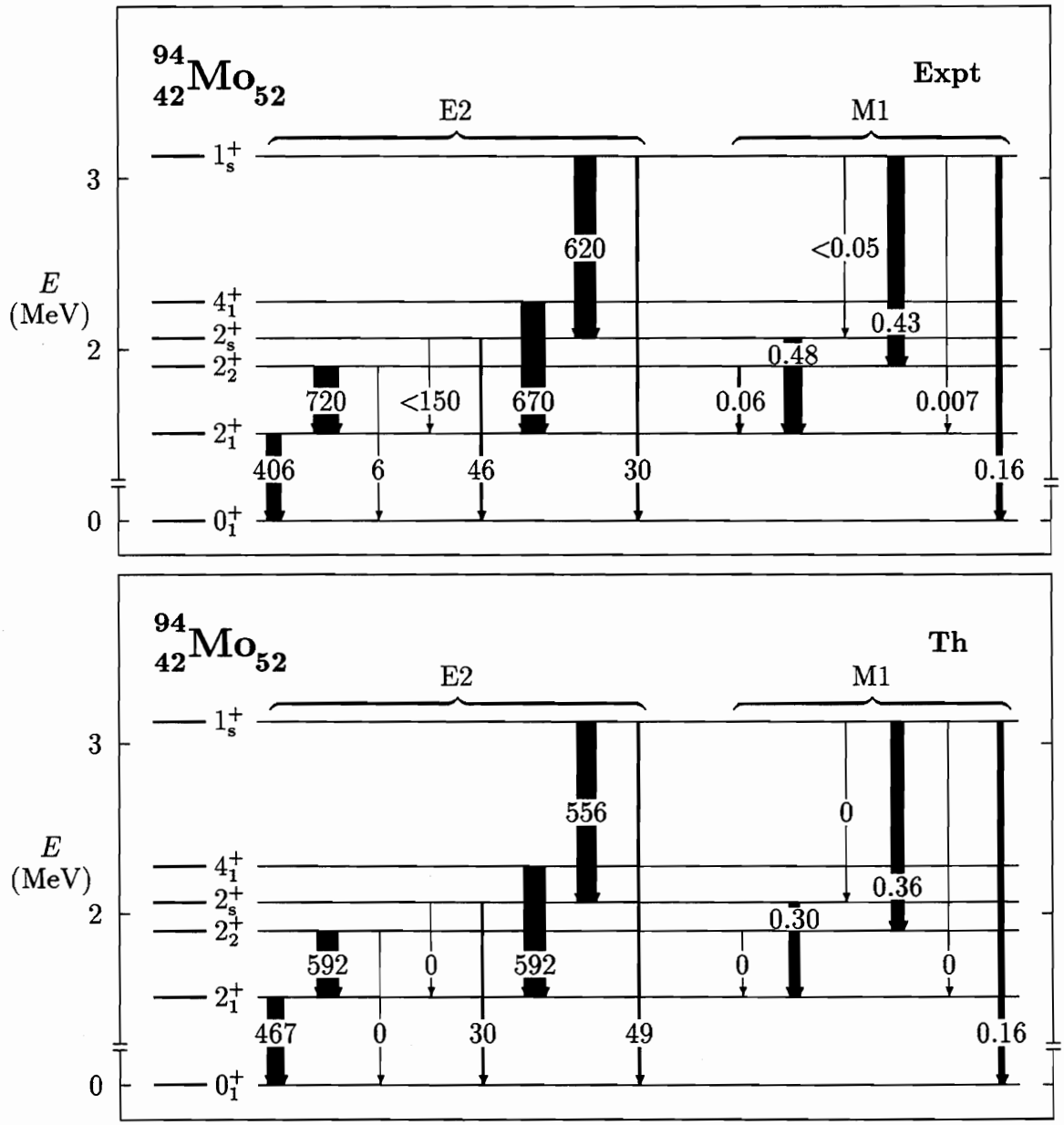

Figure 16. Observed and calculated E2 and M1 transition rates in ^{94}Mo. Levels are labelled by their angular momentum and parity J^π. The B(E2) values are in units of $e^2\text{fm}^4$; the B(M1) values are in units of μ_N^2.

transition matrix elements depend solely on the electromagnetic operators, which in this particular example involve an effective proton boson charge, $e_\pi = 9\ e\text{fm}^2$, and an effective proton boson gyromagnetic ratio, $g_\pi = 1\ \mu_\text{N}$, the latter value being suggested by microscopic considerations.

Many of the ideas and algebraic techniques developed for IBM-2 are applicable to two-component systems in general. Numerous applications in physics exist; two nuclear physics examples are discussed in the following subsections.

Example 1: Particles, holes and I spin

The microscopic interpretation of the bosons of the IBM is one of correlated pairs of particles in the valence shell of the nucleus. Consequently, all elementary versions of the model (IBM-1,2,3,4) provide a description only of (collective) excitations of particles in the valence shell and assume a totally inert core. In many nuclei this assumption is not justified in the sense that core excitations do occur at a relatively low energy comparable to that of valence excitations. This situation arises in particular in nuclei where one type of nucleon has a closed or almost closed-shell configuration while the other type is at mid shell.

Consider as an example ${}^{116}_{50}\text{Sn}_{66}$. This nucleus is magic in the protons ($Z = 50$) and is exactly in between the neutron closed-shell configurations $N = 50$ and $N = 82$. With ${}^{100}_{50}\text{Sn}_{50}$ as inert core, valence excitations correspond to rearrangements of the neutrons in the 50–82 shell. It is, however, well established [121] that this nucleus exhibits a two-particle–two-hole (2p–2h) $J^\pi = 0^+$ excitation of the *protons* at $E_\text{x} = 1.757$ MeV (the second-excited state above the $J^\pi = 2^+$ level at 1.294 MeV), which corresponds to a core-excited or *intruder* configuration. The characteristic feature of this nucleus—rather typical for nuclei in which one type of nucleon has a closed or almost closed shell configuration—is that the core-excited states occur at an energy comparable to that of the usual valence excitations.

A complete description of the low-energy states of such nuclei should thus include both particle excitations in the valence shells as well as hole excitations in the core shells. In even–even nuclei these will predominantly occur as pair excitations and, as a result, the situation can be described neatly in the IBM through the introduction of two types of bosons: particle bosons (particle pairs in the valence shells) and hole bosons (hole pairs in the core shells). One is thus again confronted with a system of interacting bosons of two types to which the formalism of IBM-2 can be applied. Specifically, one may assign an I-spin quantum number to the bosons, $I = \frac{1}{2}$, with $I_z = -\frac{1}{2}$ for a particle boson and $I_z = +\frac{1}{2}$ for a hole boson [122]. The F-spin formalism of subsection 4.2 can be adopted in its entirety to deal with I spin: the I-spin invariance and symmetry of the IBM hamiltonian can be investigated, I-spin multiplets can be defined, etc. As

an example, the action of the I-spin raising and lowering operators $\hat{I}_\pm$ (transforming a particle into a hole boson or *vice versa*) is illustrated in figure 17 and compared to the corresponding actions of $\hat{F}_\pm$ which transform a neutron into a proton boson or *vice versa*. In both cases the total number of bosons is conserved; in an F-spin multiplet

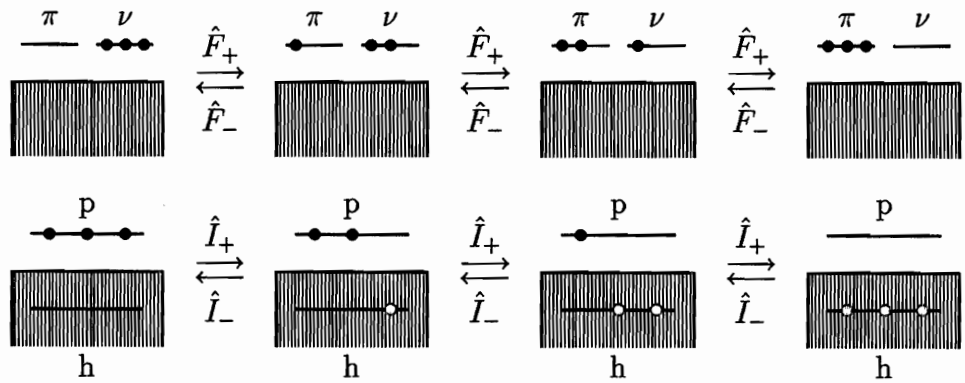

Figure 17. Schematic illustration of the action of the F-spin and I-spin raising and lowering operators on a system of $N = 3$ bosons, where N is either the number of proton plus neutron bosons, $N = N_\pi + N_\nu$, or the number of particle plus hole bosons, $N = N_{\rm p} + N_{\rm h}$.

this sum is made up from proton and neutron bosons while in an I-spin multiplet it consists of particle and hole bosons.

In spite of the formal equivalence between F-spin and I-spin multiplets, there is one difference in their application which can be made clear from figure 17. Action of $\hat{F}_\pm$ on the ground state of a nucleus in an F-spin multiplet leads to the ground state of another member of the multiplet. This is not necessarily so in an I-spin multiplet: the action of $\hat{I}_+$ on the three particle bosons on the left hand side leads to a 2p–1h boson state which is an excited configuration in a nucleus with a pair of nucleons in the valence shell. In this respect I-spin multiplets are akin to isospin multiplets.

An example of an I-spin multiplet with $I = \frac{3}{2}$ is shown in figure 18. It is seen that a (dynamical) I-spin symmetry (which gives rise to identical excitation spectra) is only approximately valid. Deviations arise due to the difference in microscopic structure between the particle and hole pairs that correspond to the bosons.

Example 2: Neutron-skin nuclei as three-component systems

One of the advantages of algebraic methods is the ease with which they can be extended to deal with more complex situations. A nice illustration of this feature concerns the physics of very neutron-rich nuclei, a topic of intense debate in today's nuclear structure research. Microscopic nuclear models predict that such nuclei develop a neutron 'skin'

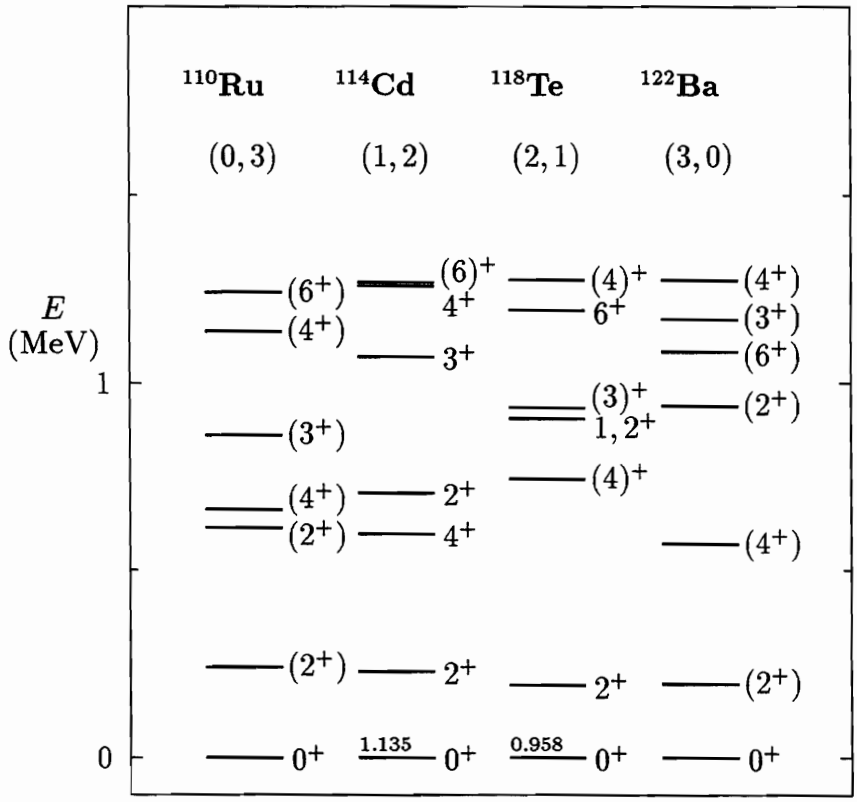

Figure 18. Level schemes of nuclei belonging to an I-spin multiplet with $N_{\rm p}^{\pi}+N_{\rm h}^{\pi}=3$. Levels are labelled by their angular momentum and parity J^{π}. Underneath each isotope are given the particle and hole proton boson numbers $(N_{\rm p}^{\pi}, N_{\rm h}^{\pi})$. The levels are drawn relative to the lowest $I=\frac{3}{2}$ state; in the $I_z=\pm\frac{1}{2}$ nuclei ^{114}Cd and ^{118}Te these are excited states with an energy as indicated on top of the level.

implying that the neutron density extends further than that of the protons. Whether this neutron skin sufficiently decouples from the rest of the nucleus for it to exhibit independent oscillations is not clear at the moment, but if one assumes it does, the consequences can be studied straightforwardly in an algebraic approach [123].

The starting point is a *triple* product of U(6) algebras associated with the protons π, the core neutrons $\nu_{\rm c}$ and the skin neutrons $\nu_{\rm s}$, respectively,

$$\begin{array}{ccccc} {\rm U}_{\pi}(6) & \otimes & {\rm U}_{\nu_{\rm c}}(6) & \otimes & {\rm U}_{\nu_{\rm s}}(6) \\ \downarrow & & \downarrow & & \downarrow \\ [N_{\pi}] & & [N_{\nu_{\rm c}}] & & [N_{\nu_{\rm s}}] \end{array} , \tag{4.32}$$

where each U(6) algebra is characterised by a number N_{ρ} of symmetrically coupled bosons. The fact that the skin neutrons are assumed to interact weakly with the core protons and neutrons, which interact strongly with each other, is represented in the reduction of (4.32) by coupling the corresponding U(6) algebra of the neutron skin *after*

those describing the core nucleons. The reduction thus proceeds as

$$\mathrm{U}_\pi(6) \otimes \mathrm{U}_{\nu_c}(6) \otimes \mathrm{U}_{\nu_s}(6) \supset \mathrm{U}_{\pi\nu_c}(6) \otimes \mathrm{U}_{\nu_s}(6) \supset \mathrm{U}_{\pi\nu_c\nu_s}(6). \tag{4.33}$$

The algebra $\mathrm{U}_{\pi\nu_c\nu_s}(6)$, obtained by summing the generators of the different U(6) algebras, has a subalgebra structure familiar from IBM-1. By virtue of the presence of $\mathrm{U}_{\pi\nu_c\nu_s}(6)$ it is assumed that an identical mixture of U(5), SU(3) and O(6) is valid for all three subsystems π, ν_c and ν_s. More general situations, where the deformations of the subsystems are different, can also be envisaged in an algebraic treatment.

In the reduction (4.33), $\mathrm{U}_{\pi\nu_c}(6)$ is characterised by representations $[N_c - f, f]$, where $N_c = N_\pi + N_{\nu_c}$ is the number of nucleon pairs in the core. The lowest states are contained in $[N_c, 0]$ which denotes the totally symmetric coupling. The lowest non-symmetric states are in the next representation $[N_c - 1, 1]$. The algebra $\mathrm{U}_{\pi\nu_c\nu_s}(6)$ is characterised by up to three rows; the lowest couplings arising from $[N_c, 0] \times [N_{\nu_s}]$ are $[N, 0, 0]$ and $[N-1, 1, 0]$, where N denotes the total number of bosons. Hence the first non-symmetric representation of $\mathrm{U}_{\pi\nu_c\nu_s}(6)$ describes the *symmetric* coupling of the core nucleons and *non-symmetric* coupling of the skin neutrons. However, the non-symmetric representation $[N-1, 1, 0]$ of $\mathrm{U}_{\pi\nu_c\nu_s}(6)$ may also arise from the product $[N_c - 1, 1] \times [N_{\nu_s}]$. In this case, the core nucleons are coupled non-symmetrically. The result is that there are now *two* scissors modes, one representing out-of-phase motion of the protons and neutrons in the core and the other denoting an oscillation of the core against the skin neutrons, a 'soft scissors' mode where the core protons carry the core neutrons with them.

The magnetic dipole strength to the soft scissors state can be obtained by considering the separate contributions to the M1 operator from the core and the skin neutrons. This gives

$$B(\mathrm{M1}; 0_1^+ \to 1_{\mathrm{ss}}^+) = \frac{3}{4\pi}(g_\pi - g_\nu)^2 f(N) \frac{N_\pi^2 N_{\nu_s}}{N_\pi + N_{\nu_c}}, \tag{4.34}$$

where $f(N)$ is the function (4.31). From (4.30) and (4.34) one finds the following simple result for the ratio of $B(\mathrm{M1})$ values to the soft and normal scissors modes:

$$\frac{B(\mathrm{M1}; 0_1^+ \to 1_{\mathrm{ss}}^+)}{B(\mathrm{M1}; 0_1^+ \to 1_{\mathrm{s}}^+)} = \frac{N_\pi N_{\nu_s}}{(N_\pi + N_{\nu_c}) N_\nu}. \tag{4.35}$$

By way of qualitative arguments it can be shown [123] that the soft scissors state is expected to occur at about half the excitation energy of the normal scissors state. The resulting prediction is thus as illustrated schematically in figure 19.

This algebraic analysis can be applied to any three-component system of which the internal degrees of freedom of each component can be described by a unitary algebra $\mathrm{U}(n)$. Many of the features obtained for U(6), such as the appearance of two classes of non-symmetric $[N-1, 1, 0]$ states and the associated fragmentation of transition

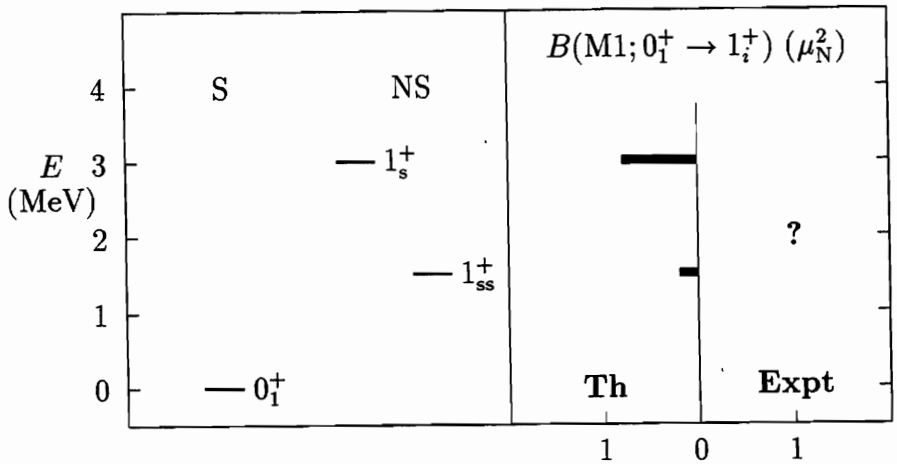

Figure 19. Schematic representation of calculated normal and soft scissors M1 strength in a very neutron-rich deformed nucleus. There is at present no experimental confirmation of this conjectured mode.

strength, remain valid generally. Also, the generalisation towards non-compact algebras can be considered with the purpose of describing 'borromean' systems, that is, systems that consist of pairwise unbound components but which themselves are bound.

4.3. Isospin invariant forms of the interacting boson model

The IBM-2 is applicable to medium-mass and heavy nuclei, more specifically, to nuclei where protons and neutrons occupy different valence shells. In such nuclei it suffices to assume correlated proton–proton and neutron–neutron pairs, and to include the effect of the proton–neutron interaction by means of a quadrupole term in the hamiltonian. When protons and neutrons occupy the same valence shell, this approach is no longer valid since there is no reason not to include the proton–neutron pair with isospin $T = 1$. If such pairs are included, an extended version of the model is obtained which is referred to as IBM-3 [124].

The IBM-3 has three kinds of bosons as 'elementary' building blocks: the usual proton (π) and neutron (ν) bosons, which correspond to fermion pairs with isospin $t = 1$ and projection $m_t = \pm 1$, and the proton–neutron (δ) boson with $t = 1$ and $m_t = 0$. Just as π and ν bosons, the δ boson is assumed to occupy two angular momentum states, $l = 0$ and $l = 2$. There are thus three kinds of bosons each with six components and, as a result, an N-boson state belongs to the symmetric representation $[N]$ of U(18). It is possible to construct IBM-3 states that have good total angular momentum (denoted

by L, as in IBM-1 and IBM-2) and good total isospin T via the classification

$$\begin{array}{ccccccc} \mathrm{U}(18) & \supset & \big(\mathrm{U}(6) & \supset \cdots \supset & \mathrm{O}(3)\big) \\ \downarrow & & \downarrow & & \downarrow \\ [N] & & [N_1N_2N_3] & & L \end{array} ,$$

$$\otimes \quad \begin{array}{ccccc} \big(\mathrm{U}(3) & \supset & \mathrm{SU}(3) & \supset & \mathrm{O}(3)\big) \\ \downarrow & & \downarrow & & \downarrow \\ [N_1N_2N_3] & & (\lambda\mu) & & T \end{array} , \tag{4.36}$$

where the dots refer to one of the possible reductions of U(6). Overall symmetry of the N-boson wavefunction requires the representations of U(6) and U(3) to be identical. Consequently, the allowed U(6) representations can have up to three rows, in contrast to IBM-1 where the representations necessarily are symmetric, and IBM-2 where only up to two-rowed representations are allowed. The SU(3) representations are denoted in Elliott's notation, $\lambda = N_1 - N_2, \mu = N_2 - N_3$, and determine the allowed values of the isospin T of the bosons. Thus the choice of a particular spatial boson symmetry $[N_1N_2N_3]$ determines the allowed isospin values T.

The elementary bosons of IBM-3 are summarised in table 8, together with those of IBM-4 discussed below. The essential property is that complete multiplets are present

Table 8. Enumeration of bosons in IBM-3 and IBM-4.

angular momentum				isospin	
'orbital'	'spin'	total	isospin	projection	operator
l	s	j	t	m_t	
0,2	0	0,2	1	+1	$s_\pi^\dagger, d_\pi^\dagger$
0,2	0	0,2	1	0	$s_\delta^\dagger, d_\delta^\dagger$
0,2	0	0,2	1	−1	$s_\nu^\dagger, d_\nu^\dagger$
0	1	1	0	0	$\theta_1'^\dagger$ $(^3s_1)$
2	1	1	0	0	$\theta_1^\dagger$ $(^3d_1)$
2	1	2	0	0	$\theta_2^\dagger$ $(^3d_2)$
2	1	3	0	0	$\theta_3^\dagger$ $(^3d_3)$

both in angular momentum and isospin space, allowing the construction of operators with good tensor character in the associated quantum numbers.

The restoration of the isospin quantum number in the boson space has the advantage that the boson hamiltonian acquires a simpler form. Without isospin (dynamical) symmetry the introduction of the δ boson would dramatically increase the number of interactions, just as this number increases from IBM-1 to IBM-2. The condition of isospin symmetry reduces the number of independent matrix elements and, as a consequence, IBM-2 and IBM-3 have comparable numbers of interactions.

A second, more important consequence of isospin symmetry in IBM-3 concerns its mapping onto the shell model. In the mapping procedure which is most commonly used, known as OAI mapping [125], the central idea is to identify the various quantum numbers in the two models and to connect, on this basis, states in both models. The great advantage of IBM-3 (as compared to IBM-1 or IBM-2) is that, besides angular momentum and seniority, also isospin can be used in establishing a mapping. For small numbers of particles these three quantum numbers uniquely characterise all states of interest and hence define a one-to-one correspondence between shell model and IBM-3. Given an input shell-model hamiltonian, its matrix elements between states with good angular momentum, seniority and isospin can be calculated and equated to corresponding matrix elements in the boson space. In this way a boson hamiltonian is deduced from a shell-model interaction. Several studies of this type (mostly carried out by Elliott and coworkers) are reviewed in [126]. They can be regarded as tests of the boson approximation in terms of the shell model with applications to nuclei in the $1f_{7/2}$ shell [127] as well as isotopes just beyond $^{56}_{28}\mathrm{Ni}_{28}$ [128].

The classification of dynamical symmetries of IBM-3, of which (4.36) is only an example, is rather complex and as yet their analysis is incomplete. The cases with dynamical SU(3) charge symmetry [corresponding to (4.36)] are studied in detail in [129]. Other classifications that conserve L and T [but not charge SU(3)] are proposed and analysed in [130, 131].

In IBM-3 all bosons have $t = 1$ and no $t = 0$ bosons are considered in the construction of the basis. Comparisons with shell-model calculations show this to be adequate for even–even but insufficient for odd–odd nuclei. In the latter case one needs to include some of the $t = 0$ bosons. A possible way to do so is to assume that $t = 0$ bosons can be in an (orbital) s or d state but in addition assign to them an intrinsic spin $s = 1$ [132]. The total set of bosons in IBM-4 consisting of these $t = 0$ bosons together with the usual ones with $t = 1$ which are now assigned an intrinsic spin of $s = 0$, is shown in table 8. The conventional spectroscopic notation $^{2s+1}l_j$ is also indicated.

There are several reasons for choosing the $t = 0$ bosons in this particular fashion. One justification is found in the LS coupling limit for fermions where the two-particle states of lowest energy have orbital angular momenta $L = 0$ and $L = 2$ with $T = 1$, $S = 0$ or $T = 0$, $S = 1$ (see table 1). In addition, the choice of bosons in IBM-4 allows a boson classification containing Wigner's supermultiplet algebra SU(4). These qualitative arguments in favour of IBM-4 have been corroborated quantitatively by the studies of Halse *et al.*, who have shown IBM-4 to be a good approximation both in even–even [133] and odd–odd [134] *sd*-shell nuclei.

The IBM-4 classification that conserves 'orbital' angular momentum L, 'intrinsic'

spin S and isospin T, reads

$$\begin{array}{ccccccccc} \mathrm{U}(36) & \supset & \big(\mathrm{U}(6) & \supset & \cdots & \supset & \mathrm{O}(3)\big) & & \\ \downarrow & & \downarrow & & & & \downarrow & & \\ [N] & & [N_1 \ldots N_6] & & & & L & & \\ & \otimes & \big(\mathrm{U}(6) & \supset & \mathrm{SU}(4) & \supset & \mathrm{O}(3) & \otimes & \mathrm{O}(3)\big) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & [N_1 \ldots N_6] & & (\lambda\mu\nu) & & S & & T \end{array} \quad . \tag{4.37}$$

To analyse this coupling scheme, it is helpful to compare it to the *fermion* supermultiplet classification, which is given in table 1 for one and two particles in the *sd* shell. The corresponding problem for *bosons* is worked out in table 9. The essential point to

Table 9. Classification of one and two boson(s) in IBM-4.

boson number	spatial symmetry	L	spin-isospin symmetry	$(\lambda\mu\nu)$	(S,T)
1	$\square$	$0,2$	$\square$	(010)	(0,1) (1,0)
2	$\square\square$ (S)	$0^2, 2^2, 4$	$\square\square$ (S)	(020)	(0,0) (0,2) (1,1) (2,0)
				(000)	(0,0)
	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$ (A)	$1,2,3$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$ (A)	(101)	(0,1) (1,0) (1,1)

Note: S stands for symmetric, A for antisymmetric.

note here is that, of the two SU(4) representations (010) and (200) that occur for two fermions, the first one is also the fundamental (or one-boson) representation of SU(4). The starting point of the IBM-4 is thus a truncation of the shell model which preserves the labels $(\lambda\mu\nu)$, S, L, J and T of a fermion LS coupling scheme. These quantum numbers can be used to establish a mapping between the shell model and IBM-4.

A crucial aspect of the IBM-4 is that it does not require the exact validity of such quantum numbers in the shell model for carrying out the mapping to the boson space. An illustrative example of this flexibility is provided by the pseudo LS coupling scheme developed in subsection 3.4. Neither the orbital angular momentum L nor the spin S are good quantum numbers in this coupling scheme; nevertheless, the *pseudo* orbital angular momentum $\tilde{L}$ and the *pseudo* spin $\tilde{S}$ are conserved and *they* can be mapped onto the corresponding boson classification (4.37). This provides an explicit example in which the labels L and S in (4.37) do *not* correspond to the physical orbital angular momentum and intrinsic spin. In particular, the l and s associated with a single boson should *not* be thought of as the orbital angular momentum and the spin of a pair of fermions but rather as effective labels in fermion space which acquire an exact significance in terms

of bosons. A similar argument holds for the Wigner supermultiplet labels $(\lambda\mu\nu)$. In contrast, the label T in (4.36) and (4.37) does correspond to the total isospin, and this is so as long as isospin is an exact dynamical symmetry in the fermion space. The feature of carrying over broken symmetries of a fermion model into exact (dynamical) symmetries of a boson model—of which the IBM-4 application is an example—arguably is one of the most important arguments in favour of boson models. A general understanding of this process of symmetry restoration after mapping is, however, still lacking at present.

5. Symmetries of interacting boson–fermion models

One particularly important extension of the IBM concerns odd-mass nuclei, achieved by considering, in addition to the bosons, a fermion coupled to the core through an appropriate boson–fermion interaction [69, 135]. The resulting interacting boson–fermion model (IBFM) lends itself very well to a study based on symmetry considerations whereby certain classes of model hamiltonians can be solved analytically. Many of these cases—comprehensively reviewed in [136]—have been analysed in detail during the eighties but lately the activity in this field has subsided somewhat. This is partly due to a lack of understanding of the boson–fermion interaction that simulates exchange effects between the odd particle and the core. This problem notwithstanding the IBFM has proven to be a useful approach for the treatment of odd-mass nuclei. A particularly attractive feature is the similarity in the description of even–even and odd-mass nuclei which has given rise to the development of a supersymmetric model, discussed in subsection 5.2. In view of the dearth of recent advances in the IBFM the subject is only summarily treated here.

5.1. The interacting boson–fermion model for odd-mass nuclei

If one omits technical details, the algebraic treatment of odd-mass nuclei in the context of the IBFM proceeds along the same lines as the one of even–even nuclei with the IBM. First, the algebra of the model must be identified. For the bosons this algebra remains $\mathrm{U^B}(6)$, where the superscript B is added to indicate its boson character. The fermion algebra is $\mathrm{U^F}(\Omega)$ where Ω is the degeneracy of the orbits available to the fermion (e.g., $\Omega = 4$ for a $j = \frac{3}{2}$ orbit, $\Omega = 12$ for $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, etc.). Since the bosons and fermions are assumed to commute, the total algebra of an odd-mass nucleus is $\mathrm{U^B}(6) \otimes \mathrm{U^F}(\Omega)$. (The error resulting from the assumption of commuting bosons and fermions is corrected approximately by a boson–fermion exchange interaction.) Next, degeneracy breaking terms are introduced by considering Casimir operators of subalgebras of the total algebra, just as this is done for even–even nuclei.

Details of this approach as applied to odd-mass nuclei are not given here but rather illustrated with the example of the nucleus ${}^{195}_{78}\mathrm{Pt}_{117}$.

Example: Excitation energies of and electromagnetic transitions in ^{195}Pt

The nucleus ^{195}Pt is situated in a spherical-to-deformed transitional region. Its core, ^{196}Pt, can be reasonably well described as an O(6) nucleus. The dominant natural-parity orbits for the neutrons are $3p_{1/2}$, $3p_{3/2}$ and $2f_{5/2}$, which can be decomposed into pseudo orbital angular momenta $\tilde{l} = 0$ and $\tilde{l} = 2$, and pseudo spin $\tilde{s} = \frac{1}{2}$.

Because of the similarity in the angular momentum structure of the bosons and that of the pseudo-orbital part of the fermion, an analytical decomposition is possible [137]. The corresponding hamiltonian contains five parameters which must be adjusted to reproduce the experimental energies [138, 139]. The resulting fit is shown in figure 20. Levels are labelled by their angular momentum and parity J^π, and also by a quantum

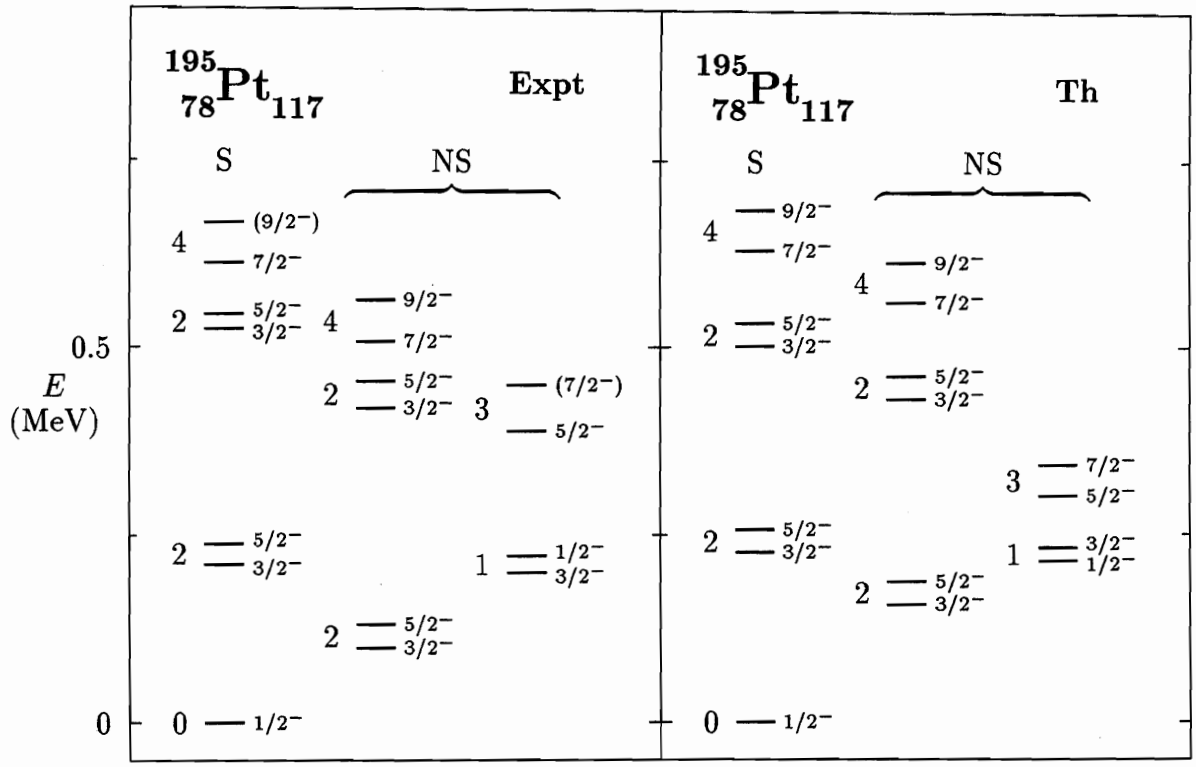

Figure 20. The experimental and calculated energy spectrum of negative-parity states in ^{195}Pt obtained by coupling one fermion to $N = 6$ bosons. Levels are labelled by their angular momentum and parity J^π (right), by the quantum number $\tilde{L}$ (left) and by S (symmetric) and NS (non-symmetric), as explained in the text.

number $\tilde{L}$ which results from the coupling of the angular momentum of the bosons to the pseudo orbital angular momentum of the fermion. Futhermore, in complete analogy with the IBM-2 (see subsection 4.2), the pseudo orbital part of the fermion can be coupled either symmetrically or non-symmetrically to the boson core; this is indicated in figure 20 by S and NS, respectively.

One should not underestimate the difficulty of obtaining such level of agreement, even with five parameters: all states are calculated at about the correct energy and for each J^π the correct number of levels is found. The question of assigning quantum numbers to observed levels is closely related to the goodness of these quantum numbers, which can be probed by means of selection rules in electromagnetic transitions and transfer reactions. An example is given in table 10, where 25 measured $B(\mathrm{E2})$ values in ^{195}Pt [138] are compared with the predictions of the symmetry classification for this nucleus obtained with the E2 operator

$$\hat{T}_\mu(\mathrm{E2}) = e_\mathrm{b}\hat{Q}_{\mathrm{b},\mu} + e_\mathrm{f}\hat{Q}_{\mathrm{f},\mu}, \tag{5.1}$$

where $\hat{Q}_{\mathrm{b},\mu}$ and $\hat{Q}_{\mathrm{f},\mu}$ are the boson and fermion quadrupole operators, and e_b and e_f the effective boson and fermion charges. From microscopic considerations one expects

Table 10. Observed and calculated E2 transition rates in ^{195}Pt.

E_i[a]	J_i	E_f[a]	J_f	$B(\mathrm{E2}; J_\mathrm{i} \to J_\mathrm{f})$[b] Expt	Th	E_i[a]	J_i	E_f[a]	J_f	$B(\mathrm{E2}; J_\mathrm{i} \to J_\mathrm{f})$[b] Expt	Th
211	$\frac{3}{2}$	0	$\frac{1}{2}$	190(10)	179	667	$\frac{9}{2}$	239	$\frac{5}{2}$	200(40)	239
239	$\frac{5}{2}$	0	$\frac{1}{2}$	170(10)	179	563	$\frac{9}{2}$	239	$\frac{5}{2}$	91(22)	22
525	$\frac{3}{2}$	0	$\frac{1}{2}$	17(1)	0	239	$\frac{5}{2}$	99	$\frac{3}{2}$	60(20)	0
544	$\frac{5}{2}$	0	$\frac{1}{2}$	8(4)	0	525	$\frac{3}{2}$	99	$\frac{3}{2}$	≤ 33	7
99	$\frac{3}{2}$	0	$\frac{1}{2}$	38(6)	35	613	$\frac{7}{2}$	99	$\frac{3}{2}$	5(3)	9
130	$\frac{5}{2}$	0	$\frac{1}{2}$	66(4)	35	420	$\frac{3}{2}$	99	$\frac{3}{2}$	5(4)	177
420	$\frac{3}{2}$	0	$\frac{1}{2}$	15(1)	0	508	$\frac{7}{2}$	99	$\frac{3}{2}$	240(50)	228
455	$\frac{5}{2}$	0	$\frac{1}{2}$	≤ 0.04	0	389	$\frac{5}{2}$	99	$\frac{3}{2}$	200(70)	219
199	$\frac{3}{2}$	0	$\frac{1}{2}$	25(2)	0	525	$\frac{3}{2}$	130	$\frac{5}{2}$	9(5)	3
389	$\frac{5}{2}$	0	$\frac{1}{2}$	7(1)	0	667	$\frac{9}{2}$	130	$\frac{5}{2}$	12(3)	10
613	$\frac{7}{2}$	211	$\frac{3}{2}$	170(70)	215	563	$\frac{9}{2}$	130	$\frac{5}{2}$	240(40)	253
508	$\frac{7}{2}$	211	$\frac{3}{2}$	55(17)	20	389	$\frac{5}{2}$	130	$\frac{5}{2}$	≤ 14	55
525	$\frac{3}{2}$	239	$\frac{5}{2}$	≤ 19	72						

[a] In units of keV.
[b] In units of 10^{-1} $e^2\mathrm{fm}^2$.

$e_\mathrm{b} \approx -e_\mathrm{f}$, which is taken here. The level assignment is identical to the one in figure 20. The results shown in table 10 do not depend on the parameters entering the hamiltonian. This is so because, whatever the values of these parameters, the wavefunction of a given state invariably is characterised by the same set of quantum numbers. As a result, the calculated transition rates only depend on the effective charges in the transition operator and, in the case of table 10, on the single value $e_\mathrm{b} = -e_\mathrm{f} = 151$ $e$fm in the E2 operator. The selection rules in the E2 transitions are clear from the table: several of the transitions are predicted zero. One generally finds that the corresponding measured

values are small indeed (typically, a factor 10 smaller than the large allowed ones) confirming the approximate validity of the quantum numbers in this symmetry scheme.

The structure of the levels of ^{195}Pt can also be probed by magnetic dipole properties and, specifically, by magnetic dipole moments μ which are particularly sensitive to the single-particle structure. The magnetic moment operator is proportional to the M1 operator and consists of a collective (boson) and a single-particle (fermion) part,

$$\hat{T}_\mu(\mathrm{M1}) = \sqrt{\frac{3}{4\pi}}\left(g_\mathrm{b}\hat{L}_{\mathrm{b},\mu} + \sum_j g_j \hat{L}_{\mathrm{f},\mu}(j)\right), \tag{5.2}$$

where $\hat{L}_{\mathrm{b},\mu}$ and $\hat{L}_{\mathrm{f},\mu}(j)$ are the boson and fermion angular momentum operators, respectively. The effective boson gyromagnetic ratio is determined from the magnetic dipole moment $\mu(2_1^+)$ of the neighbouring even–even Pt isotopes, $g_\mathrm{b} = 0.3\ \mu_\mathrm{N}$, while for the g_j factors Schmidt values are taken with an appropriate quenching (0.6) of the spin part. Magnetic dipole moments of states in the odd-mass nucleus ^{195}Pt are then entirely determined and, in fact, closed expressions can be derived for them [140]. Results for the moments of yrast symmetric and non-symmetric states are shown in figure 21. Considering that no free parameters are involved in this calculation, the level

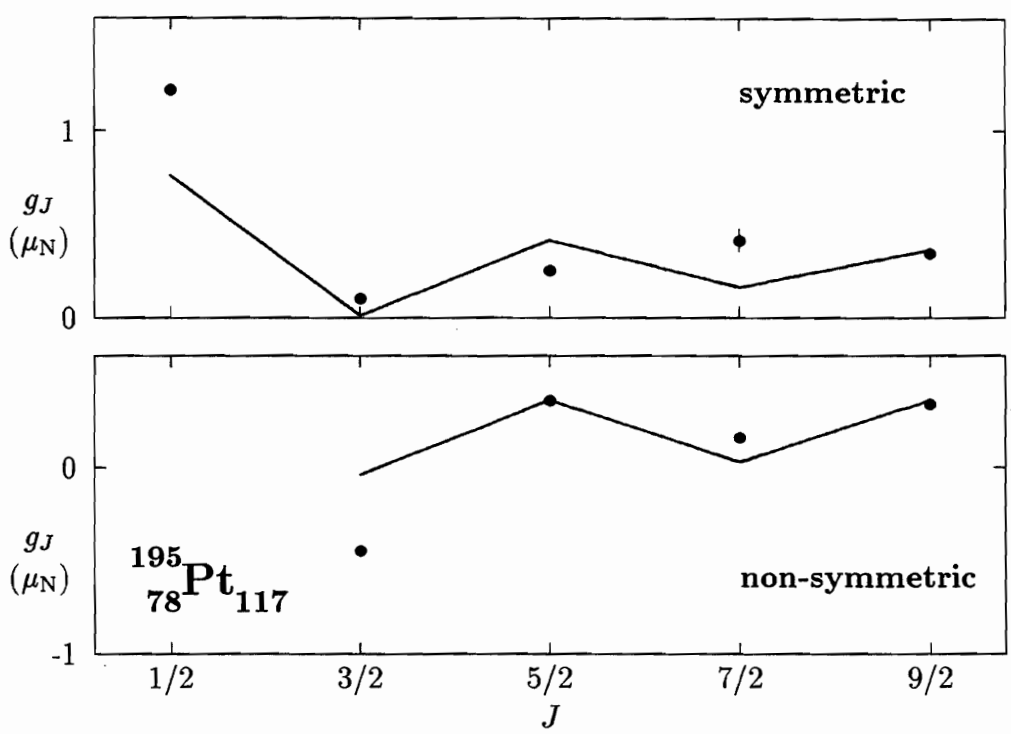

Figure 21. Observed magnetic dipole moments (dots) of the yrast symmetric and non-symmetric levels in ^{195}Pt compared with the prediction of the symmetry calculation (line).

of agreement can be regarded as remarkable.

Results of similar quality are obtained [141] for intensities of one-neutron transfer reactions starting from and leading to ^{195}Pt and confirm the proposed assignment of quantum numbers.

5.2. Supersymmetry

According to the preceding sections, even–even and odd-mass nuclei can be treated in a unified framework using symmetry ideas of the IBM and IBFM, respectively. Schematically, states in such nuclei are connected by the generators

$$\begin{pmatrix} b^\dagger b & 0 \\ 0 & a^\dagger a \end{pmatrix}, \tag{5.3}$$

where a (b) refers to fermions (bosons) and indices are omitted for simplicity. Even–even nuclei involve operators in the upper left-hand corner of (5.3) while odd-mass nuclei require both sets of generators. The operators (5.3) provide a *separate* description of even–even and odd-mass nuclei; although the treatment is similar in both cases, no operator exists that *connects* even–even and odd-mass states.

An extension of the algebraic structure (5.3), proposed by Iachello [142], considers in addition operators that transform a boson into a fermion or *vice versa*:

$$\begin{pmatrix} b^\dagger b & b^\dagger a \\ a^\dagger b & a^\dagger a \end{pmatrix}. \tag{5.4}$$

This set does not any longer form a classical Lie algebra which is defined in terms of commutation relations. Instead, to define a closed algebraic structure, one needs to introduce an internal operation that corresponds to a mixture of commutation and anticommutation and the resulting algebra is called a *graded* or *super*algebra, denoted by $\mathrm{U}(n/m)$ where n and m are the dimensions of the boson and fermion algebras.

To understand the purpose of the introduction of the supersymmetric generators $a^\dagger b$ or $b^\dagger a$, it is best to inspect their action on an even–even nucleus, say $^{194}_{78}\mathrm{Pt}_{116}$, in the context of the $\mathrm{U}^{\mathrm{B}}(6) \otimes \mathrm{U}^{\mathrm{F}}(12)$ model discussed in subsection 5.1,

$$a^\dagger b\ {}^{194}_{78}\mathrm{Pt}_{116} \longrightarrow a^\dagger\ {}^{196}_{78}\mathrm{Pt}_{118} \longrightarrow {}^{195}_{78}\mathrm{Pt}_{117}, \tag{5.5}$$

where bosons and fermions are assumed to have a neutron-hole character. The supersymmetric generators thus induce a connection between even–even and odd-mass nuclei; a description with the superalgebra $\mathrm{U}(6/12)$ leads to a *simultaneous* treatment of such pairs of nuclei. This idea is illustrated schematically in figure 22 for the case of a particular $\mathrm{U}(6/12)$ supermultiplet. The supermultiplet containing ^{194}Pt also contains ^{195}Pt, since the two nuclei are connected by the supersymmetric generator (5.5). Further action of $a^\dagger b$ on ^{195}Pt leads to configurations whereby two neutron-holes are coupled

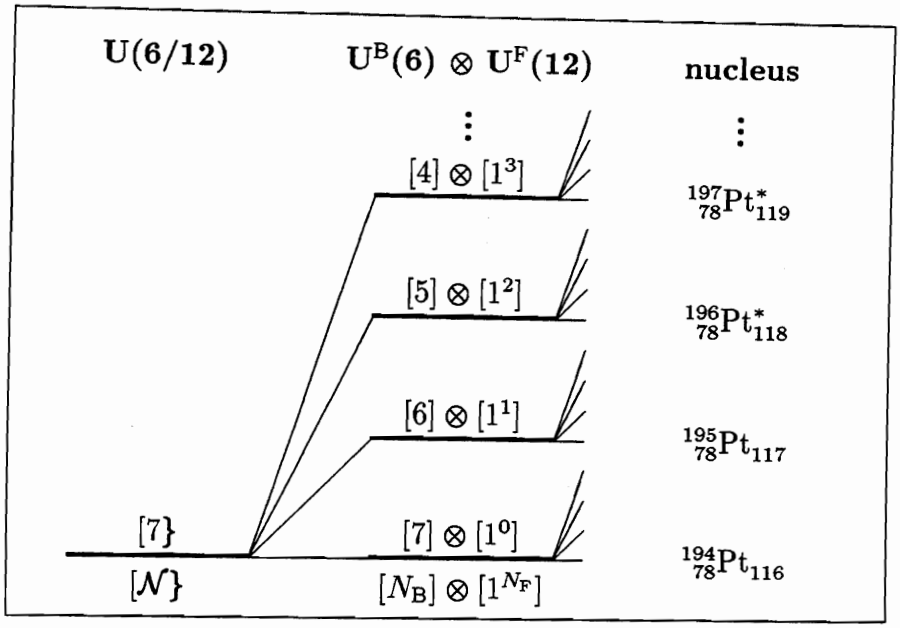

Figure 22. Schematic illustration of part of a U(6/12) supermultiplet in the Pt region. The supermultiplet is characterised by the supersymmetric representation $[\mathcal{N}\}$ with $\mathcal{N} = N_B + N_F = 7$. A breaking of the U(6/12) symmetry in a dynamical manner leads to a splitting in the binding energies of the different nuclei.

to a ^{198}Pt core, that is, to two-quasiparticle excitations in ^{196}Pt. This action of $a^\dagger b$ may continue indefinitely until no more neutrons are available. A U(6/12) symmetry predicts all states of all nuclei belonging to the supermultiplet degenerate in energy; this degeneracy is first lifted by including $U^B(6) \otimes U^F(12)$ invariants which correspond to nuclear binding energy terms. The analysis then proceeds as in subsection 5.1 with the inclusion of Casimir operators of the lower algebras, as schematically indicated in figure 22.

Example: Supermultiplets of nuclei in the Pt–Au region

Supersymmetry predicts a connection between the energy splittings in the spectra of the members of the supermultiplet because their spectra are calculated with the *same* hamiltonian. This is illustrated in figure 23 with the example of the supermultiplet containing $^{194}_{78}Pt_{116}$ and $^{195}_{78}Pt_{117}$. The figure shows the spectra of the first two members of the U(6/12) supermultiplet of figure 22 which are seen to satisfy the structural relations implied by supersymmetry. The multi-quasiparticle excitations in the other members of the supermultiplet are not shown since there is at present not sufficient evidence to link their properties to those of the levels shown.

This process of enlargement of the dynamical algebra can be continued and from

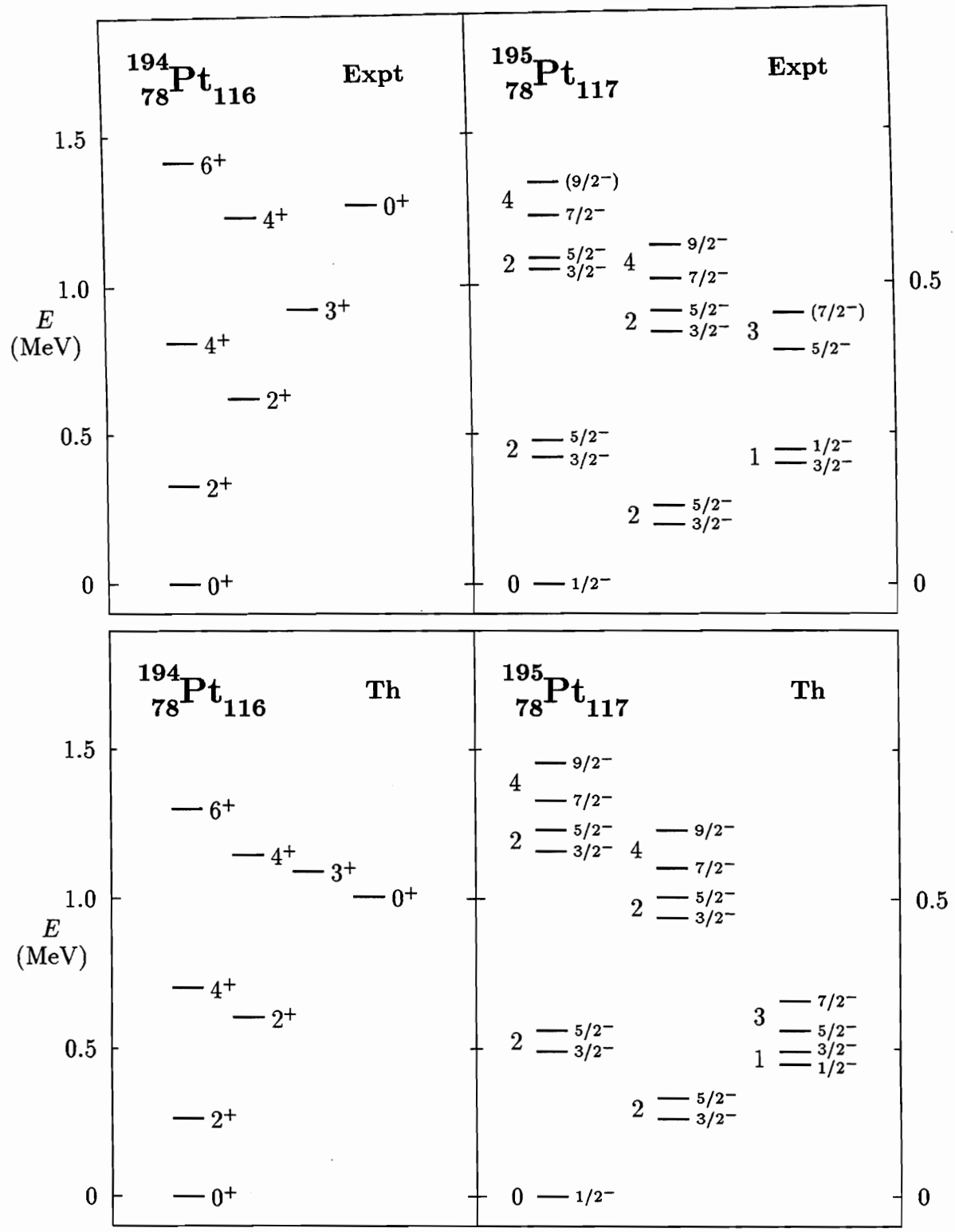

Figure 23. The experimental and calculated energy spectra of the nuclei ^{194}Pt and ^{195}Pt, two of the nuclei belonging to the $\mathcal{N} = 7$ supermultiplet of U(6/12). Levels are labelled by their angular momentum and parity J^{π} (right) and by the quantum number $\tilde{L}$ (left). The left (right) scale applies to ^{194}Pt (^{195}Pt).

it results a unified description of ever higher numbers of nuclei. A further example of this mechanism is obtained if a distinction is made between protons and neutrons, both for fermions and for bosons. The action of such operators is illustrated in figure 24. It shows that now a quartet of nuclei (even–even, even–odd, odd–even and odd–odd)

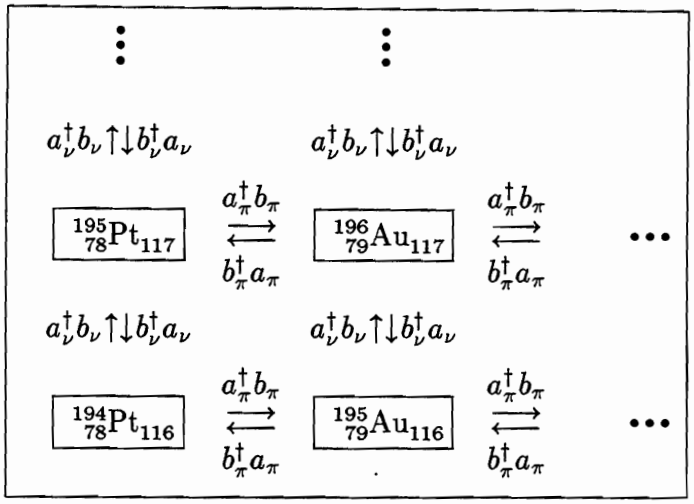

Figure 24. An example of $\mathrm{U}_\pi(6/4)\otimes\mathrm{U}_\nu(6/12)$ multiplet of nuclei: ^{194}Pt, ^{195}Pt, ^{195}Au and ^{196}Au.

is connected by operators of the type $b_\rho^\dagger a_\rho$ or $a_\rho^\dagger b_\rho$ ($\rho = \pi, \nu$) which belong to the product of algebras $\mathrm{U}_\pi(6/4) \otimes \mathrm{U}_\nu(6/12)$, the first referring to protons and the second to neutrons. Consequently, this quartet can be described simultaneously with a single hamiltonian [143]. The interest of this particular example is that one essentially predicts the spectrum of the odd–odd nucleus, on the basis of a fit to the spectra of the three other members of the multiplet. This may yield an insight into the structure of such nuclei which are difficult to interpret otherwise.

6. Epilogue

Symmetry is where the old and the new meet. This, in one sentence, has been the main theme of this review. Ideas of symmetry have been employed in nuclear physics as soon as the subject itself had seen the light, as the example of isospin symmetry illustrates. What is perhaps less well known but equally true is that the idea of *dynamical* symmetry also has a venerable history in nuclear physics. Wigner's supermultiplet model, Racah's pairing model and Elliott's rotation model can all be understood from a common framework in terms of spectrum generating algebras. One of the purposes of this review has been to present those early nuclear structure models from that common perspective and at the same time to demonstrate that they naturally lead into more

recent developments such as the interacting boson model of Arima and Iachello. The second *raison d'être* of this review has been to show convincing evidence that, even if these ideas of symmetry date back to the very beginnings of the subject, they continue to inspire experiments at the forefront of today's research in nuclear physics.

Acknowledgements

I wish to thank all colleagues who have contributed to my understanding (however limited) of this subject and, in particular, to express my gratitude to my teachers: to Kris Heyde for introducing me to nuclear physics and to Phil Elliott, Franco Iachello and Marcos Moshinsky for showing me the beauty of symmetries. Most of all, I am indebted to my long-term collaborator Alejandro Frank who shared many miles of my journey towards an understanding of symmetries.

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