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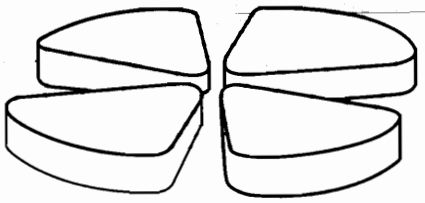

Generalised Wigner surmise for (2×2) random matrices

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LETTER TO THE EDITOR

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Abstract. We present new analytical results concerning the spectral distributions for (2×2) random real symmetric matrices which generalise the Wigner surmise.

1. Introduction

More than forty years ago Wigner initiated the use of Random Matrix Theory [1] to investigate the statistical properties of eigenvalues and eigenvectors of many-body quantum systems. Particular attention was always paid to Gaussian ensembles. In fact, assuming that (i) the elements of the Hamiltonian matrix are *independent real variables* and (ii) the matrix distribution is *invariant* under an *orthogonal* transformation of the basis states, it is possible to show [2] (see also [3]) that the matrix elements are independent Gaussian variables with zero mean and with variance satisfying the conditions $\sigma_{ij}^2 = (1 + \delta_{ij})\sigma^2$. Since then, many analytical results have been derived for such matrices (see e.g. [4]), and the Gaussian Orthogonal, Unitary or Symplectic Ensembles (GOE, GUE or GSE, respectively) have been widely and successfully applied in many fields of physics (see e.g. reviews [5, 6]).

Realistic interactions used in many-body nuclear, molecular or atomic problems however are predominantly of one- and two-body nature, implying that the elements of the Hamiltonian matrix are *not independent* and that the distribution of the matrix is *not invariant* under an orthogonal (unitary) transformation of the basis. In this context, French and Wong [7] and Bohigas and Flores [8] independently introduced the Two-Body Random Ensemble (TBRE) which is more relevant for many-body physics. Indeed, the level distributions of experimental nuclear spectra resemble much more a Gaussian

distribution, given by a TBRE, rather than a semi-circle provided by a GOE [7]. The TBRE has been intensively studied numerically [7, 8, 9], but to our knowledge the only analytical result has been obtained up to now by Gervois who proved [10] that the cumulative energy distribution for the TBRE was Gaussian.

A random matrix ensemble different from the GOE can arise in a many-body problem provided that only interactions up to a certain rank are used. Given the importance of the TBRE in realistic calculations and, in particular, the interest initiated by recent studies in the context of the shell model [11] and the interacting boson model [12], we present in this letter some analytical results concerning the properties of (2×2) random symmetric matrices for which the assumption (ii) mentioned above is not satisfied. We justify our choice of ensemble taking as an example a two-particle system interacting via one- and two-body forces (a typical situation found in nuclear, atomic and molecular physics). First, we derive the Hamiltonian distribution as a function of its eigenvalues and we calculate the nearest-neighbour spacing distribution which generalises the well-known "Wigner surmise" [1]. Then, for a particular case, we give the analytical expressions for the first moments of this distribution. Finally, we propose a method to derive the moments of the eigenvalues distribution without knowledge of an explicit expression for the distribution.

2. One- and two-body random interactions for a two-particle system

We first illustrate how the problem solved in the next section can arise. We consider two simple examples provided by typical nuclear structure models, namely, the nuclear shell model (see e.g. [13]) and the interacting boson model [14].

In both cases, the most general one- and two-body (hermitian, rotational invariant, particle-number and parity conserving) Hamiltonian reads

$$\hat{H} = \sum_j \varepsilon_j \sqrt{2j+1}\left[a_j^\dagger \otimes \tilde{a}_j\right]^{(0)} \pm \sum_{\substack{j_1\le j_2\\ j_3\le j_4\\ J}} v^J_{j_1j_2j_3j_4}\sqrt{\frac{2J+1}{(1+\delta_{j_1j_2})(1+\delta_{j_3j_4})}}\left[\left[a^\dagger_{j_1}\otimes a^\dagger_{j_2}\right]^{(J)}\otimes\left[\tilde{a}_{j_3}\otimes\tilde{a}_{j_4}\right]^{(J)}\right]^{(0)}, \tag{1}$$

where $a^\dagger_{jm}$ and a_{jm} are boson (fermion) creation and annihilation operators, $\tilde{a}_{jm} \equiv (-1)^{j-m}a_{j-m}$, j and m are the particle angular momentum and its projection (integer for bosons or half-integer for fermions) and J is the total angular momentum. The sign "+" in front of the second term in the r.h.s. of (1) applies to bosons, while "−" stands for fermions and $\otimes$ denotes a tensor product. The parameters ε_j are the single-particle energies, while $v^J_{j_1j_2j_3j_4}$ are the interaction matrix elements between normalised two-particle states (properly symmetrised for bosons or antisymmetrised for fermions) ,

$$v^J_{j_1j_2j_3j_4} = \langle j_1j_2; J \mid \hat{H} \mid j_3j_4; J\rangle.$$

Let us consider a system of two identical fermions in two different j-orbitals. This can be ^{18}O represented as ^{16}O plus two neutrons in the $0d_{5/2}$ and $1s_{1/2}$ orbitals, or

^{30}Si modeled as a ^{28}Si core with two neutrons in the $1s_{1/2}$ and $0d_{3/2}$ orbitals. In ^{18}O the configuration space contains six states: two states with $J^\pi = 0^+$, two states with $J^\pi = 2^+$, a $J^\pi = 3^+$ and a $J^\pi = 4^+$ state. The eigenvalues for $J^\pi = 0^+, 2^+$ are determined from the diagonalisation of the following (2×2) real symmetric matrices

$$H\left(0^+\right) = \begin{pmatrix} 2\varepsilon_{\frac{1}{2}} + v^0_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}} & v^0_{\frac{1}{2}\frac{1}{2}\frac{5}{2}\frac{5}{2}} \\ v^0_{\frac{1}{2}\frac{1}{2}\frac{5}{2}\frac{5}{2}} & 2\varepsilon_{\frac{5}{2}} + v^0_{\frac{5}{2}\frac{5}{2}\frac{5}{2}\frac{5}{2}} \end{pmatrix}, \tag{2}$$

and

$$H\left(2^+\right) = \begin{pmatrix} \varepsilon_{\frac{1}{2}} + \varepsilon_{\frac{5}{2}} + v^0_{\frac{1}{2}\frac{1}{2}\frac{5}{2}\frac{5}{2}} & v^0_{\frac{1}{2}\frac{5}{2}\frac{5}{2}\frac{5}{2}} \\ v^0_{\frac{1}{2}\frac{5}{2}\frac{5}{2}\frac{5}{2}} & 2\varepsilon_{\frac{5}{2}} + v^0_{\frac{5}{2}\frac{5}{2}\frac{5}{2}\frac{5}{2}} \end{pmatrix}. \tag{3}$$

A similar situation can arise in a system of identical bosons. For example, the spectrum of two bosons with $j = 0, 2$ contains five states: two $J^\pi = 0^+$ states, two $J^\pi = 2^+$ states and a $J^\pi = 4^+$ state. The corresponding (2×2) matrices are

$$H\left(0^+\right) = \begin{pmatrix} 2\varepsilon_0 + v^0_{0000} & v^0_{0022} \\ v^0_{0022} & 2\varepsilon_2 + v^0_{2222} \end{pmatrix}, \tag{4}$$

and

$$H\left(2^+\right) = \begin{pmatrix} \varepsilon_0 + \varepsilon_2 + v^2_{0202} & v^2_{0222} \\ v^2_{0222} & 2\varepsilon_2 + v^2_{2222} \end{pmatrix}. \tag{5}$$

The explicit form of these matrices is identical to those obtained in two-fermion system above. If we assume that the parameters of the Hamiltonian (1) are independent Gaussian random variables, it is clear that the random matrices (2)–(5) in general do not belong to the conventional GOE, although the matrix elements are still independent. The latter statement is obvious in case of $H\left(0^+\right)$, (2) or (4). For the matrices $H\left(2^+\right)$, (3) or (5), we note that they can be represented as a sum of the unity matrix multiplied by $\varepsilon_{\frac{5}{2}}$ or ε_2, respectively, and a (2×2)-matrix with completely independent matrix elements.

Similar matrices may appear for specific cases of an n-particle system provided low rank interactions are used. Therefore, in order to study the statistical features of the many-body spectra, one is required to generalise the Wigner surmise.

3. Generalised Wigner surmise

Let us consider a (2×2) real symmetric matrix whose elements are independent Gaussian variables with zero mean and variance σ^2_{ij}. The probability density of the matrix H is then given by

$$p(H) = \frac{1}{(2\pi)^{3/2}\sqrt{\sigma^2_{11}\sigma^2_{12}\sigma^2_{22}}} \exp\left[-\left(\frac{H^2_{11}}{2\sigma^2_{11}} + \frac{H^2_{12}}{2\sigma^2_{12}} + \frac{H^2_{22}}{2\sigma^2_{22}}\right)\right]. \tag{6}$$

Each matrix H can be diagonalised in an orthogonal basis and therefore $H = O^t D O$, with

$$O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and

$$D = \begin{pmatrix} E_\alpha & 0 \\ 0 & E_\beta \end{pmatrix}.$$

Similar to the case of GOE [3], we find that in the general case

$$\begin{aligned} H_{11} &= E_\alpha \cos^2\theta + E_\beta \sin^2\theta \\ H_{12} &= (E_\alpha - E_\beta)\cos\theta\sin\theta \\ H_{22} &= E_\alpha \sin^2\theta + E_\beta \cos^2\theta \end{aligned} .$$

We deduce that the probability density expressed in terms of the eigenvalues and the angle θ is

$$\begin{aligned} p(E_\alpha, E_\beta, \theta) &= \frac{E_\alpha - E_\beta}{(2\pi)^{3/2}\sqrt{\sigma_{11}^2\sigma_{22}^2\sigma_{12}^2}} \exp\left\{ -\frac{\left[E_\alpha\Sigma^2 - (E_\alpha - E_\beta)\left(\sigma_{11}^2\cos^2\theta + \sigma_{22}^2\sin^2\theta\right)\right]^2}{2\sigma_{11}^2\sigma_{22}^2\Sigma^2} \right\} \\ &\quad \exp\left[-\frac{1}{2}(E_\alpha - E_\beta)^2 \left(\frac{\cos^2(2\theta)}{\Sigma^2} + \frac{\sin^2(2\theta)}{4\sigma_{12}^2} \right) \right] \end{aligned} \tag{7}$$

where $\Sigma^2 = \sigma_{11}^2 + \sigma_{22}^2$ and $E_\alpha - E_\beta \geq 0$.

The nearest-neighbour spacing distribution for the variable $\varepsilon = E_\alpha - E_\beta$ is given by the following integral

$$\tilde{p}(\varepsilon) = \int_{-\pi/2}^{\pi/2} d\theta \int_{-\infty}^{\infty} dE_\alpha \int_{-\infty}^{E_\alpha} dE_\beta \; p(E_\alpha, E_\beta, \theta)\, \delta(\varepsilon - E_\alpha + E_\beta), \tag{8}$$

from which we obtain

$$\tilde{p}(\varepsilon) = \frac{\varepsilon}{2\sqrt{\Sigma^2\sigma_{12}^2}} \exp\left[-\frac{\varepsilon^2\left(\Sigma^2 + 4\sigma_{12}^2\right)}{16\Sigma^2\sigma_{12}^2} \right] I_0\left(\frac{\varepsilon^2\left(\Sigma^2 - 4\sigma_{12}^2\right)}{16\Sigma^2\sigma_{12}^2} \right) \tag{9}$$

where I_0 is a modified Bessel function of the first kind.

The expression (9) looks like a Rayleigh-Rice distribution, well known in signal theory [15], except for the argument of I_0, which is not linear as in the usual Rayleigh-Rice distribution but quadratic. This is why we will refer to $\tilde{p}(\varepsilon)$ as to a *quadratic* Rayleigh-Rice distribution.

Let us consider a particular case when the diagonal matrix elements have the same variance $\sigma_{11}^2 = \sigma_{22}^2$, which is χ times larger than the variance, $\sigma_{12}^2 = \sigma^2$, of the non-diagonal matrix elements, i.e. $\chi = \sigma_{11}^2/\sigma_{12}^2$. Then the eigenvalue distribution (7) reduces to

$$p_\chi(E_\alpha, E_\beta, \theta) = \frac{E_\alpha - E_\beta}{(2\pi\sigma^2)^{3/2}\chi} \exp\left[-\frac{E_\alpha^2 + E_\beta^2 + \frac{1}{4}(E_\alpha - E_\beta)^2(\chi - 2)\sin^2(2\theta)}{2\chi\sigma^2} \right], \tag{10}$$

while for the nearest-neighbour spacing we get

$$\tilde{p}_\chi(\varepsilon) = \frac{\varepsilon}{\sqrt{2\chi}2\sigma^2} \exp\left(-\frac{(\chi + 2)\,\varepsilon^2}{16\chi\sigma^2} \right) I_0\left(\frac{(\chi - 2)\varepsilon^2}{16\chi\sigma^2} \right). \tag{11}$$

For $\chi = 2$, expression (11) reduces to the Wigner surmise. The distributions $\tilde{p}_\chi(\varepsilon)$ are plotted in figure 1 for $\chi = 1$, $\chi = 2$ and $\chi = 5$. For the two-particle systems considered

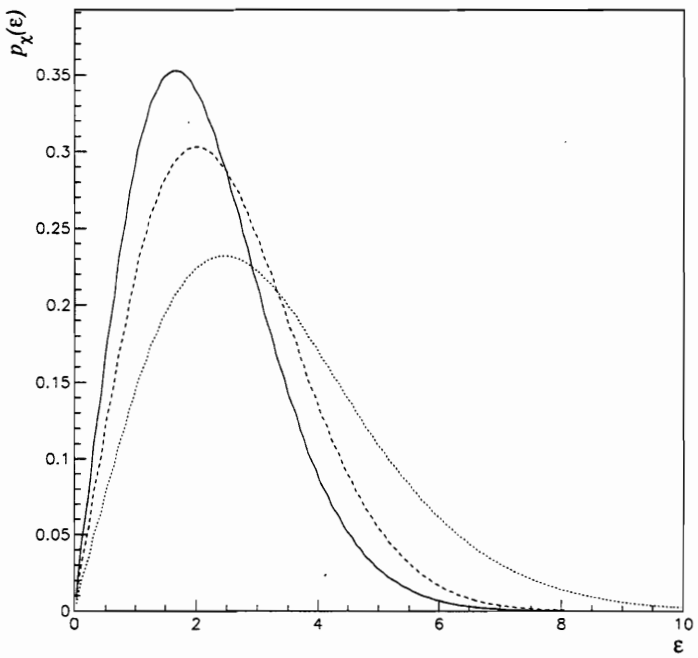

Figure 1. Quadratic Rayleigh-Rice distributions for $\chi = 1$ (solid line), $\chi = 2$ (dashed line) and $\chi = 5$ (dotted line) with $\sigma^2 = 1$.

above, $\chi = 5$ corresponds to the matrix $H(0^+)$ and $\chi = 2$ corresponds to the matrix $H(2^+)$, assuming that all parameters in Hamiltonian (1) have the same variance.

To calculate various statistical characteristics, it is often required to know certain moments of the distribution. Thus, we have derived analytical expressions for some moments of the distribution (11), and the lowest are given in table 1.

Table 1. The moments up to n=5 of the quadratic Rayleigh-Rice distribution (11) for $\sigma^2 = 1$. F is the hypergeometric function [16].

	$M_n = \int_0^\infty \varepsilon^n \tilde{p}_\chi(\varepsilon) d\varepsilon$	values for $\chi = 2$	for $\chi = 5$
$n = 0$	1	1	1
$n = 1$	$4\chi\sqrt{2\pi}(\chi+2)^{-3/2} F\left[\frac{3}{4}, \frac{5}{4}, 1, \left(\frac{\chi-2}{\chi+2}\right)^2\right]$	$\sqrt{2\pi}$	$1.3\sqrt{2\pi}$
$n = 2$	$2(2+\chi)$	8	14
$n = 3$	$96\chi^2\sqrt{2\pi}(\chi+2)^{-5/2} F\left[\frac{5}{4}, \frac{7}{4}, 1, \left(\frac{\chi-2}{\chi+2}\right)^2\right]$	$12\sqrt{2\pi}$	$28.7\sqrt{2\pi}$
$n = 4$	$4(12+4\chi+3\chi^2)$	128	428
$n = 5$	$3840\chi^3\sqrt{2\pi}(\chi+2)^{-7/2} F\left[\frac{7}{4}, \frac{9}{4}, 1, \left(\frac{\chi-2}{\chi+2}\right)^2\right]$	$240\sqrt{2\pi}$	$1139.2\sqrt{2\pi}$

Integrating p_χ in (10) over E_β from $-\infty$ to E_α and over E_α from $-\infty$ to $+\infty$, we obtain the angular distribution

$$r_\chi(\theta) = \frac{1}{\pi}\sqrt{\frac{\chi}{2}}\frac{1}{\left[1+\frac{1}{4}(\chi-2)\sin^2(2\theta)\right]}. \tag{12}$$

This distribution is represented in figure 2 for different values of χ. For $\chi = 2$ it is an exactly uniform distribution, which means that there is no privileged basis (the orthogonal invariance holds). For high values of χ, the initial basis is nearly the eigenbasis (the diagonal elements are much larger than the non-diagonal ones), thus r_χ takes its maximum absolute values for $\theta = 0$ and $\theta = \pi/2$, whereas for small values of χ, the eigenstates are more likely obtained after a rotation of $\pi/4$ of the initial basis and r_χ is maximum for $\theta = \pi/4$.

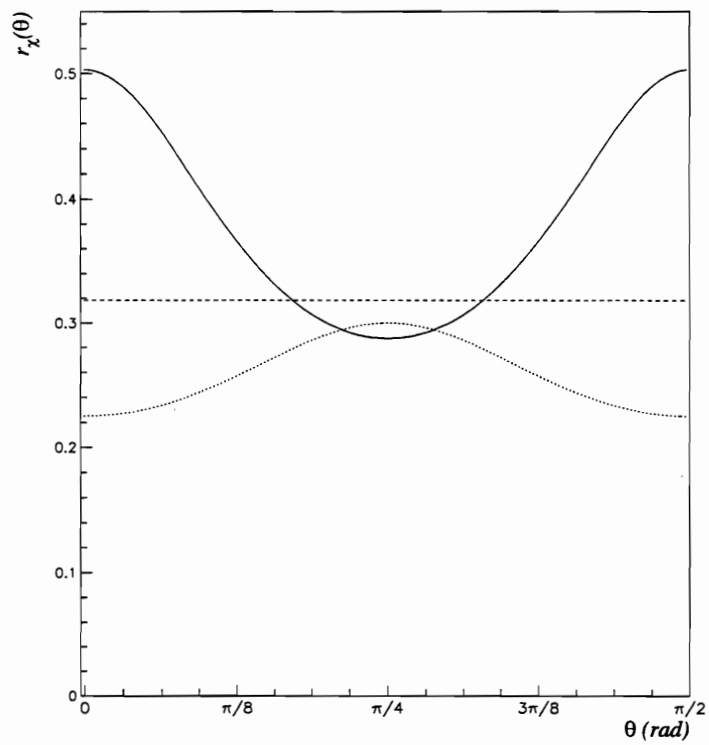

Figure 2. Angular distributions for $\chi = 1$ (dotted line), $\chi = 2$ (dashed line) and $\chi = 5$ (solid line).

We can re-express $p_\chi(E_\alpha, E_\beta)$ as a function of ε and $S = E_\alpha + E_\beta$. Then p_χ can be factorised into a function depending on ε times a function depending on S, i.e. these variables are independent. Moreover since S is the trace of the matrix it is a Gaussian variable with zero mean and all its odd moments are zero. From the independence of ε and S we deduce that the moments of the eigenvalues fulfill

$$\langle E_\alpha^n\rangle = (-1)^n\langle E_\beta^n\rangle \tag{13}$$

$$\langle E_\alpha^n\rangle = \frac{1}{2^n}\sum_{p=0}^{n/2}\binom{n}{2p}\langle S^{2p}\rangle\langle\varepsilon^{n-2p}\rangle \tag{14}$$

From (13) and (14), we can quite easily derive the moments of the eigenvalues whose distributions are difficult to compute. Note that from (13), we deduce that the highest eigenvalue and the lowest one have opposite mean values and the same variance.

4. Conclusion

The study of the statistical properties of spectra of realistic many-body Hamiltonians requires consideration of a random matrix ensemble whose elements are not independent or whose distribution is not invariant under orthogonal transformation of a chosen basis. In this letter we have concentrated on the properties of (2×2) real symmetric matrices whose elements are independent Gaussian variables with zero means but do not belong to the GOE. We have derived the distribution of eigenvalues for such a matrix, the nearest-neighbour spacing distribution which generalises the Wigner surmise and we have calculated some important moments. We believe that these expressions hold for certain matrices of high dimensions which would justify the use of (11) instead of the Brody distribution [17] to fit the data. We also think that these results can be extended to hermitian matrices.

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