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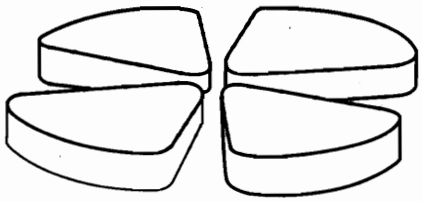

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PARTIAL DYNAMICAL SYMMETRIES IN QUANTAL MANY-BODY SYSTEMS

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Partial dynamical symmetries are associated with hamiltonians that are *partially* solvable. A brief review is given of some recent developments.

The determination of the properties of a quantal system of N interacting particles moving in an external potential requires the solution of the eigenvalue equation associated with the second-quantised hamiltonian

$$\hat{H} = \sum_i \epsilon_i a_i^\dagger a_i + \sum_{ijkl} v_{ijkl} a_i^\dagger a_j^\dagger a_k a_l + \cdots. \tag{1}$$

The operators $a_i^\dagger$ and a_i are creation and annihilation operators for the particles, characterised by an index i which labels a set of single-particle states in the confining potential and which also includes possible intrinsic quantum numbers such as spin, isospin, colour... The coefficients ϵ_i are single-particle energies and the v_{ijkl} are two-body interactions; higher-order interactions can be included in the expansion, if needed. The (unitary) Lie algebra generated by the operators $a_i^\dagger a_j$ over an appropriate set of single-particle states is called the ***spectrum generating*** (or also ***dynamical***) algebra $G_{\rm SGA}$ of the system.

The solution of the eigenvalue problem for N particles associated with (1) requires the diagonalisation of $\hat{H}$ in the symmetric representation $[N]$ of $G_{\rm SGA}$ in case of bosons or in the anti-symmetric representation $[1^N]$ of $G_{\rm SGA}$ in case of fermions. In many situations of interest, the hamiltonian will have a lower symmetry $G_{\rm SYM} \subset G_{\rm SGA}$, that is, $\hat{H}$ will commute with transformations that constitute a symmetry algebra $G_{\rm SYM}$. The enumeration of all hamiltonians of the type (1) that are ***analytically solvable*** and that conserve the symmetry $G_{\rm SYM}$ requires the knowledge of all nested algebraic chains of the type

$$G_{\rm SGA} \supset G \supset G' \supset \cdots \supset G_{\rm SYM}. \tag{2}$$

To appreciate the relevance of this classification, one notes that associated with each chain (2) is a hamiltonian

$$\hat{H} = \sum_n \kappa_n \hat{C}_n[G] + \sum_n \kappa'_n \hat{C}_n[G'] + \cdots, \tag{3}$$

where κ_n are coefficients and $\hat{C}_n[G]$ denotes a Casimir invariant of the algebra G of order n in its generators. Since $\hat{C}_n[G]$ commutes with all elements of G (by definition), one can easily convince oneself that $\hat{H}$ in (3) is written as a sum of commuting operators; as a result its eigenstates are labelled by the quantum numbers associated with these commuting operators. Note that the condition of the *nesting* of the algebras is crucial for constructing a set of commuting operators and hence for obtaining an analytical solution. Also, although (3) is written in an abstract form in terms of Casimir invariants, it should be emphasised that it corresponds to a particular choice of the general hamiltonian (1) with the order of the interactions determined by the maximal order n of the invariants. Thus a generic scheme is established for finding all analytically solvable hamiltonians (1).

This approach to find analytical eigensolutions for a system of interacting bosons and/or fermions has received prominence with the work of Arima and Iachello [1] who proposed a U(6) dynamical algebra for the description of collective nuclear excitations and showed that the subalgebraic structure of U(6) allows three types of analytical solutions, corresponding to vibrational, rotational and γ-soft nuclei. Similar ideas were later explored in molecular [2] and hadronic [3] physics. It should not be forgotten, however, that, although these symmetry methods certainly crystalised with the work of Arima and Iachello, they had been used before in different contexts and different models of physics in general. A few of these early examples are given in Table 1, where they are ordered according to increasing size of the dynamical algebra. The well-known Isobaric Multiplet Mass Equation (IMME) and the equally known Gell-Mann–Okubo mass formula can be viewed as immediate examples of dynamical symmetry breaking of the type (3). A beautiful example in nuclear physics is Elliott's rotational SU(3) model in which Wigner's supermultiplet [13] degeneracy associated with SU(4) is lifted dynamically by the quadrupole interaction. Note also the existence of F-spin multiplets in the context of the neutron-proton interacting boson model (IBM-2) as proposed by von Brentano *et al.* [8] which allows, in analogy with isospin, to define an F-spin Multiplet Mass Equation (FMME). All nuclear physics examples of Table 1 are discussed in detail in a recent review [14].

Since the work of Arima and Iachello, the theory of dynamical symmetries has witnessed a further development due to Alhassid and Leviatan [15] who showed that, although classifications (2) enumerate all *completely* solvable hamiltonians, it is possible to construct additional interactions that preserve solvability for *part* of the eigenstates. Alhassid and Leviatan developed their idea using an intrinsic-state formalism but the same results can also be obtained by a tensor decomposition of the interaction as the following example

Table 1: Some examples of spectrum generating algebras

Spectrum generating algebra	Origin of symmetry breaking	Application	Reference
Isopin SU(2)	Coulomb interaction	Isobaric multiplets, IMME	Heisenberg [4] Wigner [5]
Quasi-spin SU(2)	Pairing interaction	Seniority spectra	Racah [6] Kerman [7]
F-spin SU(2)	$V_{\nu\nu} \neq V_{\pi\pi}$	F-spin multiplets, FMME	Brentano [8]
Flavour SU(3)	Strong interaction	Gell-Mann–Okubo mass formula	Gell-Mann [9] Okubo [10]
Supermultiplet SU(4)	Quadrupole interaction	Rotational bands	Elliott [11]
sd-boson U(6)	Collective interactions	Collective spectra	Arima and Iachello [12]

illustrates. The starting point in this example is the SU(3) hamiltonian of the IBM with eigenstates $|[N](\lambda\mu)KLM_L\rangle$. The structure of the spectrum with SU(3) dynamical symmetry is illustrated in the left-hand panel of Figure 1 where the lowest (ground, β and γ) bands are shown. This spectrum is obtained with two Casimir operators, $\hat{C}_2[\mathrm{SU}(3)]$ and $\hat{C}_2[\mathrm{SO}(3)]$; any additional one- or two-body term leads to a non-solvable hamiltonian. It is, however, possible to find an interaction that preserves the analyticity of *some* of the states. To construct this interaction, it is necessary to perform an SU(3) tensor decomposition of the two-body interaction. The two-boson operators $b^\dagger_{lm} b^\dagger_{l'm'}$ can be coupled to definite SU(3) quantum numbers,

$$B^\dagger_{(\lambda\mu)LM_L} \propto \sum_{ll'} \langle (20)l\,(20)l' | (\lambda\mu)L\rangle (b^\dagger_l \times b^\dagger_{l'})^{(L)}_{M_L}, \tag{4}$$

where $\langle \cdots \cdots | \cdot\cdot \rangle$ is a generalised coupling coefficient associated with SU(3) $\supset$ SO(3). Any interaction between two-boson states can now be written as an interaction between the $B^\dagger_{(\lambda\mu)LM_L}$. In addition, it is a simple matter of SU(3) coupling to show that

$$B^\dagger_{(02)0} \cdot \tilde{B}_{(02)0} |[N](2N,0)LM_L\rangle = B^\dagger_{(02)2} \cdot \tilde{B}_{(02)2} |[N](2N,0)LM_L\rangle = 0, \tag{5}$$

that is, these interactions have the property of giving zero acting on specific SU(3) eigenstates such as the ground-band states $|[N](2N,0)LM_L\rangle$. From this

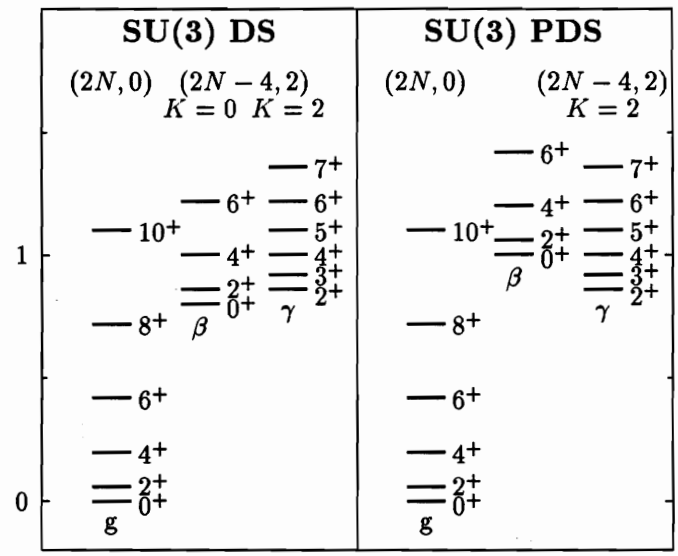

Figure 1: Partial eigenspectrum of a hamiltonian with SU(3) dynamical symmetry (left) and with partial SU(3) dynamical symmetry (right).

property and from the identity

$$B^{\dagger}_{(02)0} \cdot \tilde{B}_{(02)0} + B^{\dagger}_{(02)2} \cdot \tilde{B}_{(02)2} = \frac{1}{18}\left(2\hat{N}(2\hat{N}+3) - \hat{C}_2[\mathrm{SU}(3)]\right), \qquad (6)$$

one concludes that the hamiltonian

$$\kappa\hat{C}_2[\mathrm{SU}(3)] + \kappa'\hat{C}_2[\mathrm{SO}(3)] + \kappa''\left(B^{\dagger}_{(02)0} \cdot \tilde{B}_{(02)0} - B^{\dagger}_{(02)2} \cdot \tilde{B}_{(02)2}\right) \qquad (7)$$

has $|[N](2N,0)LM_L\rangle$ as eigenstates. In fact, it can be shown [15] that all states of the type $|[N](2N-2k,k)K=k\ LM_L\rangle, k=0,2,4,\ldots$ are eigenstates of (7); other SU(3) states, however, are not. For the lowest bands of an SU(3) spectrum it means that the ground and the γ bands are analytically solvable but that the β band is not (see the right-hand panel of Figure 1).

It should be emphasised that the above procedure is *generic*: given a certain classification (2) any interaction can be decomposed into tensors associated with it, and conditions of the type (5) can be verified as a matter of routine.

The property of this type of partial dynamical symmetry (PDS) can be summarised by stating that *all* labels associated with (2) remain good quantum

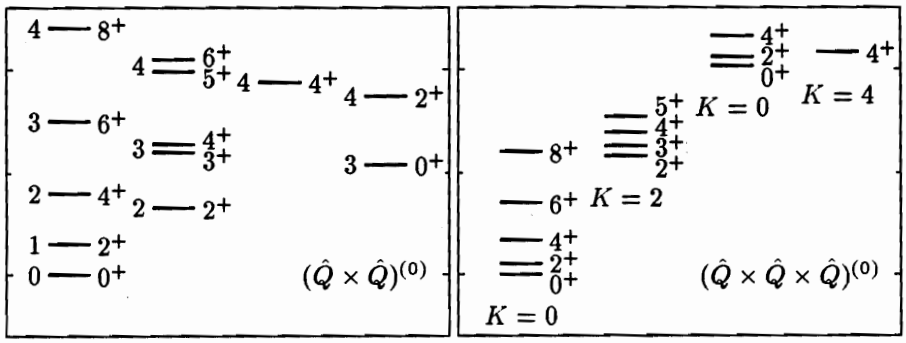

Figure 2: Partial spectra generated by the quadratic and cubic SO(6)-conserving interactions $(\hat{Q}\times\hat{Q})^{(0)}$ and $(\hat{Q}\times\hat{Q}\times\hat{Q})^{(0)}$. All levels have SO(6) dynamical symmetry and those shown have $\sigma = N$. Boson numbers are $N = 6$ and $N = 15$, respectively, and an $\hat{L}^2$ term is added to lift the degeneracy in L.

numbers for *part* of the eigenstates. Recently, another possibility was pointed out [16] in which *part* of the labels remain good quantum numbers for *all* of eigenstates. The idea can be explained with reference to the general classification (2). Let $G_1 \supset G_2 \supset G_3$ be a set of nested algebras which may occur anywhere in the reduction (2). The observation is now simply that any hamiltonian which has G_1 and G_3 as dynamical symmetries does not necessarily also have G_2 as a dynamical symmetry. It suffices to construct combinations of generators $\hat{g}_i$ that belong to G_1 but not to G_2 and which are scalar in G_3 (and hence conserve the associated labels Γ_3) but admix representations Γ_2 of G_2. Note that the resulting hamiltonian is in general not analytically solvable. Nevertheless, the occurrence of the quantum numbers Γ_1 and Γ_3 carries with it the existence of selection rules, hallmark of dynamical symmetries.

Consider as an example the sequence SO(6) $\supset$ SO(5) $\supset$ SO(3) which occurs in the SO(6) limit of the IBM with eigenstates $|[N]\sigma\tau LM_L\rangle$. The above analysis provides a procedure for generating σ-conserving, τ-violating interactions. The generators $\hat{g}_i$ contained in SO(6) but not in SO(5) are the five components of the quadrupole operator $\hat{Q}_\mu \equiv (s^\dagger \times \tilde{d} + d^\dagger \times \tilde{s})^{(2)}_\mu$. On the basis of SO(5) multiplication rules it can be shown [16] that the quadratic SO(3)-scalar $(\hat{Q}\times\hat{Q})^{(0)}$ conserves SO(6) and SO(5). However, the cubic SO(3)-scalar $(\hat{Q}\times\hat{Q}\times\hat{Q})^{(0)}$ is an example of a σ-conserving, τ-violating interaction. The spectra generated by the $(\hat{Q}\times\hat{Q})^{(0)}$ and $(\hat{Q}\times\hat{Q}\times\hat{Q})^{(0)}$ interactions are different as is illustrated in Figure 2. Nevertheless, all states shown in the figure have good SO(6) symmetry. Another example of this type of PDS concerns the

reduction U(6) $\supset G \supset$ SO(5). It is possible to construct an IBM-1 hamiltonian that conserves neither U(5) nor SO(6) but still keeps the SO(5) label as was pointed out some time ago[17]. Other examples have been found in IBM-2[18,19].

As a final remark it should be noted that PDS may occur anywhere in physics. The concept transcends the IBM and examples in other branches of physics should be investigated.

Acknowledgements

It is a great pleasure for me to participate at this celebration of Peter von Brentano's scientific career. I hope to continue collaborating with him for many years not in the least to enjoy his lively discussions.

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