33

08/26/9

FUB - HEP/94 - 1

CLASSICAL AND QUANTUM BEHAVIOUR OF MULTIDIMENSIONAL INTEGRABLE COSMOLOGI

FUB-HEP-94-1

Fermilab Library
0 1160 0052336 9

FERMILAB
SEP 19 1994
LIBRARY

U. BLEYER [1]
WIP-Gravitationsprojekt, Universität Potsdam
An der Sternwarte 16
D-14482 Potsdam, Germany

and

A. ZHUK [2 3]
WIP-Gravitationsprojekt, Universität Potsdam
An der Sternwarte 16
D-14482 Potsdam, Germany
and
Fachbereich Physik, Freie Universität Berlin
Arnimallee 14
D-14195 , Germany

Abstract

Multidimensional cosmological models with n ($n > 1$) Einstein spaces are discussed classically and with respect to canonical quantization. These models are integrable in the case of Ricci flat internal spaces. For negative curvature of the external space we find exact classical solutions modelling dynamical as well as spontaneous compactification of the internal spaces. Spontaneous compactification turns out to be an attractor solution. Solutions of the quantum Wheeler-DeWitt equation are also obtained. Some of them describe the tunneling process to be interpreted as the birth of the universe from "nothing".

[1]This work was supported by WIP grant 016659

[2]Permanent address: Department of Physics, University of Odessa, 2 Petra Velikogo, Odessa 270100, Ukraine

[3]This work was supported in part by DAAD and by DFG grant 436 UKR - 17/7/93

FUB - HEP/94 - 1

CLASSICAL AND QUANTUM BEHAVIOUR OF MULTIDIMENSIONAL INTEGRABLE COSMOLOGIES

U. BLEYER [1]
WIP-Gravitationsprojekt, Universität Potsdam
An der Sternwarte 16
D-14482 Potsdam, Germany

and

A. ZHUK [2 3]
WIP-Gravitationsprojekt, Universität Potsdam
An der Sternwarte 16
D-14482 Potsdam, Germany
and
Fachbereich Physik, Freie Universität Berlin
Arnimallee 14
D-14195 , Germany

Abstract

Multidimensional cosmological models with n ($n > 1$) Einstein spaces are discussed classically and with respect to canonical quantization. These models are integrable in the case of Ricci flat internal spaces. For negative curvature of the external space we find exact classical solutions modelling dynamical as well as spontaneous compactification of the internal spaces. Spontaneous compactification turns out to be an attractor solution. Solutions of the quantum Wheeler-DeWitt equation are also obtained. Some of them describe the tunneling process to be interpreted as the birth of the universe from "nothing".

[1]This work was supported by WIP grant 016659

[2]Permanent address: Department of Physics, University of Odessa, 2 Petra Velikogo, Odessa 270100, Ukraine

[3]This work was supported in part by DAAD and by DFG grant 436 UKR - 17/7/93

1 INTRODUCTION

Everyday experience seems to show evidently that we are living in a four dimensional space-time. Why should we speak about a multidimensional universe? There are good reasons to do so. First of all, we know that consistent theories unifying fundamental interactions take place in multidimensional spaces only, and may be this is not purely a question of mathematical methods and extra dimensions are a physical reality. Second, extra dimensions are actually not observable, because they are extremely small at present time. If they are of the scale of Planck's length ($L_{PL} \sim 10^{-33}$cm) their observation is impossible due to the super high frequency (energy) necessary to observe internal dimensions, and behind Planck's lenght quantum uncertainties forbid the observation. Nevertheless, during the evolution of the universe all dimensions including ours and internal ones might have been of the same scale at early stages of the universe. Moreover, extra dimensions could be much larger then ours at that time. Thus, there is a reason to investigate multidimensional cosmological models and the observable consequences of the possible existence of extra dimensions.

In all multidimensional cosmological models (MCM) a mechanism of dimensional reduction or, in other words, compactification of the extra dimensions should be present. There are two approaches to realize compactification. In the first case the internal dimensions become much smaller than our external ones during the evolution of the universe. These are the MCM with *dynamical compactification*. Observable consequences of extra dimensions are in this case possible variations of effective constants of nature (like the gravitation constant)[1] - [4], imprints in cosmic rays of ultrahigh energy [5] or in the spectrum of gravitational waves [6]. Another possibility consists in the proposal that all extra dimensions are static and small from the very beginning. Such MCM are called models with *spontaneous compactification*. The presence of extra dimensions leads in this case to the generation of particle masses [7] - [9]. In both of these approaches the presence of extra dimensions has very strong influence on the evolution of our external space-time. Compactification takes place for pure gravity as well as for gravity coupled to different matter fields. There is a large amount of papers devoted to these questions (see e.g. the references in [10]).

In our paper we shall consider MCM which consist of n ($n > 1$) spaces of constant curvature. This model was investigated from the classical as well as from quantum points

of view in papers [11] - [20]. The model can be generalized to the case of all spaces being Einstein spaces. The multidimensional Einstein equations as well as the quantum Wheeler-DeWitt equation (WDW) can be integrated for this model if at most one of the spaces of constant curvature M_i $(i = 1, \ldots, n)$ is not Ricci flat [11], [14], [17]. This property is not changed if the model contains in addition a massless minimally coupled scalar field. From the point of view of dynamical and spontaneous compactification this integrable model was considered in [10] in the case when the non Ricci flat space is of positive constant curvature. This space, let it be M_1, was considered there as our external space and all other factor spaces are internal ones. Both types of compactification were found. In the case of a real scalar field as matter source in the Lorentzian region the solution with spontaneous compactification permits an intersting continuation to the Euclidean region describing Euclidean wormholes.

In the present paper we shall investigate this integrable models on the classical as well as on the quantum levels for the case when the non Ricci flat space M_1 is of constant negative curvature. The main problem consists in the investigation of the dynamical and spontaneous compactification. In the case of positive curvature of M_1 the parameter playing the role of energy may in the Lorentzian region adopt positive values only [14]. In contrast to this case, the model with negative curvature permits positive as well as non positive values of this parameter. This feature of the models with negative curvature leads to a more rich and interesting picture than in the former case.

It will be shown that the MCM investigated here have solutions describing spontaneous and dynamical compactification. The paper is organized as follows. In Section 2. we describe our MCM and represent the classical Einstein equations for this model in appropriate coordinates. Section 3. is devoted to the investigation of the dynamical as well as spontaneous compacdtification on the classical level. In Section 4. we consider the quantum properties of the model. The results of the paper are summarized in the Conclusions. A particular case of the model with dynamical compactification is presented in an Appendix. References complete the paper.

2 MINISUPERSPACE COSMOLOGICAL MODELS

Let us consider the metric

$$ds^2 = -d\tau^2 exp(2\gamma(\tau)) + \sum_{i=1}^{n} e^{2\beta^i(\tau)} ds_i^2 \tag{1}$$

on a D-dimensional space-time manifold

$$M = R \times M_1 \times \ldots \times M_n \tag{2}$$

where the M_i are d_i-dimensional compact spaces of constant curvature with line elements ds_i^2. The connection to the scale factors a_i is given by $a_i = e^{\beta^i}$. The scalar curvature of M_i is normalized in such a way that we can write

$$R[g_{(i)}] = \theta_i = k_i d_i (d_i - 1), \qquad i = 1, \ldots, n. \tag{3}$$

where $k_i = 0, \pm 1$. In the case of non positive curvature the compactness condition for the internal spaces can be achieved by appropriate periodicity conditions for the coordinates [21].

As mentioned in the Introduction, this model can be generalized to the case of Einstein spaces M_i for which $R[g_{(i)}] = \lambda_i d_i$ instead of (3) and λ_i are arbitrary numbers. In formulas obtained later on this generalization is achieved by the trivial substitution $\theta_i \to \lambda_i d_i$.

We restrict our consideration to the important case when only one of the spaces M_i is not Ricci flat with negative curvature: $\theta_1 < 0$, $\theta_i = 0$, $i = 2, \ldots, n$. In this case the cosmological model is a completely integrable system [14,17]. This can be generalized by taking into account a minimally coupled free scalar field φ.

The action S for the model with the metric (1) and a minimally coupled scalar field can be written in the form [11]

$$S = \int \mathcal{L} dt \tag{4}$$

where the Lagrangian has the form

$$\mathcal{L} = \frac{\mu}{2} e^{-\gamma + \sum_{i=1}^{n} d_i \beta^i} \left\{ \sum_{j=1}^{n} d_j (\dot{\beta}^j)^2 - (\sum_{j=1}^{n} d_j \dot{\beta}^j)^2 + \kappa^2 \dot{\varphi}^2 \right\} - \frac{\mu}{2} e^{\gamma + \sum_{i=1}^{n} d_i \beta^i} \mid \theta_1 \mid e^{-2\beta^1} \tag{5}$$

Here κ^2 denotes the gravitational constant and $\mu = \prod_{i=1}^{n} V_i / \kappa^2$ where V_i is the volume of M_i: $V_i = \int_{M_i} d^{d_i} y (det(g_{m_i n_i}))^{1/2}$. The metric (1) can be normalized in such a way that

$\mu = 1$. After normalizing μ we may also consider noncompact spaces. We shall also use natural units with $\kappa^2 = 1$.

The analysis of this system will be done mostly in two time gauges, in the gauge of harmonic time τ [11] where $\gamma = \sum_{i=1}^{n} d_i\beta^i$, and in the gauge of synchronous time t with $\gamma = 0$.

The possibility of the free choice of gauge implies the following constraint equation

$$e^{-\gamma}\left[\sum_{i=1}^{n} d_i(\dot{\beta}^i)^2 - (\sum_{i=1}^{n} d_i\dot{\beta}^i)^2 + \dot{\varphi}^2\right] + e^{-\gamma} \mid \theta_1 \mid e^{-2\beta^1} = 0 \tag{6}$$

It was shown in [14] that the field equations for this model can be integrated most easily using the following coordinates:

$$\begin{aligned} qv^0 &= (d_1 - 1)\beta^1 + \sum_{i=2}^{n} d_i\beta^i, \\ qv^1 &= \left[(D-2)/(d_1\textstyle\sum_2\right]^{\frac{1}{2}} \sum_{i=2}^{n} d_i\beta^i, \\ qv^i &= \left[(d_1-1)d_i/(d_1\textstyle\sum_i\sum_{i+1}\right]^{\frac{1}{2}} \sum_{j=i+1}^{n} d_j(\beta^j - \beta^i), \qquad i = 2,\ldots,n-1 \end{aligned} \tag{7}$$

Here we used the notation $D = 1 + \sum_{i=1}^{n} d_i$, $q^2 = (d_1 - 1)/d_1$, and $\sum_i = \sum_{j=i}^{n} d_j$. In the Lorentzian region and in harmonic time gauge the Lagrangian and the constraint equation take the form

$$\mathcal{L} = \frac{1}{2}\left(-(\dot{v}^0)^2 + \sum_{i=1}^{n-1}(\dot{v}^i)^2 + \dot{\varphi}^2\right) - \frac{1}{2} \mid \theta_1 \mid e^{2qv^0} \tag{8}$$

and

$$-(\dot{v}^0)^2 + \sum_{i=1}^{n-1}(\dot{v}^i)^2 + \dot{\varphi}^2 + \mid \theta_1 \mid e^{2qv^0} = 0 \tag{9}$$

The dot denotes the derivative with respect to the harmonic time τ. The consideration of the system can be generalized to the Euclidean region by analytic continuation.

The equations of motion corresponding to the Langrangian (8) read

$$\begin{aligned} \ddot{v}^0 - q \mid \theta_1 \mid e^{2qv^0} &= 0 \\ \ddot{v}^i &= 0, \qquad i = 1,\ldots,n-1 \\ \ddot{\varphi} &= 0 \end{aligned} \tag{10}$$

The last two equations are easy to integrate. We find

$$\begin{aligned} v^i &= \nu^i\tau + c^i, \qquad i = 1,\ldots,n-1 \\ \varphi &= \nu^n\tau + c^n \end{aligned} \tag{11}$$

where the ν^i and c^i are constants of integration. In minisuperspace of vectors $\vec{v} = (v^0, v^1, \ldots, v^{n-1}, v^n \equiv \varphi)$ the indices are raised and lowered by the diagonal metric $\eta = (-1, +1, \ldots, +1)$ [14]. Thus, we have $v^0 = -v_0, v^i = v_i, \nu^i = \nu_i$ and $c^i = c_i, i = 1,\ldots,n$. Now the constraint equation may be rewritten

$$(\dot{v}^0)^2 - |\theta_1|e^{2qv^0} = \epsilon \tag{12}$$

with

$$\epsilon = \sum_{i=1}^{n}(\nu^i)^2 \tag{13}$$

It can be seen from (12) that ϵ can be treated as an energy. This was shown in more detail in [14].

3 CASSICAL SOLUTIONS

Equations (10 - 13) are written in the Lorentzian region. The parameters ν_i $(i = 1,\ldots,n)$ are the momenta in minisuperspace. Thus, $E \equiv \frac{1}{2}\epsilon$ plays the role of energy [14]. Equation (12) shows that we can consider the case $\epsilon \leq 0$ as well as $\epsilon > 0$. We have to demand the metric to be real in the Lorentzian region. In what follows, the momenta ν_i $(i = 1,\ldots,n-1)$ should be real there (see eqn. (11). The case $\epsilon = 0$ is treated as ground state where all momenta are putted equal to zero: $\nu_i = 0$ $(i = 1,\ldots,n)$. For $\epsilon > 0$ all ν_i $(i = 1,\ldots,n)$ are considered to be arbitrary real numbers. The case $\epsilon < 0$ is a little bit more complicated. Here the demand of a real metric in the Lorentzian region leads to the condition that all ν_i $(i = 1,\ldots,n-1)$ are real and the condition $\epsilon < 0$ and eqn. (13) are compatible only for a purely imaginary ν_n. This means that the scalar field has in this case to be imaginary in the Lorentzian region and we have the following condition

$$\sum_{i=1}^{n-1}(\nu_i)^2 - \mid \nu_n \mid^2 < 0 \tag{14}$$

Let us consider the three special cases $\epsilon = 0$, $\epsilon < 0$, and $\epsilon > 0$ separately.

3.1 The case $\epsilon = 0$

As shown above in this case we have $\nu_i = 0$ $(n = 1,\ldots,n)$. Then it can be seen from eqns. (11) and (7) that all scale factors are frozen ($a_i = e^{\beta^i} = a_{0(i)}$, $i = 2,\ldots,n$. $\varphi = \Phi_0$) with the exception of $a_1 = e^{\beta^1}$. Such solutions we call *solutions with spontaneous compactification.*

In this case we have only one scale factor with dynamical behaviour (in our case a_1) and the corresponding factor space M_1 will be associated with the external (our real) space. The fixed scale factors are assumed to be of the order of Planck length $a_{(0)i} \sim L_{Pl} \sim 10^{-33}$cm ($i = 2, \ldots, n$) and the associated factor spaces are looked at as unobservalbe internal dimensions.

The dynamical behaviour of the remaining developing scale factor can be defined from the solution of eqn. (12)

$$e^{\gamma v^0} = ((d_1 - 1) \mid \tau \mid)^{-1}, \qquad -\infty < \tau < \infty \tag{15}$$

With the help of transformation (7) we find the expression for the scale factor a_1 in hamonic time gauge

$$a_1(\tau) = [1/C(d_1 - 1) \mid \tau \mid]^{1/(d_1-1)} \tag{16}$$

where

$$C = \prod_{i=2}^{n} a_{(0)i}^{d_i} \tag{17}$$

Then the metric in harmonic time takes the form

$$g = -e^{2\gamma(\tau)} d\tau \otimes d\tau + a_1^2(\tau) g_{(1)} + \sum_{i=2}^{n} a_{(0)i}^2 g_{(i)} \tag{18}$$

where

$$e^{\gamma(\tau)} = C a_1^{d_1}(\tau) \tag{19}$$

It is worth to rewrite the metric in synchronous time coordinate t. In this case we have $a_1(t) = \mid t \mid$ and the metric is given by

$$g = -dt \otimes dt + t^2 g_{(1)} + \sum_{i=2}^{n} a_{(0)i}^2 g_{(i)} \tag{20}$$

From this expression we see that the dynamical part of the universe is described the Milne model. [[22]]. In this way we find for the physically interesting case of Kaluza-Klein theory with $d_1 = 3$ and M_1 being an open hyperbolic space that the solution with $\epsilon = 0$ describes the following topology of a spontaneously compactified universe

$$M^4 \times T^{d_2} \times \ldots \times T^{d_n} \tag{21}$$

Here M^4 is the four dimensional Minkowski space-time and the T^{d_i} are d_i- dimensional Tori (or other compact spaces of constant zero curvature). The Tori are frozen and asumed to have scales of Planck size.

3.2 The case $\epsilon > 0$

As explained above, in this case all parameters $\nu_i, (i = 1, \ldots, n)$ are considered to be real. The solution of eqn. (12) takes the form

$$e^{\gamma v^0} = \frac{\sqrt{\epsilon / \mid \theta_1 \mid}}{\mid \sinh \left[(d_1 - 1)\sqrt{\epsilon / \mid \theta_1 \mid}(\tau - \tau_0)\right] \mid}, \qquad -\infty < \tau < +\infty \tag{22}$$

where τ_0 is a constant of integration. Choosing the initial value of the harmonic time coordinate in an appropriate way we can put $\tau_0 = 0$.

The eqns. (22) and (11) give a general solution of the system (10) with constraint (12). But in cosmology usually the synchronous time coordinate t is used and it is quite difficult to find the dependence of the scale factors $a_i = e^{\beta^i}$ on this time coordinate. But the explicit dependence on t can be found in some interesting special cases which we now consider.

3.2.1 The 2-component universe. Dynamical compactification

Let us consider the special case where only two factor spaces are included in the model. We shall show that in this case solutions with dynamical compactification occur, that means solutions with both scale factors depending on time but with one, let it be a_1, increasing and the other one (a_2) shrinking to Planck scales. In this case M_1 is treated as our external space and M_2 describes an unobservable internal space. For two component cosmological models (i.e. for $n = 2$ in eqns. (11) and (22)) it is easy to get the explicit expressions for the scale factors as functions of harmonic time:

$$a_1^{d_1-1} = \frac{2 a_{(0)_1}^{d_1-1}}{e^{\sqrt{\frac{d_2(d_1-1)}{D-2}}\,\nu_1 \tau}\left[e^{\sqrt{\frac{(d_1-1)(\nu_1^2+\nu_2^2)}{d_1}}\,|\tau|} - e^{-\sqrt{\frac{(d_1-1)(\nu_1^2+\nu_2^2)}{d_1}}\,|\tau|}\right]} \tag{23}$$

$$a_2^{d_2} = a_{(0)_2}^{d_2} e^{\sqrt{\frac{d_2(d_1-1)}{D-2}}\,\nu_1 \tau} \tag{24}$$

Here $a_{(0)_1}$ and $a_{(0)_2}$ are connected with the constant of integration c_1 in (11) by the expressions

$$a_{(0)_1}^{d_1-1} = \sqrt{\frac{\nu_1^2 + \nu_2^2}{d_1(d_1 - 1)}}\, e^{-\sqrt{\frac{d_2(d_1-1)}{D-2}}\, c_1} \tag{25}$$

$$a_{(0)_2}^{d_2} = e^{\sqrt{\frac{d_2(d_1-1)}{D-2}}\, c_1} \tag{26}$$

From this we get the connection between $a_{(0)_1}$ and $a_{(0)_2}$

$$a_{(0)_1}^{d_1-1} a_{(0)_2}^{d_2} = \sqrt{\frac{\nu_1^2+\nu_2^2}{d_1(d_1-1)}} \tag{27}$$

It can be seen from (23) that a_1 has a discontinuity at $\tau = 0$. Therefore, we have different solutions in the region $-\infty < \tau \leq 0$ and $0 \leq \tau < \infty$.

It follows from eqn. (23) that we have three different types of development of the scale factors a_1 and a_2 in dependence from the sign of $\sqrt{d_1 d_2 \nu_1^2} - \sqrt{(d_1+d_2-1)(\nu_1^2+\nu_2^2)}$. Let us consider these cases in more detail.

1. $\sqrt{d_1 d_2 \nu_1^2} - \sqrt{(d_1+d_2-1)(\nu_1^2+\nu_2^2)} > 0$

Here, in dependence on the sign of ν_1 there two types of solutions exist for the scale factors (see qualitative pictures fig. 1 and fig. 2). From fig. 1 ($\nu_1 > 0$) we see that for $\tau > 0$ dynamical compactification occurs when $\tau \to +\infty$. For $\tau < 0$ the dynamical behaviour of the scale factors is more complicated. Nevertheless, there are also time intervals where one of the scale factors is much bigger than the other one, for example in the limit $\tau \to -0$. For the case of fig. 2 ($\nu_1 < 0$) dynamical compactification occurs for $\tau < 0$ when $\tau \to -0$ or for $\tau > 0$ if $\tau \to +\infty$. Moreover, depending on the sign of ν_1 the space M_1 or the space M_2 can play the role of the external space. We have also to mention that the case 1. can be realized if no scalar field is present ($\nu_2 = 0$). It is clear that the condition 1. is not valid for $d_2 = 1$.

2. $\sqrt{d_1 d_2 \nu_1^2} - \sqrt{(d_1+d_2-1)(\nu_1^2+\nu_2^2)} < 0$

The qualitative behaviour of the scale factors is shown in figs. 3, 4. It can be seen that there are also in this case regions with dynamical compactification. Because of $d_1, d_2 \geq 1$ the case 2. can not be realized if $\nu_2 = 0$, i.e. without a scalar field being present.

3. $\sqrt{d_1 d_2 \nu_1^2} - \sqrt{(d_1+d_2-1)(\nu_1^2+\nu_2^2)} = 0$

In this case we have a connection between the two constants of integration ν_1 and ν_2:

$$\nu_2^2 = \left(\frac{d_1 d_2}{d_1+d_2-1} - 1\right)\nu_1^2 \tag{28}$$

From this expression we see that in the case $d_2 = 1$ there is no scalar field: $\nu_2 = 0$. Using (28), eqn. (27) reads

$$a_{(0)1}^{d_1-1} a_{(0)2}^{d_2} = \sqrt{\frac{d_2}{(d_1-1)(d_1+d_2-1)}}\,|\,\nu_1\,| \tag{29}$$

In this case we find for the scale factors the following expressions

$$a_1^{d_1-1} = \frac{2a_{(0)1}^{d_1-1}}{|\,e^{\pm 2\sqrt{\frac{d_2(d_1-1)}{D-2}}\,|\nu_1|\tau} - 1\,|} \tag{30}$$

$$a_2^{d_2} = a_{(0)2}^{d_2} \exp\left\{\sqrt{\frac{d_2(d_1-1)}{D-2}}\,\nu_1\tau\right\} \tag{31}$$

where the upper sign in the expression for a_1 corresponds to $\nu_1 > 0$ and the lower one to $\nu_1 < 0$. The qualitative behaviour of the scale factors in the case $2^{1/(d_1-1)} a_{(0)1} > a_{(0)2}$ is given in figs. 5, 6 showing the existence of regions with dynamical compactification, too.

Because of the simple form of eqn. (30) we can find the explicit connection between the conformal time τ and the synchronous time t. This connection is given by [11]

$$dt = \pm e^{\gamma} d\tau = \pm a_1^{d_1} a_2^{d_2} d\tau \tag{32}$$

Putting (30) and (31) into (32) we find

$$t = \pm\tilde{c}\int \frac{dy}{|\,y^2-1\,|^{\frac{d_1}{d_1-1}}} + \tilde{\tilde{c}} \tag{33}$$

where

$$y = e^{\pm\sqrt{\frac{d_2(d_1-1)}{D-2}}\,|\nu_1|\tau} \tag{34}$$

The upper sign corresponds to $\nu_1 > 0$ the lower one to $\nu_1 < 0$. The constant $\tilde{c}$ is defined by

$$\tilde{c} = 2^{\frac{d_1}{d_1-1}} \sqrt{\frac{d_1+d_2-1}{d_2(d_1-1)}}\,\frac{1}{|\,\nu_1\,|}\, a_{(0)1}^{d_1} a_{(0)2}^{d_2} \tag{35}$$

and the initial value of the synchronous time will be taken so that $\tilde{\tilde{c}} = 0$.

In the case $d_1 = 2, d_2 \geq 1$ the integration of (34) gives

$$t = \pm\frac{\tilde{c}}{2}\begin{cases} \frac{y}{1-y^2} + \text{arcoth}y, & |\,y\,| > 1 \\ \frac{y}{1-y^2} + \text{artanh}y, & |\,y\,| < 1 \end{cases} \tag{36}$$

This case is of special interest because for $d_2 = 1$ it describes a 3-dimensional anisotropic Bianchi III universe without scalar field [25]. As we could see above there is no isotropization in this case.

The integral (33) can be easyly calculated in the mostly important for Kaluza-Klein theory case $d_1 = 3$ (M_1 can be looked at as our external space). Then we have two types

of solutions:

$$a_1 = \frac{2^{1/2}a_{(0)1}}{\tilde{c}}(t^2-\tilde{c}^2)^{1/2} \tag{37}$$

$$a_2 = a_{(0)2}\left[\frac{t^2}{t^2-\tilde{c}^2}\right]^{\frac{1}{2d_2}} \tag{38}$$

$$\varphi = \pm\frac{1}{2}\sqrt{\frac{d_2-1}{d_2}}\ln\frac{t^2}{t^2-\tilde{c}^2} \tag{39}$$

where $\mid t \mid \geq \tilde{c}$ and

$$a_1 = \frac{2^{1/2}a_{(0)1}}{\tilde{c}}(t^2+\tilde{c}^2)^{1/2} \tag{40}$$

$$a_2 = a_{(0)2}\left[\frac{t^2}{t^2+\tilde{c}^2}\right]^{\frac{1}{2d_2}} \tag{41}$$

$$\varphi = \pm\frac{1}{2}\sqrt{\frac{d_2-1}{d_2}}\ln\frac{t^2}{t^2+\tilde{c}^2} \tag{42}$$

where $-\infty < t < +\infty$. In figures 7, 8 the behaviour of the scale factors corresponding to (37), (38) and (40), (41) is presented. In both of these cases there are regions with dynamical compactification. For the solutions (37), (38) the scale factors are everytimes in opposite phases and the role of the exterior space can be played by M_1 as well as by M_2. For instance, in the region $t \geq \tilde{c}$ the scale factor a_2 shrinks from $+\infty$ to $a_{(0)2}$ (asymptotically). For $a_{(0)2} \sim L_{Pl}$ M_2 becomes unobservable. At the same time $a_1 \sim t (t \gg \tilde{c})$ and M_1 describes the exterior space of Milne type. In this way the topology of the universe tends asymptotically to $M^4 \times T^{d_2}$.

Fig. 8 is drawn for the case $\sqrt{2}a_{(0)1} > a_{(0)2}$. For $\mid t \mid \gg \tilde{c}$ we have $a_2 \to a_{(0)2}$ and $a_1 \sim |t|$, i.e. the factor space M_2 becomes static and the factor space M_1 has asymptotically the behaviour of a Milne universe. Thus, for $|t| \gg \tilde{c}$ the topology of the universe tends asymptotically again to $M^4 \times T^{d_2}$.

Figures 9, 10 show the behaviour of the scalar field for the upper sign in formulas (39) and (42) correspondingly. It is interesting to note that $|\varphi| \to 0$ if $|t| \gg \tilde{c}$.

Here, we have investigated the two-component model ($n = 2$) in the case $\epsilon > 0$. There exists one particular case for $n > 2$ which may be reduced to the two-component model considered above (s. Appendix).

3.2.2 The n-component universe. Spontaneous compactification

Like in the case $\epsilon = 0$ we have a solution with spontaneous compactification also for $\epsilon > 0$. Obviously, this special case corresponds to the following choice of contants of integration: $\nu_i = 0$ $(i = 1, \ldots, n-1)$, $\nu_n \neq 0$. Therefore, this case is only realized in the presence of a real scalar field. All scale factors are frozen ($e^{\beta^i} = a_i = a_{(o)i}, i = 2, \ldots, n$) except one ($e^{\beta^1} = a_1$). Then the metric in harmonic time τ takes the form (18) where the scale factor a_1 is

$$a_1(\tau) = \left(\sqrt{\epsilon/\mid\theta_1\mid}/C\right)^{1/(d_1-1)}\left\{\sinh\left[(d_1-1)\sqrt{\epsilon/\mid\theta_1\mid}\mid\tau\mid\right]\right\}^{-1/(d_1-1)} \tag{43}$$

where C is defined by (17). The interval $(-\infty, -0]$ describes the expanding universe and the interval $[+0, +\infty)$ describes the contracting universe. It is not difficult to obtain for the modulus of φ (taking $c_n = 0$ in (11)) the formula

$$\mid\varphi\mid = \left[\mid\nu_n\mid\sqrt{\mid\theta_1\mid/\epsilon}/(d_1-1)\right]\operatorname{arsinh}\left(\sqrt{\epsilon/\mid\theta_1\mid}/Ca_1^{d_1-1}\right) \tag{44}$$

It is convenient to rewrite the metric in conformal time η which is connected with harmonic time τ by

$$\sinh\left[(d_1-1)\sqrt{\epsilon/\mid\theta_1\mid}(-\tau)\right] = \{\sinh[(d_1-1)\eta]\}^{-1} \tag{45}$$

Then

$$g = a_1^2(\eta)\left[-d\eta\otimes d\eta + g_{(1)}\right] + a_{(0)2}^2 g_{(2)} + \ldots + a_{(0)n}^2 g_{(n)} \tag{46}$$

and the scale factor a_1 as a function of the conformal time is given by

$$a_1(\eta) = \left(\sqrt{\epsilon/\mid\theta_1\mid}/C\right)^{1/(d_1-1)}\{\sinh[(d_1-1)\mid\eta\mid]\}^{1/(d_1-1)} \tag{47}$$

In synchronous time t the metric takes the form

$$g = -dt\otimes dt + a_1^2(t)g_{(1)} + \sum_{i=2}^{n} a_{(0)i}^2 g_{(i)} \tag{48}$$

where the scale factor a_1 and the time coordinate t are connected by

$$t = \int\frac{a_1^{d_1-1}da_1}{\sqrt{\epsilon/(|\theta_1|C^2) + a_1^{2(d_1-1)}}} + const \tag{49}$$

If $a_1 \ll \left[\sqrt{\epsilon/\mid\theta_1\mid}/C\right]^{1/(d_1-1)}$ the scale factor a_1 has the asymptotic behaviour $a_1 \sim \mid t\mid^{1/d_1}$. It corresponds to the open Friedmann universe filled with radiation for $d_1 = 2$ and filled

with ultrastiff matter for $d_1 = 3$. If $a_1 \gg \left[\sqrt{\epsilon/\mid\theta_1\mid}/C\right]^{1/(d_1-1)}$ the scale factor a_1 has the asymptotic behaviour corresponding to a Milne universe $a_1 \sim \mid t \mid$ for all d_1. Therefore, in the case of spontaneous compactification the topology of the universe asymptotically tends to $M^{d_1+1} \times T^{d_2} \times \ldots \times T^{d_n}$, where M^{d_1+1} denotes the d_1+1 dimensional observable Milne universe (for $d_1 = 3$ this is the Minkowky space-time) and the T^{d_i} represent the frozen (unobservable) internal spaces which are d_i dimensional tori or other compact spaces of constant zero curvature (later on we shall call this topology $M \times T$ topology).

For $d_1 = 2$ the integral (49) can be expressed by elementary functions

$$a_1 = \left[\sqrt{\epsilon/\mid\theta_1\mid}/C\right]\left[\left(C\sqrt{\mid\theta_1\mid/\epsilon}\mid t\mid+1\right)^2 - 1\right]^{1/2} \tag{50}$$

but for $d_1 > 2$ we get elliptic integrals.

3.3 The case $\epsilon < 0$

In this case, as we can see from equation (12), there are classically allowed and forbidden regions. Classically forbidden regions are usually treated as regions with Euclidean signature and classically allowed ones as regions with Lorentzian signature. As we shall see in the next chapter, on the quantum level there are tunneling solutions which describe processes with metric signature alteration [18], for example, universe nucleation from "nothing"[26]. If $\epsilon < 0$ at least some of the ν_i should be imaginary. This leads to complex metric and scalar field. As stressed above it is possible to have real metric and imaginary scalar field in Lorentzian region if we demand that ν_i $(i = 1, \ldots, n-1)$ are real and ν_n is imaginary and the ν_i should satisfy the condition (14).

Let us consider the equations (10), (12) in the classically allowed region $\exp(2qv^0) \geq \epsilon/\theta_1$. The solutions of the equations of motion in harmonic time gauge are

$$\exp(qv^0) = \frac{\sqrt{\epsilon/\theta_1}}{cos[(d_1-1)\sqrt{\epsilon/\theta_1}\tau]}, \qquad \mid\tau\mid \leq \frac{\pi}{2(d_1-1)}\sqrt{\frac{\theta_1}{\epsilon}} \equiv \tau_1 \tag{51}$$

where the constant of integration τ_1 is fixed by a proper choice of the origin of the time coordinate τ, and

$$\begin{aligned} v^i &= \nu_i\tau + c_i, \qquad i = 1, \ldots, n-1 \\ \varphi &= \nu_n\tau + c_n \end{aligned} \tag{52}$$

The solution in the classically forbidden region can be found with the help of an analytic continuation $\tau \to -i\tau$ in the expressions (51), (52).
As in the case $\epsilon > 0$ we consider here two special cases.

3.3.1 The 2-component universe. Dynamical compactification

In this case the scale factors as functions of the harmonic time coordinate read

$$\begin{aligned} a_1^{d_1-1} &= \frac{a_{(0)1}^{d_1-1}}{\exp\left[\sqrt{\frac{d_2(d_1-1)}{D-2}}\,\nu_1\tau\right]\cos\left[\sqrt{\frac{d_1-1}{d_1}(\mid\nu_2\mid^2-\nu_1^2)}\,\tau\right]}, \\ a_2^{d_2} &= a_{(0)2}^{d_2}\exp\left[\sqrt{\frac{d_2(d_1-1)}{D-2}}\,\nu_1\tau\right] \end{aligned} \tag{53}$$

where $\mid\tau\mid \leq \tau_1$ and the constants $a_{(0)1}$ and $a_{(0)2}$ are defined by eqns. (25), (26) with the substitution $\nu_1^2 + \nu_2^2 \to |\nu_2|^2 - \nu_1^2$. Corresponding to (11) the scalar field is given by $\varphi = i \mid \nu_2 \mid \tau + c_2$. It can be seen from (53) that there are different types of development of the scale factors in dependence from the sign of ν_1. The qualitative picture is shown in fig. 11 and fig. 12. Fig. 11 corresponds to $\nu_1 > 0$ where the scale factor a_2 increases monotonically from the minimal value at $\tau = -\tau_1$ to the maximal value at $\tau = \tau_1$, while a_1 decreases from $+\infty$ at $\tau = -\tau_1$ down to $a_{1\min}$ and after that tends to $+\infty$ at $\tau = \tau_1$. If the value $a_{2\max}$ does not exceed L_{Pl} too much for $\tau \to \tau_1$ dynamical compactification arises. Fig. 12 corresponds to $\nu_1 < 0$. Here, a_2 monotonically decreases from $a_{2\max}$ at $\tau = -\tau_1$ to $a_{2\min}$ at $\tau = \tau_1$ and a_1 once more decreases from $+\infty$ at $\tau = -\tau_1$ down to $a_{1\min}$ and after that tends to $+\infty$ at $\tau = \tau_1$. In this case the region $0 \leq \tau \leq \tau_1$ gives an example of dynamical compactification for $\tau \to \tau_1$.

In contrast to the case $\epsilon > 0$ for $\epsilon < 0$ we are not able to give the explicit expression for the scale factors as functions of the synchronous time coordinate if more than one scale factors have a dynamical behaviour.

As it was shown in [18] the quantum solutions with fixed value of $\epsilon < 0$ describe transitions from the classically forbidden Euclidean region to the classically allowed Lorentzian one (s. next section). The universe is created by quantum tunneling with the scale $e^{qv_0} = \sqrt{\epsilon/\theta_1}$ what cooresponds to the time $\tau = 0$ in eqns. (51) - (53). After that the evolution of the universe is described by the classical equations (51) - (53). Figs. 11, 12 show that we have two possible pictures for the evolution of the universe. Due to the first possibility (fig. 11) the universe is created by quantum tunnelling with the scale

factors $a_1 = a_{(0)1}$ and $a_2 = a_{(0)2}$. After that the space M_1 shrinks to $a_1 = a_{1\min}$ and then increases to $+\infty$. The space M_2 expands monotonically to $a_2 = a_{2\max}$. The second possibility (fig. 12) describes a universe created with $a_1 = a_{(0)1}$ and $a_2 = a_{(0)2}$. Here M_1 expands monotonically to $+\infty$ while M_2 contracts to $a_2 = a_{2\min}$.

In conclusion of this section we would like to mention the existence of the special case of a n $(n > 2)$ component model which may be reduced to the two-component model considered above (s. the Appendix).

3.3.2 The n- component universe. Spontaneous compactification

From the considerations above we conclude that spontaneous compactification corresponds to the case $\nu_i = 0$ $(i = 1, \ldots, n-1), \nu_n \neq 0$. The condition $\epsilon < 0$ restricts spontaneous compactification to the presense of a purely imaginary scalar field (ν_n imaginary). All the scale factors $a_i = a_{(0)i}$ $(i = 2, \ldots, n)$ are frozen and $a_1 = e^{\beta^1}$ has a dynamical behaviour, only. From (51), (52) and the transformation (7) we find in the Lorentzian region the scale factor a_1 as a function of the harmonic time coordinate

$$a_1(\tau) = \left(\sqrt{\epsilon/\theta_1}/C\right)^{1/(d_1-1)} \left\{\cos\left[(d_1-1)\sqrt{\epsilon/\theta_1}\tau\right]\right\}^{-1/(d_1-1)} \tag{54}$$

where C is defined by (17). In the interval $\left[-\frac{\pi}{2(d_1-1)}\sqrt{\theta_1/\epsilon}, 0\right]$ the universe contracts from infinity to the classical turning point where $a_1(\tau = 0) = \left(\sqrt{\epsilon/\theta_1}/C\right)^{1/(d_1-1)}$ and after that in the interval $\left[0, \frac{\pi}{2(d_1-1)}\sqrt{\theta_1/\epsilon}\right]$ it expands to infinity again.

It is not difficult to obtain the absolute value of the scalar field in dependence of the scale factor a_1 (taking $c_n = 0$ in (52))

$$|\varphi| = \left[|\nu_n| \frac{\sqrt{\theta_1/\epsilon}}{d_1 - 1}\right] \arccos\left[\frac{\sqrt{\epsilon/\theta_1}}{C a_1^{d_1-1}}\right] \tag{55}$$

This formula shows that $|\varphi|$ has it's minimum at the turning point and tends asymptotically to $\frac{\pi|\nu_n|}{2(d_1-1)}\sqrt{\theta_1/\epsilon}$ when $a_1 \to \infty$. The space-time metric takes in harmonic time gauge the form (18), where $a_1(\tau)$ is defined by (54).

The harmonic time τ and the conformal time η are connected to each other by the expression

$$\cos\left[(d_1-1)\sqrt{\epsilon/\theta_1}\tau\right] = \{\cosh[(d_1-1)\eta]\}^{-1}, \qquad -\infty < \eta < +\infty \tag{56}$$

and the metric in conformal time takes the form (46) where a_1 depemds on η as

$$a_1(\eta) = \left[\sqrt{\epsilon/\theta_1}/C\right]^{1/(d_1-1)} \{\cosh[(d_1-1)\eta]\}^{1/(d_1-1)}, \qquad -\infty < \eta < +\infty \tag{57}$$

In order to investigate the asymptotic behaviour of the scale factor at large times let us consider synchronous coordinates whith the metric (48). We find for the time dependence of the scale factor a_1

$$t = \int \frac{a_1^{d_1-1} da_1}{\sqrt{a_1^{2(d_1-1)} - \epsilon/(\theta_1 C^2)}} + const \tag{58}$$

Then asymptotically, when $a_1 \gg \left[\sqrt{\epsilon/\theta_1}/C\right]^{1/(d_1-1)}$, we have $a_1 \sim |t|$. Thus, the universe behaves asymptotically like a Milne Universe with respect to the scale factor a_1. Therefore, as in the case of spontaneous compactification for $\epsilon > 0$ (paragraph 3.2.2) the topology of the universe asymptotically tends to $M \times T$.

In the particular case $d_1 = 2$ we have from (58)

$$a_1^2(t) = t^2 + \epsilon/(\theta_1 C^2), \qquad -\infty < t < +\infty \tag{59}$$

For $d_1 > 2$ this integral can be expressed with the help of elliptic integrals. For example, in the case $d_1 = 3$ we have

$$t = \left[\frac{1}{C}\sqrt{\frac{\epsilon}{\theta_1}}\right]^{1/2} \left\{\frac{1}{\sqrt{2}} F\left(\Psi, \frac{\sqrt{2}}{2}\right) - \sqrt{2} E\left(\Psi, \frac{\sqrt{2}}{2}\right)\right\} + \frac{1}{a_1}\left[a_1^4 - \frac{\epsilon}{\theta_1 C^2}\right]^{1/2} \tag{60}$$

where

$$\Psi = \arccos\left[\left(\sqrt{\frac{\epsilon}{\theta_1}}/C\right)^{1/2} / a_1\right] \tag{61}$$

and F and E are the elliptic integrals of first and second kind repectively.

The classical expressions for the Euclidean region $e^{q\nu^0} < (\epsilon/\theta_1)^{1/2}$ can be found by analytic continuation of the formulas obtained here. Then the point

$$a_1 = \left[\sqrt{\epsilon/\theta_1}/C\right]^{1/(d_1-1)} \tag{62}$$

is the classical turning point. The nucleation of the universe can be considered as the quantum tunneling process through the potential barrier [26]. The universe is nucleated with finite size $a_1 = \left[\sqrt{\epsilon/\theta_1}/C\right]^{1/(d_1-1)}$ and zero speed $(da_1/dt = 0)$ and its further evolution is described by classical formulas (54), (57), and (58).

4 SOLUTIONS TO THE QUANTIZED MODEL

At the quantum level the constraint equation (9) turns over to the Wheeler-De Witt equation (WDW). The WDW equation is covariant with respect to minisuperspace coordinate transformations and can be written in the harmonic time gauge in the following form [14]

$$\left(-\frac{\partial^2}{\partial v^{0^2}}+\frac{\partial^2}{\partial v^{1^2}}+\ldots+\frac{\partial^2}{\partial v^{n-1^2}}+\frac{\partial^2}{\partial \varphi^2}-\mid\theta_1\mid e^{2qv^0}\right)\Psi=0 \tag{63}$$

It is easy to obtain solutions of the WDW equation (63) by separation of variables

$$\Psi=\Psi_0(v^0)\ldots\Psi_{n-1}(v^{n-1})\Psi_n(\varphi) \tag{64}$$

where

$$\Psi_i(v^i)=e^{i\nu_i v^i}, i=1,\ldots,n-1; \qquad \Psi_n(\varphi)=e^{i\nu_n\varphi} \tag{65}$$

and Ψ_0 satisfies the equation

$$\left(-\frac{d^2}{dv_0{}^2}-\mid\theta_1\mid e^{2qv^0}\right)\Psi_0=\epsilon\Psi_0 \tag{66}$$

Here, ϵ and the arbitrary numbers ν_i are related to each other by

$$\epsilon=\sum_{i=1}^{n}\nu_i^2 \tag{67}$$

The solutions of equation (66) are

$$C^{(m)}_{i\sqrt{\epsilon}/q}\left[\left(\sqrt{\mid\theta_1\mid}/q\right)e^{qv^0}\right] \tag{68}$$

where $C^{(m)}$ denotes the Bessel function of the first ($m=1$), second ($m=2$) or third ($m=3$) kind. It was shown in [14] that ϵ can be interpreted as an energy. From this point of view the states with $\epsilon=0$ are treated as ground states. The general solution can be written in the form

$$\Psi=\sum_{m=1}^{3}\int d^n\nu B^{(m)}(\nu)C^{(m)}_{i\sqrt{\epsilon}/q}\prod_{j=1}^{n}e^{i\nu_j v^j} \tag{69}$$

where the $B^{(m)}(\nu)$ are arbitrary coefficients depending on the quantum numbers ν_i.

The solutions (64) are eigenstates of the quantum mechanical operators

$$\Pi_{v^i}=-\frac{i}{N}\frac{\partial}{\partial v^i}, i=1,\ldots,n-1, \qquad \Pi_\varphi=-\frac{i}{N}\frac{\partial}{\partial\varphi} \tag{70}$$

with the eigenvalues ν_n/N, where we have $N=1$ for the Lorentzian space-time and $N=i$ for the Euclidean one. The classical equations corresponding to the states (64) are [14]

$$\dot{v}^i=N^{-1}\nu_i, \qquad i=1,\ldots,n-1, \qquad \dot{\varphi}=N^{-1}\nu_n \tag{71}$$

where the dot denotes the differentiation with respect to harmonic time τ. Evidently, equations (71) coinside with (10), (11). Thus, for the classical equations corresponding to (64) the constants of integration ν_i in (11) should coinside with the quantum numbers ν_i in (65).

Let us consider the wave function (64) in more detail. In the same way as in the classical case we distinguish three special cases: $\epsilon=0,>0,<0$.

4.1 The case $\epsilon=0$

This case was considered earlier in [14], where the state with $\epsilon=0$ was treated as the ground state. For all ν_i to be real the condition $\epsilon=0$ leads to the demand $\nu_i=0$ ($i=1,\ldots,n$). As shown above, this corresponds in the classical limit to an universe with spontaneous compactification and the $M\times T$ topology. Once more, this describes the product of an d_1+1 dimensional Milne Universe and static d_i dimensional tori or other compact spaces of constant zero curvature. The wave function

$$\Psi_0^{(0)}=J_0\left[\left(\sqrt{\mid\theta_1\mid}/q\right)e^{qv^0}\right]=J_0\left[\left(\sqrt{\mid\theta_1\mid}/q\right)a_1^{d_1-1}a_{(0)2}^{d_2}\ldots a_{(0)n}^{d_n}\right] \tag{72}$$

is related to the Hartle-Hawking boundary conditions [27] and describes the superposition of expanding and contracting universes [14]. The wave function

$$\Psi_0^{(2)}=H_0^{(2)}\left[\left(\sqrt{\mid\theta_1\mid}/q\right)e^{qv^0}\right] \tag{73}$$

describes the expanding universe and satisfies the Vilenkin boundary conditions [14,28]. The wave function

$$\Psi_0^{(1)}=H_0^{(1)}\left[\left(\sqrt{\mid\theta_1\mid}/q\right)e^{qv^0}\right] \tag{74}$$

describes the contracting universe. Here J_0 and $H_0^{(1,2)}$ are the Bessel functions of first and third kind respectively.

The Hartle-Hawking ground state wave function (72) is non-singular. The vacuum wave functions (73) and (74) go to infinity as $\ln a_1$ when $a_1\to 0$.

4.2 The case $\epsilon > 0$

Let us assume all ν_i to be real. This corresponds to real metric and scalar field in the Lorentzian region.

The excited states (64) can be written in the form

$$\Psi^{(0)}_{\nu_1,\ldots,\nu_n} = J_{ik}\left[\left(\sqrt{|\theta_1|}/q\right)e^{qv^0}\right]\prod_{j=1}^{n} e^{i\nu_j(v^j-c^j)} \tag{75}$$

$$\Psi^{(1,2)}_{\nu_1,\ldots,\nu_n} = H^{(1,2)}_{ik}\left[\left(\sqrt{|\theta_1|}/q\right)e^{qv^0}\right]\prod_{j=1}^{n} e^{i\nu_j(v^j-c^j)} \tag{76}$$

where the energy $\epsilon = \sum_1^n \nu_i^2 \equiv q^2k^2, -\infty < k < +\infty$, $\varphi \equiv v^n$, and the translation invariance of the WDW equation (63) was used for the v^i $(i = 1,\ldots,n)$.

All these wave functions oscillate an infinite number of times when the spacial geometry degenerates ($a_i \to 0$). This singular behaviour reflects the initial and final singularities of the classical solutions.

As shown above the solutions of the classical Lorentzian equations have the following asymptotic behaviour. When $\tau \to 0$ we have $|t| \to \infty$ and $v^i \to c^i$ so that $a_i \to a_{(0)i}$ $(i = 2,\ldots,n)$ and $a_1 \sim |t|$. This corresponds asymptotically to a freezing of the internal dimensions, and the limit of dynamical compactification is spontaneous compactification. Therefore, spontaneous compactification acts as an attractor solution for solutions with dynamical compactification. The corresponding limit for the wave functions (75), (76) may be achieved for $v^0 \to \infty, v^i \to c^i (i = 1,\ldots,n)$. In this case the asymptotic bahaviour of the solutions (75), (76) is given by

$$\Psi^{(0)} = \cos\left[\left(\sqrt{|\theta_1|}/q\right)a_1^{d_1-1}a_{(0)2}^{d_2}\cdots a_{(0)n}^{d_n}\right] \tag{77}$$

$$\Psi^{(j)} = \exp\left[(-1)^{j+1}i\left(\sqrt{|\theta_1|}/q\right)a_1^{d_1-1}a_{(0)2}^{d_2}\cdots a_{(0)n}^{d_n}\right] \tag{78}$$

where $j = 1,2$ and (77), (78) describe in the classical limit asymptotically a universe with the $M \times T$ topology.

Evidently, the solutions (75, 76) are not the unique solutions of equation (63), other wave functions can also be found. Some of these wave functions may be free of the above mentioned singularities. For the one component model with scalar field this was analysed in [29]. In analogy to the papers [15,16], where the case $\theta > 0$ was considered, we can give

here an example of such a solution. We write the wave function (76) in the form

$$\Psi^{(j)}_{k,\gamma_1,\ldots,\gamma_{n-1}} = \exp\left[ik\sum_{l=1}^{n} q(v^l - c^l)\Gamma_l\right]H^{(j)}_{ik}\left[\left(\frac{\sqrt{|\theta_1|}}{q}\right)e^{qv^0}\right] \tag{79}$$

where the quantum numbers ν_i are written in the form

$$\nu_i = kq\Gamma_i \tag{80}$$

and the n dimensional unit vector Γ_i is defined by

$$\Gamma_i = \begin{pmatrix} \cos\gamma_1 & & & \\ \sin\gamma_1 & \cos\gamma_2 & & \\ \sin\gamma_1 & \sin\gamma_2 & \cos\gamma_3 & \\ \vdots & & & \\ \sin\gamma_1 & \ldots & \sin\gamma_{n-2} & \cos\gamma_{n-1} \\ \sin\gamma_1 & \ldots & \sin\gamma_{n-2} & \sin\gamma_{n-1} \end{pmatrix} \tag{81}$$

Using the integral transformation [30]

$$\Psi^{(j)}_{\lambda,\gamma_1,\ldots,\gamma_{n-1}} = \int_{-\infty}^{+\infty} dk C^{(j)}_k(\lambda)\Psi^{(j)}_{k,\gamma_1,\ldots,\gamma_{n-1}} \tag{82}$$

with

$$C^{(j)}_k(\lambda) = \frac{i}{2}(-1)^{j+1}\exp\left[(-1)^j\frac{\pi k}{2}\right]\exp(-ik\lambda) \tag{83}$$

where $-\infty < \lambda < \infty$, we get

$$\Psi^{(j)}_{\lambda,\gamma_1,\ldots,\gamma_{n-1}} = \exp\left\{(-1)^{j+1}i\left(\frac{\sqrt{|\theta_1|}}{q}\right)e^{qv^0}\cosh\left[\sum_{i=1}^{n} q(v^i - c^i)\Gamma_i - \lambda\right]\right\} \tag{84}$$

This wave function has the same asymptotic behaviour at $v^i \to c^i$ $(i = 1,\ldots,n)$, $v^0 \to +\infty$, as (79). It corresponds asymptotically to the contracting ($j = 1$) and expandig ($j = 2$) $(d_i + 1)$ dimensional Milne universe. The wave function (84) can already not descibe a state of fixed energy ϵ.

4.3 The case $\epsilon < 0$

In this case the wave functions (64) can be written in the form

$$\Psi^{(0)}_{\nu_1,\ldots,\nu_n} = J_k\left[\left(\frac{\sqrt{|\theta_1|}}{q}\right)e^{qv^0}\right]\prod_{j=1}^{n} e^{i\nu_j(v^j-\tilde{c}^j)} \tag{85}$$

$$\Psi^{(1,2)}_{\nu_1,\ldots,\nu_n} = H^{(1,2)}_k \left[\left(\frac{\sqrt{|\theta_1|}}{q} \right) e^{qv^0} \right] \prod_{j=1}^{n} e^{i\nu_j(v^j - \bar{c}^j)} \tag{86}$$

where $\varphi \equiv v^n, j = 1,2, \epsilon \equiv -q^2k^2, -\infty < k < \infty$ and for v^j $(j = 1,\ldots,n)$ the translation invariance of the WDW equation (63) was used.

As a consequence of the condition $\epsilon < 0$ the quantum numbers ν_i (or part of them) became imaginary. Then, in the classically allowed region we would get a complex metric. In order to avoid this we demand in analogy to the classical case all ν_i $(i = 1,\ldots,n-1)$ to be real and ν_n to be imaginary and the condition (14) to be valid. Consequently, with these conditions the metric of the classically accessible region is a real Lorentz metric, while the scalar field in this region is imaginary.

It can be seen from equation (66) (the quantum analog of (12)) that in the case $\epsilon < 0$ there exist classically accessible as well as forbidden regions. The solutions (85), (86) are solutions with fixed energy ϵ and describes transitions between the classically allowed and forbidden regions due to tunneling processes. As a consequence, it becomes possible to analyze processes with changes of the metric signature [18]. For instance, the solution $\Psi^{(2)}$ in (86) describes quantum tunneling through a potential barrier and is usually interpreted as creation of the universe from "nothing" [14,18,26]. In this way for $\nu_i = 0$ $(i = 1,\ldots,n-1)$ a n component universe with spontaneous compactification will be created, while for $\nu_1 \neq 0, \nu_i = 0$ $(i = 2,\ldots,n-1)$ the creation of a n-component universe with dynamical compactification is described. The solution $\Psi^{(1)}$ describes the opposite process, the transition into "nothing" as the final stage of the evolution of the universe.

In analogy to the case $\epsilon > 0$ we have also solutions to equation (63), which already do not describe wave functions for fixed energy. For instance, with an integral transformation of $\Psi^{(0)}$ we find [30]

$$\begin{aligned} \Psi^{(0)}_{\lambda,\gamma_1,\ldots,\gamma_{n-1}} &= \int_{-\infty}^{+\infty} dk C_k(\lambda) \Psi^{(0)}_{k,\gamma_1,\ldots,\gamma_{n-1}} \\ &= \exp\left\{ i \left(\frac{\sqrt{|\theta_1|}}{q} \right) e^{qv^0} \sin\left[\sum_{i=1}^{n-1} q(v^i - \bar{c}^i)\Gamma_i \sinh\gamma_{n-1} + \right.\right. \\ &\quad \left.\left. + qi(v^n - \bar{c}^n)\cosh\gamma_{n-1} + \lambda \right] \right\} \end{aligned} \tag{87}$$

where $C_k(\lambda) = \exp(ik\lambda)$, λ has a continuous spectrum of width 2π and the wave function (85) $\Psi^{(0)}_{\nu_1\ldots\nu_n}$ was refwritten in the form

$$\Psi^{(0)}_{k,\gamma_1,\ldots,\gamma_{n-1}} = \exp\left\{ ik \left[\sum_{i=1}^{n-1} q(v^i - \bar{c}^i)\Gamma_i \sinh\gamma_{n-1} + \right.\right.$$

$$\left.\left. + qi(v^n - \bar{c}^n)\cosh\gamma_{n-1} \right] \right\} \times J_k \left[\frac{\sqrt{|\theta_1|}}{q} e^{qv^0} \right] \tag{88}$$

where we used for the quantum numbers ν_i the representation

$$\nu_i = kq\Gamma_i \sinh\gamma_{n-1}, \qquad i = 1,\ldots,n-1 \tag{89}$$

$$\nu_n = ikq\cosh\gamma_{n-1} \tag{90}$$

Here Γ_i represents a $(n-1)$ dimensional unit vector given by a formula similar to (81) and the arbitrary constants $\bar{c}^i$ can be taken best as the classical limits for the v^i $(i = 1,\ldots,n)$ in equation (52) for $|\tau| \to \tau_1$, i.e. $\bar{c}^i = \nu^i\tau_1 + c^i$ or $\bar{c}^i = -\nu^i\tau_1 + c^i$. In this limit the v^i $(i = 1,\ldots,n)$ tend to their maximal (or minimal) fixed values and the corresponding scale factors a_i $(i = 2,\ldots,n)$ are frozen out and $a_1 \sim |t| \to \infty$. So it can be seen that in the limit $v^i \to \bar{c}^i$ $(i = 1,\ldots,n), v^0 \to \infty$, the wave functions $\Psi^{(0)}_{k,\ldots}$ and $\Psi^{(0)}_{\lambda,\ldots}$ have the same asymptotic behaviour and describe asymptotically in the classical limit an universe with the same $M \times T$ topology like in the case $\epsilon > 0$.

5 CONCLUSIONS

We investigated multidimensional cosmological models (MCM) with n $(n > 1)$ Einstein spaces for the case these Einstein spaces were of constant curvature. The integrable case was considered where only one of these spaces was of negative constant curvature, while all others were assumed to be Ricci flat. As a matter source we introduced a massless minimally coupled homogeneous scalar field. The main attention was paid to the problem of campactification of extra dimensions. The non Ricci flat space M_1 was considered as our external space while all the other spaces with zero curvature described internal spaces. But our general solutions do not exclude the possiblility that one of the Ricci flat spaces may play the role of our external space.

The problem of compactification for the model with positive curvature of the non Ricci flat space was considered in detail in our previous paper [10] where solutions with dynamical and spontaneous compactification were found. In the case of positive curvature of the non Ricci flat space the parameter ϵ defined by formula (13) plays the role of energy [14] and should be positive in the Lorentzian region. For the model with negative curvature of the non Ricci flat space considered in the present paper we have in the Lorentzian region for this

parameter $\epsilon \geq 0$ as well as $\epsilon < 0$. This feature leads to a more complex picture than in the former case. For all values of the parameter ϵ solutions with dynamical and spontaneous compactification were found. It is not difficult to see that the solution with spontaneous compactification in the case $\epsilon = 0$ is an attractor solution for all kinds of solutions with $\epsilon > 0$ as well as $\epsilon < 0$. In the limit of large goemetry all solutions tend to $a_1 \sim |t| \to \infty$ for the scale factor of the space M_1 while all other scale factors and the scalar field become freezed (see figures 7 - 10, 13). Thus, asymptotically all solutions describe the universe with the topology $M^{d_1+1} \times T^{d_2} \times \ldots \times T^{d_n}$ where M^{d_1+1} is the (d_1+1)- dimensional Milne universe and the T^{d_i} are d_i-dimensional frozen tori or other compact spaces of constant zero curvature. Solutions to the quantum Wheeler-DeWitt equation were obtained also. In the case $\epsilon < 0$ some of them describe the process of tunneling from the Euclidean region to the Lorentzian one. often called the birth of the universe from "nothing" [28]. For all values of ϵ these wave functions asymptotically describe in the classical limit an universe with the topology $M^{d_1+1} \times T^{d_2} \times \ldots \times T^{d_n}$.

Acknowledgment

The work was sponsored by the WIP project 016659/p. One of us (A. Z.) was supported by DAAD and by DFG grant 436UKR-17/7/93. A. Z. also thanks Prof. Kleinert and the Freie Universität Berlin for their hospitality.

6 APPENDIX

It is easy to see that there exist a special n-component case ($n > 2$) which can be reduced to the two component universe with dynamical compactification considered above for $\epsilon > 0$ as well as for $\epsilon < 0$. This exceptional case corresponds to the special choice of the integration constants in equation (11)

$$\begin{aligned} \nu_1 &\neq 0 \\ \nu_2 &= \ldots = \nu_{n-1} = 0 \end{aligned} \tag{A.1}$$

Than it follows from the coordinate transformation (7) that

$$a_i = e^{B_i} a_2, \qquad i = 3, \ldots, n, \tag{A.2}$$

where B_i is an arbitrary constant. Therefore, all factor spaces M_i ($i = 3, \ldots, n$) have identical dynamical behaviour, the same as M_2 in the two component universe. Further, the equations (23) - (42) and (53) remain the same and define the dynamics of the factor spaces M_1 and M_2 with the only difference that we have to make the change $d_2 \to \sum_2 = \sum_{i=2}^{n} d_i$.

Figures

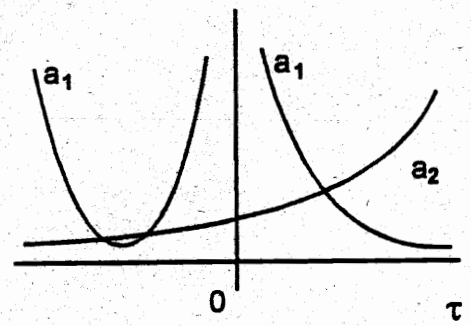

Figure 1 The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon > 0$ in the case 1. if $\nu_1 > 0$.

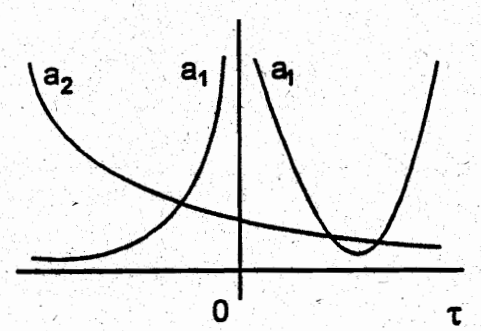

Figure 2 The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon > 0$ in the case 1. if $\nu_1 < 0$.

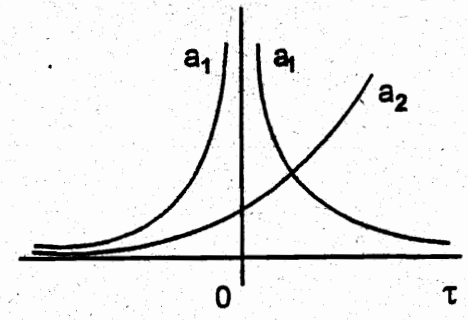

Figure 3 The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon > 0$ in the case 2. if $\nu_1 > 0$.

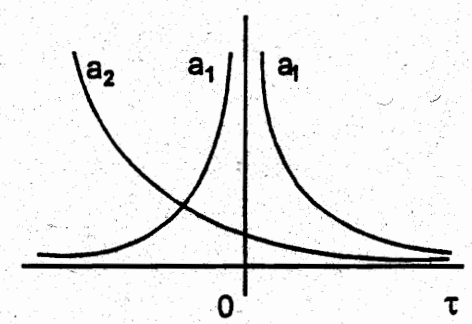

Figure 4 The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon > 0$ in the case 2. if $\nu_1 < 0$.

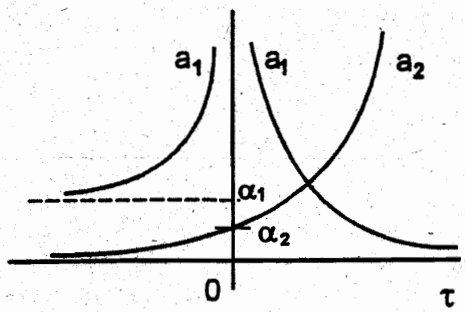

Figure 5 The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon > 0$ in the case 3. if $\nu_1 > 0$.

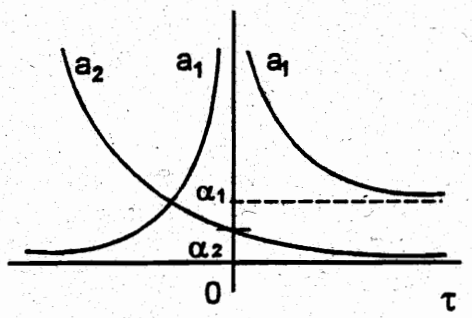

Figure 6 The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon > 0$ in the case 3. if $\nu_1 < 0$.

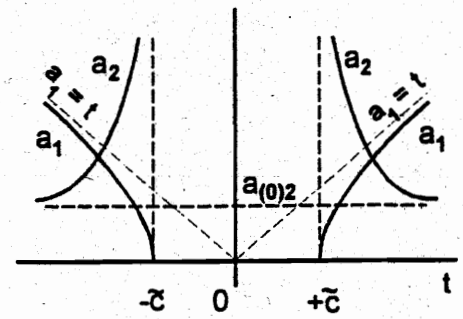

Figure 7 The dynamical behaviour of the scale factors a_1 and a_2 in synchronous time for $\epsilon > 0$ in the case 3. if $d_1 = 3$ (formulas (37), (38). The line $a_1 = |t|$ is the attractor for the scale factor a_1.

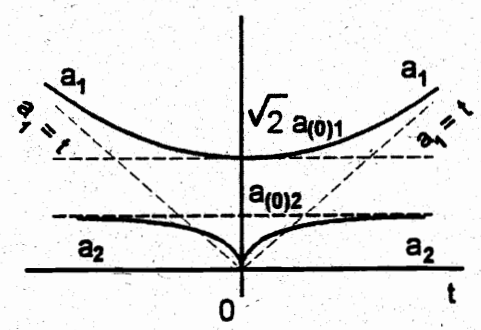

Figure 8 The dynamical behaviour of the scale factors a_1 and a_2 in synchronous time for $\epsilon > 0$ in the case 3. if $d_1 = 3$ (formulas (40), (41). The line $a_1 = |t|$ is the attractor for the scale factor a_1.

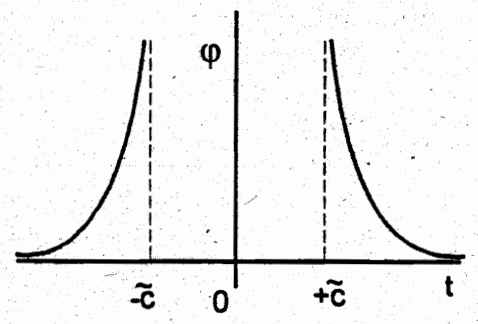

Figure 9 The case 3. ($d_1 = 3$). The behaviour of the scalar field for equation (39) (with positive sign).

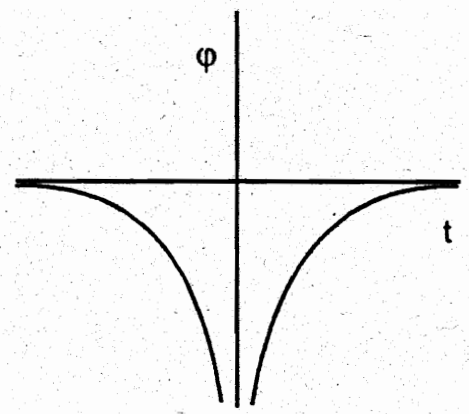

Figure 10 The case 3. ($d_1 = 3$). The behaviour of the scalar field for equation (42) (with positive sign).

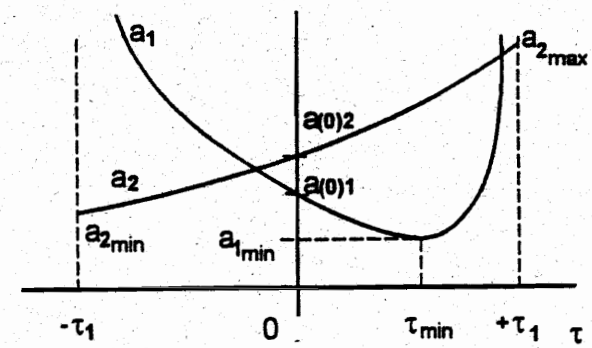

Figure 11 The qualitative behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon < 0$ (equation (53) for $\nu_1 > 0$).

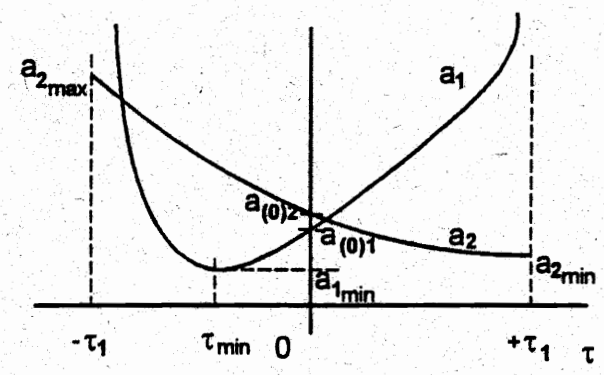

Figure 12 The qualitative behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon < 0$ (equation (53) for $\nu_1 < 0$). It can be seen that in the case $a_{2min} \sim L_{Pl}$ for $\tau \to \tau_1$ dynamical compactification takes place.

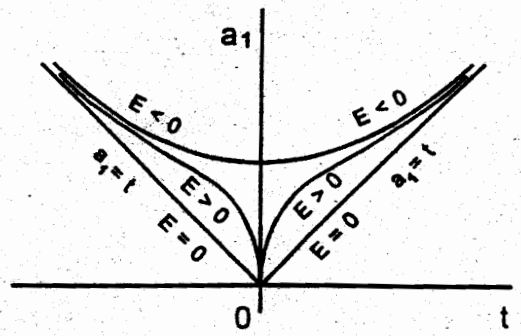

Figure 13 For solutions with spontaneou compactification of all internal spaces and negative curvature of the external space it is shown how the solution $a_1 = |t|$ for $\epsilon = 0$ acts as an attractor. The curves are drawn for the simple case $d_1 = 2$. Here we have $a_1 = \left[t^2 + \frac{\epsilon}{\theta_1}\frac{1}{C^2}\right]^{1/2}$ for $\epsilon < 0$ and $a_1 = \sqrt{\frac{\epsilon}{|\theta_1|}}\frac{1}{C}\left[\left(C\sqrt{\frac{|\theta_1|}{\epsilon}}|t|+1\right)^2 - 1\right]^{1/2}$ for $\epsilon > 0$. For all dynamical solutions $\epsilon \gtrless 0$ we have asymptotically $a_1 \sim |t| \to +\infty$, $a_2, \ldots, a_n \to$ const, $\varphi \to$ const and, therefore, the solution $a_1 = |t|$, $\epsilon = 0$ is an attractor, too (see e.g. figures 7 - 10).

REFERENCES

References

[1] W. J. Marciano, *Phys. Rev. Lett.*, **52** (1984) 489.

[2] M. Gleiser, J.G. Taylor, *Phys. Rev.*, **D33** (1986) 570.

[3] E.W. Kolb, M.J. Perry, and T.P. Walker, *Phys. Rev.*, **D33** (1986) 869.

[4] V.N. Melnikov, *Multidimensional cosmology and the problem of variation of constants*, in *Classical field theory and theory of gravitation*, vol. 1, ed. V. N. Melnikov, VINITI, Moscow 1991, p. 49 (in russ.).

[5] T. Koikawa, and M. Yoshimura, *Phys. Lett.*, **B150** (1985) 107.

[6] M. Demianski, A. Polnarev, P. Naselsky, *Phys. Rev.*, **D47** (1993) 5275.

[7] J. Scherk, J.H. Schwarz, *Phys. Lett.*, **B82** (1979) 60.

[8] A. Salam, J. Strathdee, *Ann. of Phys.*, **141** (1982) 316.

[9] E.W. Kolb, R. Slansky, *Phys. Lett.*, **B135** (1984) 378.

[10] U. Bleyer, A. Zhuk, *Multidimensional integrable cosmological models with dynamical and spontaneous compactification*, Preprint of Free University Berlin, FUB-HEP/93-19.

[11] V. D. Ivashchuk, V. N. Melnikov, and A. I. Zhuk, *Nuov. Cim.*, **B104** (1989) 575.

[12] V. D. Ivashchuk, V. N. Melnikov, *Phys. Lett.*, **A 136** (1989) 465.

[13] U. Bleyer, D.-E. Liebscher, H.-J. Schmidt, A.I. Zhuk, *Wiss. Zeitschr. Univ. Jena*, **39** (1990) 20.

[14] A.I. Zhuk, *Class. Quant. Grav.*, **9** (1992) 2029.

[15] A. Zhuk, *Phys. Rev.*, **D45** (1992) 1192.

[16] A. Zhuk, *Sov. Journ. Nucl. Phys.*, **55** (1992) 149.

[17] V.D. Ivashchuk, *Phys. Lett.*. **A170** (1992) 16.

[18] A. Zhuk, *Sov. Journ. Nucl. Phys.*, **56** (1993) 223.

[19] V.D. Ivashchuk, V.N. Melnikov, *Multidimensional quantum wormholes in the theory with cosmological constant, Teor. Mat. Fiz.* (1994) (in press).

[20] V.D. Ivashchuk, V. N. Melnikov, *Multidimensional cosmology with multicomponent perfect fluid, Int. J. Mod. Phys.* (1994) (in press).

[21] H.V. Fagunders, Phys. Rev. Lett., **70** (1993) 1579.

[22] Originally, the Milne universe is the four-dimensional ($d = 3$) hyperbolic space-time with the scale factor $a(t) = t$ [23,24]. In our multidimensional model (1) we call the space-time a Milne universe in the more general case when all scale factors are freezed with the exception of one of them, for example a_1 which depends on synchronous time as $a_1(t) = t$. and the space M_1 is the space of negative constant curvature with the arbitrary dimension $d_1 > 1$.

[23] E.A. Milne, *Nature*, **130** (1932) 9.

[24] N.D. Birrell, P.C.W. Davies, *Quantum Fields in Curved Space*, (Cambridge University Press 1982)

[25] J. Louko, *Phys. Rev.*, **D 35** (1987) 3760; *Ann. Phys.*, **181** (1988) 318; J.J. Halliwell, J. Louko, *Phys. Rev.*, **D 42** (1990) 3997.

[26] A. Vilenkin, *Phys. Rev.*, **D27** (1983) 2848.

[27] J. B. Hartle, S. W. Hawking, *Phys. Rev.*, **D28** (1983) 2960.

[28] A. Vilenkin, *Phys. Rev.*, **D33** (1986) 3560.

[29] D. N. Page, *J. Math. Phys.*, **32** (1991) 3427.

[30] A.P. Prudnikov, Yu.A. Brichkov, O.I. Marichev, *Integrals and series. Special functions.* (Moscow, Nauka, 1983)(in russ.).