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Einstein's Equivalence Principle and Riemannian Geometry

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C. Y. Lo

Applied and Pure Research Institute

17 Newcastle Dr., Nashua, NH 03060, U.S.A.

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Abstract

It was believed that, for Riemannian geometry, the satisfaction of Einstein's equivalence principle is equivalent to a proper metric signature, which mathematically implies the existence of local Minkowski spaces. However, a proper metric signature is only a necessary condition. As Einstein pointed out, the equivalence principle requires further that a "free falling" of an observer must result in a local Minkowski space whose spatial coordinates are statically attached to the observer. *This is also required by the fact that the local space-time of a space-ship under the influence of only gravity, is a local Minkowski space.* Explicit examples together with detailed calculations are given to show that, for some Lorentz manifolds, a "free falling" may not result in a statically attached local Minkowski space. In particular, based on physics and/or direct calculations, three independent proofs are provided to show that the Galilean transformation is unequivocally incompatible with the equivalence principle. Moreover, there are Lorentz manifolds, any of which cannot be diffeomorphic to a physical space, where the physical principles are satisfied. Another result from this analysis is that, due to the equivalence principle, *the time-coordinate must be orthogonal to the space-coordinates* if a particle can rest relative to the frame of reference.

"As far as the prepositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality." -- A. Einstein (in 'Geometry and Experience', 1921).

1. Introduction.

A major problem in general relativity is that any Riemannian geometry with the proper metric signature would be accepted as a valid solution of Einstein's equation of 1915. Consequently, many unphysical solutions were accepted as valid [1]. This is, in part, due to the fact that the nature of the source term has been obscure since the beginning [2,3]. When a source term is given, the adequacy of this term for a physical situation is often not clear. For instance, although the electromagnetic energy-stress tensor provides an adequate source term for the Reissner-Nordstrom metric [4,5], its adequacy for gravity involved an electromagnetic wave, is questionable [6,7]. Thus, to determine whether a solution is valid and whether a given source term is adequate, it is necessary to consider general physical requirements.

In general relativity, the most crucial physical requirement is the satisfaction of Einstein's equivalence principle in a physical space [2,3]. Mathematically, however, the equivalence principle can be incompatible with a solution of Einstein's equation, even if it is a Lorentz manifold whose space-time metric has the same signature as that of the Minkowski space [7,8]. Unfortunately, some relativists [4,9,10] seem to be unaware of this. Thus, to many theorists, a proper metric signature has become almost a synonym to the satisfaction of the equivalence principle, and they believe incorrectly that this had been proven in mathematics.

To clarify this confusion, let us first review the situation. Physically, the equivalence principle requires that a "free falling" results in a local Minkowski space [3]. Mathematically, although there always exists a local Minkowski space for any point in a Lorentz manifold (which has the proper metric signature), it should be noted that ***a "free falling" may not always result in a local Minkowski space*** [8,11]. In other words, while the proper signature of the metric is a mathematical necessity, a "free falling" results in a statically attached local Minkowski space is a physical requirement.

The misconception that, in a Riemannian space with the proper signature, a "free falling" would automatically result in a local Minkowski space [12,13], has deep-rooted mathematical errors and misunderstandings from believing in the general mathematical covariance in physics (§§ 2-4 & [11]). Thus, it is necess-

ary to demonstrate this misconception through examples with detailed calculations (see §§ 4–6) in this paper.

Moreover, there are Lorentz manifolds any of which is not diffeomorphic to a physical space (see § 6 and also [11]). Thus, to accept a Lorentz manifold as valid in physics, it is necessary to verify the validity of the equivalence principle in a space–time coordinate system on which physical interpretation can be based. Then, any diffeomorphism [4] can be used to obtain new coordinates for the purpose of calculation although, due to the equivalence principle, the diffeomorphism may not be adequate for physics (see §§ 4–6).

Note that Einstein's requirement of a mathematical general covariance among all concievable coordinate systems [2], is actually an over–extended demand since the equivalence of all frames of reference does not require the equivalence of all mathematical coordinate systems (see also § 4). (Note that the gauge related to general mathematical covariance, was not accepted by Eddington [14].) Analysis shows that covariance must be restricted by requiring the satisfaction of the equivalence principle. However, ***general relativity as a physical theory is unaffected*** because the equaivalence principle is satisfied in a physical space [11].

2. Two Mathematical Theorems in Reimannian Space and Einstein's Equivalence Principle

Now let us discuss first two mathematical theorems of Reimannian space [15] because they are often confused with Einstein's equivalence principle by mathematicians and "relativists". They are:

Theorem 1. Given any point P in any Lorentz manifold (whose metric signature is the same as a Minkowski space) there always exist coordinate systems (x^j) in which $\partial g_{ik}/\partial x^j = 0$ at P.

Theorem 2. Given any time–like curve Γ there always exist a coordinate system (so–called Fermi coordinates) (x^j) in which $\partial g_{ik}/\partial x^j = 0$ along Γ.

From theorem 1, it is clear that a local Minkowski metric exists at any given point. From theorem 2, it is claimed [15] that the existence of Fermi coordinate implies the existence of freely falling i.e. inertial observers in any Lorenz manifold. It should be noted, however, that here the existence of inertial observers means only local constant metric but not local Minkowski spaces. Thus, these theorems do not constitute a

physical principle since there are insufficient specifics in physics to exclude unphysical situations.

Einstein [16] pointed out, "As far as the prepositions of mathematics refers to reality, they are not certain; and as far as they are certain, they do not refer to reality." Although it is possible to transform a local constant metric to a local Minkowski space, such a local Minkowski coordinate system may not necessarily be related to the free falling (see §§ 3 & 4). *But physics requires that a free falling must result in a local Minkowski space*[1] *because the local space-time of a space-ship under the influence of only gravity, is a Minkowski space.* Thus, the equivalence principle is necessary to ensure that [2] "special theory of relativity applies to the case of the absence of a gravitational field."

Einstein proposed that the equivalence principle is satisfied in a physical space-time. *Thus, a Riemannian space, where the equivalence principle is not satisfied, is not physically realizable; and in a physically unrealizable "space-time", the equivalence principle cannot be satisfied.* Thus, although defining a coordinate system for the purpose of calculation is only a mathematical step, choosing a space-time coordinate system, which must be physically realizable, requires physical considerations.

Although the equivalence principle does not determine the space-time coordinates, it does reject physical unrealizable coordinate systems. Whereas in special relativity the Minkowski metric limits the coordinate transformations to the Lorentz transformations; in general relativity the equivalence principle limits the coordinate transformations to be among space-time physical coordinate systems. Thus, the role played by the Minkowski metric in special relativity, is extended by the equivalence principle (see also § 6).

3. The Restriction of Covariance and the Equivalence Principle

The foundation of general relativity consists of the equivalence principle and covariance. The principle of covariance [2] states that "The general laws of nature are to be expressed by equations which hold good for all systems of coordinates, that is, are covariant with respect to any substitutions whatever (generally covariant)." The covariance principle can be considered as consisting of two features: 1) the mathematical formulation in terms of Riemannian geometry and 2) the general validity of any Gaussian coordinate system as a space-time coordinate system in physics. While feature 1) was eloquently established by Einstein, feature 2) is actually over-extended. The equivalence of all frames of reference simply does not require the equivalence

of all coordinate systems [11]. Eddington [14] pointed out that "space is not a lot of points close together; it is a lot of distances interlocked." Moreover, because of the equivalence principle, it is found that the general mathematical covariance must be restricted [7,8,11].

Kretschmann [17] pointed out that the postulate of general covariance does not make any assertions about the physical content of the physical laws, but only about their mathematical formulation, and Einstein entirely concurred with his view. Pauli [10] pointed out further that "The generally covariant formulation of the physical laws acquires a physical content only through the principle of equivalence, ..." Thus, one has to modify the mathematical general covariance to accommodate the equivalence principle if incompatibility can occur. Einstein [2] argued that "... there is no immediate reason for preferring certain systems of coordinates to others, that is to say, we arrive at the requirement of general co-variance." This is, of course, incorrect since the equivalence principle is a reason to *reject* some coordinate systems (see also §§ 5 & 6).

Moreover, a mathematical general covariance requires the indistinguishability between the time-coordinate and a space-coodinate. On the other hand, the equivalence principle is related to the Minkowski space which requires a distinction between the time-coordinate and a space-coordinate. Thus, a general covariance is inherently inconsistent with the equivalence principle. In addition, as Hawking [18] pointed out, physics requires that time has a direction. Thus, the mathematical general covariance must be restricted in physics.

If, at the earlier stage, Einstein's arguments are not so perfect, he seldom allowed such defects be used in his calculations. This is evident in his book, 'The Meaning of Relativity' which he edited in 1954. According to his book and related papers, Einstein's viewpoints on space-time coordinates are:

1) A physical (space-time) coordinate system must be physically realizable (see also 2) & 3) below).

   Einstein [19] made clear in 'What is the Theory of Relativity? (1919)' that "In physics, the body to which events are spatially referred is called the coordinate system." Furthermore, Einstein wrote "If it is necessary for the purpose of describing nature, to make use of a coordinate system arbitrarily introduced by us, then the choice of its state of motion ought to be subject to no restriction; the laws ought to be entirely independent of this choice (general principle of relativity)". Thus, Einstein's coordinate system has a state of motion and is usually referred to a physical body. Since the time coordinate is *accordingly fixed*, choosing a space-time system is not only a mathematical but also a physical step.

2) A physical coordinate system is a Gaussian system such that the equivalence principle is satisfied.

One might attempt to justify the viewpoint of accepting any Gaussian system as a space-time coordinate system by pointing out that Einstein [3] also wrote in his book that "In an analogous way (to Gaussian curvilinear coordinates) we shall introduce in the general theory of relativity arbitrary co-ordinates, x_1, x_2, x_3, x_4, which shall number uniquely the space-time points, so that neighbouring events are associated with neighbouring values of the coordinates; otherwise, the choice of co-ordinate is arbitrary." But, Einstein [3] qualified this with a physical statement that "In the immediate neighbour of an observer, falling freely in a gravitational field, there exists no gravitational field." This statement will be clarified later with a demonstration of the equivalence principle (see eqs. [7] & [8]).

3) The equivalence principle requires not only, at each point, the existence of a local Minkowski space[1)]

$$ds^2 = c^2dT^2 - dX^2 - dY^2 - dZ^2, \tag{1}$$

but a free falling ***must*** result in a local Minkowskian space (see also [5,9,10]).

Note that ***a "free falling" must result in a local Minkowski space, is a <u>physical</u> requirement.***

4. Free Falling and the Equivalence Principle

To clarify the 1916 paper [2], Einstein wrote in his book [3], "According to the principle of equi-valence, the metrical relation of the Euclidean geometry are valid relative to a Cartesian system of reference of infinitely small dimensions, and in a suitable state of motion (free falling, and without rotation)." Thus, at each point (x,y,z,t) of a physical space, a "free falling" observer P must be in a local ***Minkowski space*** (1), whose spatial coordinates are ***statically*** attached to P, whose motion is governed by the geodesic,

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \tag{2}$$

where $\Gamma^\mu{}_{\alpha\beta}=(\partial_\alpha g_{\nu\beta}+\partial_\beta g_{\nu\alpha}-\partial_\nu g_{\alpha\beta})g^{\mu\nu}/2$ and the metric of the space is

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}. \tag{3}$$

Thus, there is a clear physical distinction between a space-coordinate and the time-coordinate. This free falling is equivalent to the existence [9] of "orthogonal tetrad of arbitrarily accelerated observer." In a way, the "free falling" of an observer locally extends the Minkowski space to general relativity. Thus, according to Einstein, in a space-ship under the influence of gravity only, the local space-time is ***automatically*** Minkowski, because for a free falling observer, the local Minkowski space is statically attached to the observer.

However, some theorists mistook Einstein's [3] other statements as the equivalence principle. The quotation is "In the immediate neighbourhood of an observer, falling freely in a gravitational field, there exists no gravitational field. We can therefore always regard an infinitesimally small region of the space-time continuum as Galilean. For such an infinitely small region there will be an inertial system (with the space coordinates, X_1, X_2, X_3, and the time coordinate X_4) relatively to which we are to regard the laws of the special theory of relativity as valid." These statements are essentially the mathematical theorems in § 2, although the language is in physics. From these statements, the free falling observer, though in the neighbourhood of a local Minkowski space, may not move with the "inertial system".

However, a possible mathematical choice of coordinates is inadequate in physics, since its ***realization*** must be specific. Einstein [2] proposed that the acceleration of the system of reference must be in a free falling with the observer. This observation is echoed by Pauli [10]. He wrote that "For every infinitely small world region, there always exists a coordinate system K_0 (X_1, X_2, X_3, X_4) in which gravitation has no influence either on the motion of particles or any other physical processes." and that "We can think of the physical realization of the local coordinate system K_0 in terms of a freely floating, sufficiently small, box which is not subjected to any external forces apart from gravity, and which is freely falling under the action of the latter." and that "It is evidently natural to assume that the special theory of relativity should be valid in K_0." Weinberg [5] wrote also some equivalent statements in his book.

The existence of a Local Minkowski space alone implies only that it is always possible to construct mathematically a Cartesian coordinate system covering an infinitesimal neighberhood of a freely falling observer, i.e., the local space-time of a space-ship under gravity may not be Minkowski. In contrast to

mathematical coordinates, a space–time coordinate system cannot be just any Gaussian system. Apparently, a discussion on the possibility that the equivalence principle may not be satisfied even if the metric signature is (+,–,–,–), was over–looked by Einstein and others.

To see the need of considering beyond the metric signature, we consider a constant metric,

$$ds^2 = \alpha^2 dt^2 - dx^2 - dy^2 - dz^2, \tag{4a}$$

where α ($\geq$ 2c) is a constant. The unit of t is second, the unit of x, y, or z is centimeter and the unit of α is cm/sec. Metric (4a) is a solution of the Einstein equation $G_{\mu\nu} = 0$. Then, $ds^2 = 0$ would imply that the velocity of light is α. One might argue that metrice (4a) can be transformed to

$$ds^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2, \tag{4b}$$

by the following diffeomorphism,

$$x' = x, \quad y' = y, \; z' = z, \quad \text{and} \quad t' = t\alpha/c. \tag{4c}$$

Eq. (4c) implies, however, that the units of t and t' are distinct and the light speed remains α but ***not*** c.

Eq. (4c) is not a rescaling since all the physical units remain the same. In a rescaling only the physical units, but not the physics, are changed. For example, the light speed can be expressed as 1 lightyear per year or 3×10^{10} cm/sec. However, if α = 2c, metric (4a) implies that the light speed would be 2c, i.e., 6×10^{10} cm/sec; and metric (4a) implies that the light speed is 3×10^{10} cm/half–sec. Thus, if metric (4b) were considered as Minkowski, the diffeomorphism (4c) would amount to redefining the space.

Einstein [3] illustrated his equivalence principle in his calculation of the light bending. First, using his field equation of 1915, he justified the linear equation,

$$\frac{\partial^2 \gamma_{\mu\nu}}{\partial x_\alpha{}^2} = 2K \left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T\right), \tag{5a}$$

where $\gamma_{\mu\nu}$ (= $g_{\mu\nu}-\eta_{\mu\nu}$) is the deviation from the flat metric $\eta_{\mu\nu}$, $T_{\mu\nu}$ is the energy-stress tensor for massive matter, and K is the coupling constant. Then, from eq. (5a), he obtained the metric

$$ds^2 = c^2(1 - \frac{K}{4\pi} \int dV_0 \frac{\sigma}{r})dt^2 - (1 + \frac{K}{4\pi} \int dV_0 \frac{\sigma}{r})(dx^2 + dy^2 + dz^2). \tag{5b}$$

by using the asymptotically flat of the metric. (Note that eq. (5a) can be justified with physical considerations [20], which are independent of the Einstein equation.)

Now, although $d^2x^\mu/ds^2 \neq 0$, consider an observer P at (x_0,y_0,z_0,t_0) in a "free falling" state of

$$dx/ds = dy/ds = dz/ds = 0. \tag{6}$$

According to the equivalent principle, state (6) of P implies at (x_0,y_0,z_0,t_0)

$$c^2(1 - \frac{K}{4\pi} \int dV_0 \frac{\sigma}{r})dt^2 = ds^2 = c^2\, dT^2 \tag{7}$$

since the local coordinate system is attached to the observer P (i.e., dX = dY = dZ = 0). Because the space coordinates are orthogonal to dt, at (x_0,y_0,z_0,t_0) one has [3]

$$(dx^2 + dy^2 + dz^2)(1 + \frac{K}{4\pi} \int dV_0 \frac{\sigma}{r}) = (dX^2 + dY^2 + dZ^2) \tag{8}$$

In general relativity, the law of the propagation of light is characterized by the light-cone condition,

$$ds^2 = 0. \tag{9}$$

Then, the velocity of light is expressed in our selected coordinates by

$$\frac{[dx^2 + dy^2 + dz^2]^{1/2}}{dt} = c(1 - \frac{K}{4\pi} \int dV_0 \frac{\sigma}{r}). \tag{10}$$

Einstein wrote [3] "We can therefore draw the conclusion from this, that a ray of light passing near a large mass is deflected." Thus, Einstein has demonstated that ***the equivalence principle requires that a space-time coordinates system must have a physical meaning***. There is a definite physical difference between a space coordinate and a time coordinate. Thus, a space-time coordinate system cannot be just an arbitrary Gaussian coordinate system in mathematics.

Although Einstein emphasized the importance of satisfying the equivalence principle, he did not emphasis that this satisfaction is automatically only in a physical space. However, there are many ways to go wrong. For instance, if the requirement of asympotically flat were not used, one could obtain a solution which does not satisfy the equivalence principle. This illustrates also that to see whether the equivalence principle is satisfied, one must consider beyond the Einstein equation (see § 5).

Moreover, if the metric did not satisfy the equivalence principle, $ds^2 = 0$ would lead to an incorrect light velocity because the manifold is not a physical space. In addition, Einstein's calculational approach would lead to contradictory results. To illustrate these, it will be shown in next section that an arbitrary Gaussian system as a space-time coordinate would lead to theoretical inconsistency and errors in physics.

5. Validity of a Metric in Physics and the Equivalence Principle

A given metric defines a physical space only if the space-time coordinate system is physically realizable, i.e. the equivalence principle is satisfied. This will be illustated by a few examples of metric spaces. For clarity and simplicity, we discuss cases without gravitational forces.

Example 1, consider the metric (4) again. If the equivalence principle were valid, one would obtain

$$c^2dT^2 = \alpha^2dt^2 , \quad \text{and} \quad (dX^2 + dY^2 + dZ^2) = (dx^2 + dy^2 + dz^2). \tag{11a}$$

Eq. (11a) and $ds^2 = 0$ imply that at any point (x,y,z,t) the light speed is

$$\frac{[dX^2 + dY^2 + dZ^2]^{1/2}}{cdT} = \frac{[dx^2 + dy^2 + dz^2]^{1/2}}{\alpha dt} = 1 \tag{11b}$$

Eq. (11b) implies, however, that the light speed is c in the local Minkowski coordinate, but is α ($\geq$ 2c) in the (x,y,z,t) space. But, since there is no gravitational force for this case, we may have also

$$x = X, \quad y = Y, \quad \text{and} \quad z = Z \tag{12}$$

Eqs. (11) and (12) absurdly mean that for the <u>same</u> frame of reference, we have <u>different</u> light speeds. This certainly disagrees with Einstein's statement [19] that "In physics, the body to which events are spatially referred is called the coordinate system". In summary, metric (4) is not physically realizable.

Example 2, consider the transformation, which is a diffeomorphism,

$$t = C\{\exp(T/C) - \exp(-T/C)\}/2. \tag{13a}$$

Then

$$ds^2 = \frac{1}{4}\{\exp(T/C) + \exp(-T/C)\}^2 dT^2 - dx^2 - dy^2 - dz^2 \tag{13b}$$

repesents the Minkowski metric after the transformation. If metric (13b) is realizable, according to $ds^2 = 0$, the measured light speed would be $\{\exp(T/C) + \exp(-T/C)\}/2$.

From (13b), the Christoffel symbols $\Gamma_{\nu,\alpha\beta}$ ($=[\partial_\alpha g_{\nu\beta}+\partial_\beta g_{\nu\alpha}-\partial_\nu g_{\alpha\beta}]/2$), are zeros except

$$\Gamma_{t,tt} = \partial_t g_{tt}/2 \tag{14}$$

Then, according to the geodesic equation, the equation of motion for a particle is

$$\frac{d^2T}{ds^2} + \Gamma^t{}_{tt}\frac{dT}{ds}\frac{dT}{ds} = 0, \quad \text{and} \quad \frac{d^2x^\mu}{ds^2} = 0 \tag{15}$$

where

$$\Gamma^t{}_{tt} = \frac{d}{dT}(\ln\{\exp(T/C) + \exp(-T/C)\})$$

It follows eq. (15) that one obtains, for some constant k

$$\frac{dT}{ds} = k\{\exp(T/C) + \exp(-T/C)\}^{-1} \quad \text{and} \quad \frac{dx^{\mu}}{ds} = \text{constant} \tag{16}$$

Now, consider the case $dx/dT = dy/dT = dz/dT = 0$; and therefore $dx/ds = dy/ds = dz/ds = 0$. Thus, in such a "free falling", there is no change in the spatial position nor acceleration. Physically, this means that such an observer would have the same frame of reference, whether "free falling" or not. Since metric (13) is not Minkowski, according to Einstein, the equivalence principle is not satisfied. Otherwise. we would absurdly have two different light speeds from the same frame of reference. Thus, metric (13) is not realizable.

Now, to see further that the equivalence principle is needed for the theoretical consistency of general relativity, let us consider Example 3, a frame of reference K' with a constant metric,

$$ds^2 = [dz' + (c - v)dt'][-dz' + (c + v)dt'] - dx'^2 - dy'^2, \tag{17}$$

since any constant metric satisfies the Einstein equation $G_{\mu\nu} = 0$. Then, for light rays in the z'-direction, $ds^2 = 0$ would imply at any point the light speeds were

$$\frac{dz'}{dt'} = c + v, \quad \text{or} \quad \frac{dz'}{dt'} = -c + v. \tag{18}$$

Clearly, eq. (18) also does not give a correct light speed since (18) also violates ***coordinate relativistic causality***, i.e. no cause event can propagate faster than the velocity of light in a vacuum. (Note also those in (18) ***cannot be regarded as coordinate velocities,*** since metric (17) is not physically realizable.)

Moreover, according to the geodesic equation (2), metric (17) implies $d^2x'^{\mu}/ds^2 = 0$, and thus

$$\frac{dx'^{\mu}}{ds} = \text{constant.} \qquad \text{where } x'^{\mu} \ (= x', y', z', \text{ or } t') \tag{19}$$

at any point. Now, according to metric (17), consider the case of "free falling" at (x'_0, y'_0, z'_0, t'_0)

$$dx'/ds = dy'/ds = dz'/ds = 0, \quad \text{and} \quad dt'/ds = (c^2 - v^2)^{-1/2}. \tag{20}$$

Note that there is no acceleration nor any change in the spatial position. In other words, such a "free falling" observer carries with him the frame of reference K'. But, the K' metric (17) is not a Minkowski space.

Nevertheless, ***mathematics*** ensures the existence of a local Minkowski space, which can be obtained by choosing first the path of a particle to be the time coordinate and then the other three space coordinates by orthogonality. Let us investigate this scenario. According to condition (20), the time coordinate would remain the same dt'. But, the coordinate dz' is not orthogonal to dt'. In order to have three orthongonal space coordinates to form a local Minkowski space, it is necessary to transform dz' by

$$dT = dt', \quad dX = dx', \quad dY = dy', \quad \text{and} \quad dZ = dz' - vdt'. \tag{21}$$

But, since the observer at (x'_0, y'_0, z'_0, t'_0) is in a state of $dz' = 0$ (i.e., $dZ \neq 0$), this local system is ***not statically*** attached to the observer, and hence is unrelated to the "free falling". Thus, the equivalence principle is not applicable to Riemannian space (17) although, at any space-time point, it is always possible to have a local Minkowski coordinate system which is related to a certain free falling. This illustrates that ***not only the existence of a local Minkowski space, but also how such a local space is related to the geodesic, is crucial for a physical space.***

It has been shown in three different approaches that metric (17) is incompatible with the equivalence principle and therefore physically unrealizable. Also, (21) is clearly a Galilean transformation. Thus, it has been shown that the Galilean transformation is also not valid in general relativity. The failure of satisfying the equivalence principle should be expected since the Galilean transformation is experimentally not realizable.

6. The Equivalence Principle and Covariance

The foundation of general relativity consists of the equivalence principle and covariance. However, the principle of general relativity requires only the equivalence of all frames of reference, but not all mathematical coordinate systems as claimed [21]. The equivalence principle is a physical requirement for a valid physical space-time coordinate system. Also, a distinction between time and space is inherent, as Hawking [18] pointed out that "something that distinguished the past from the future, giving a direction to time."

In terms of mathematics, the satisfaction of the equivalence principle is the feasibility of a coordinate transformation which is subjected to the geodesic. The trajectory of a particle may relate to the metric in a manner depending the coordinate system. Also, a local coordinate system is restricted by the non-covariant requirement of being a Minkowski metric with spatial orthogonal coordinates statically attached to the geodesic. These would put a severe restriction on the possible Gaussian coordinate systems.

Thus, the equivalence principle is related to the physical meaning of the coordinates in a space-time coordinate system. For instance, as shown by metric (10), the metric element g_{tt} cannot be arbitrary. Moreover, the trajectory of a particle, being also the local time, is orthogonal to the local spatial coordinates. This means, however, when the "time" coordinate is not orthogonal to the "space" coordinates, the equivalence principle is not satisfied if a particle is allowed to rest relative to the frame of reference.

Let us illustrate this by revisiting metric (17). Consider the Galilean transformation

$$t = t', \quad x = x', \quad y = y', \quad \text{and} \quad z = z' - vt'. \tag{22a}$$

Then

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2, \tag{22b}$$

is obtained from metric (17). Now, at (x,y,z,t) the state (20) now becomes

$$dx/ds = dy/ds = 0, \; dz/ds = -vdt/ds, \quad \text{and} \quad dt/ds = (c^2 - v^2)^{-1/2}. \tag{23}$$

The trajectory of the particle has the direction (0,0,dz,dt), and its orthogonal vector is $(0,0,dt,dzc^{-2})$. Thus, the local Minkowski space can be obtained by a Lorentz transformation, which preserves orthogonality.

On the other hand, from metric (17) the direction of the particle is (0,0,0,dt') and its orthogonal vectors are (1,0,0,0), (0,1,0,0) and $b_\perp = (0,0,\alpha,\beta)$ where $\alpha^2 = 1 - v^2/c^2$, and $\beta = -v/c^2\alpha$. However, since the state of (0,0,0,dt') requires dz' = 0, but dt' $\neq$ 0, the vector $b_\perp$ is not statically attached to the particle. This is due to the fact that the local space (dx',dy',dz') which is fixed to the particle is not orthogonal to the the vector (0,0,0,dt'). This means that metric (17) is not physically realizable.

In the above analysis, it has been illustrated that the Galilean transformation is incompatible with the equivalence principle in the absence of gravity. In fact, the incompatibility is also true even when gravity is present. To illustrates this, let us consider metric (5b) and the physical situation that a particle at $(0,0,z_0,t_0)$ moving with velicity v at the z-direction. The Galilean transformation (22a) transforms metric (4b) to

$$ds^2 = c^2(1 - \frac{K}{4\pi} \int dV_0 \frac{\sigma}{r})dt'^2 - (1 + \frac{K}{4\pi} \int dV_0 \frac{\sigma}{r})(dx'^2 + dy'^2 + [dz' - vdt']^2). \quad (24)$$

If metric (24) had a physical realizable coordinate system S', the particle would be at $(0,0,z'_0,t'_0)$ in the state $(0,0,0,dt')$ and the local spatial coordinates dx', dy', and dz' would be statically attached to the particle at the instance t'_0. However, according to metric (24), the coordinate dz' is not orthogonal to dt'.

But, in a local Minkowski space, dt' would be orthogonal to a statically attached three dimensional linear subspace. This is not possible because dx', dy', and dz' can form a basis of a three dimensional subspace. Thus, the equivalence principle cannot be satisfied and metric (24) is not a physically realizable space. An interesting result from this analysis is that, due to the equivalence principle, ***the time-coordinate must be orthogonal to the space-coordinates*** if a particle is allowed to rest relative to the frame of reference.

Moreover, a Lorentz manifold may not be diffeomorphic to a physical space. In particular, many solutions of Einstein's equation are intrinsically unphysical since they fail physical requirements. For instance, consider

$$ds^2 = du\, dv + h_{ij}(u)x^i x^j\, du^2 - dx^i\, dx^j \quad (25)$$

where $u = t - z$, $v = t + z$, $h_{ii}(u) \geq 0$, and $h_{ij} = h_{ji}$. Its physical cause can be an electromagnetic plane wave [22]. This metric is incompatible with Einstein's notion of weak gravity and the correspondence principle since $h_{ij}(u)x^i x^j$ can be arbitrarily large. Moreover, the gravitational force (related to $\Gamma^z{}_{tt} = \frac{1}{2}\partial(h_{ij}x^i x^j)/\partial t$ has arbitrary parameters (the coordinate origin). This arbitrariness in the metric violates the principle of causality (i.e., the causes of phenomena are identifiable) [8].

A reason that metric (25) cannot be diffeomorphic to a physical space is that a diffeomorphism cannot eliminate any parameter. Metric (25) also does not satisfy coordinate relativistic causality and therefore the

equivalence principle because the requirement $1 \geq (1 + H)/(1 - H)$ (where $H = h_{ij}x^i x^j$) may not be satisfied. One can also show that metric (25) does not satisfy the equivalence principle by direct calulations as shown in the previous cases.

7. Conclusions and Discussions

Special relativity was proposed ten years ahead of general relativity, and the Minkowski metric is a special case of the metric in general relativity. However, it was not clear that all the principles which lead to general relativity are compatible with each other and special relativity as a special case. For instance, the equivalence principle can be considered as a generalization of the Minkowski metric, but this principle may not be compatible with the covariance principle. In fact, there is no physical need to extend the space-time physical coordinate system to an ***arbitrary*** Gaussian system [11].

Although the creation of general relativity is due to the desire to have a theory of gravity which is consistent with special relativity, the consistency between special relativity and general relativity has not been thoroughly checked. Note that, to establish special relativity, the Galilean transformation is proven to be physically ***unrealizable*** by experiments. Thus, a Galilean transformation cannot be compatible with the equivalence principle which is applicable to only a physical space. This means that the equivalence of all frames of reference is not the same as the physical equivalence of all mathematical coordinate systems. In particular, due to the equivalence principle, the Minkowski metric is the ***only*** physical constant space-time metric.

A Galilean transformation clearly leads to a violation of the equivalence principle and that $ds^2 = 0$ implies a light "velocity" larger than c [11]. However, instead of leading to further investigations, this problem was not recognized by some theorists due to their misconceptions. Their denial in terms of false arguments was supported with misunderstandings in physics and erroneous statements in mathematics. Thus, it is necessary to calculate examples which directly demonstrate a violation of the equivalence principle.

Moreover, some "Theorists" even prefer to distort Einstein's equivalence principle to fit the mathematical theorems (see §§ 3 & 4) because they are unable to tell the difference between mathematics and physics. However, unlike mathematics, physics is restricted by the physical reality. The fact that ***the local space-time of a spaceship under the influence of only gravity, is a Minkowski space requires that a***

free falling must result in a local Minkowski space. As stated by Einstein, the equivalence principle is necessary to ensure that [2] "special theory of relativity applies to the case of the absence of a gravitational field." Thus, nature unequivocally defeats any attempt to misinterprete the equivalence principle.

Einstein proposed the equivalence principle for physics [2,3]. It was misunderstood that the equivalence principle is always applicable to a Lorentz manifold [9,13,23]. However, for some of such metric spaces, a local Minkowski space may not be obtained in a "free falling". Thus, the quotation of Einstein at the beginning of this paper is valid not only for Euclidean geometry but also for Riemannian geometry.

Mathematically, a local Minkowski space for an hypothetical observer is obtained as follows:

1) Choose the path of a "free falling" hypotheical observer to be the local time coordinate.

2) Choose the other three space coordinates by orthogonality.

Thus, a local Minkowski space can always be constructed for any hypothetical observer. But, the so chosen spatial coordinates may not be statically attached to the hypothetical observer. In other words, the mathematically constructed local Minkowski space may be *unrelated* to the "free falling" (§ 5). Thus, in contrast to the suggestion of Misner et al. [9], an existence of the tetrad in a Reimannian space, is not always possible.

The fact that there is a distinction between the equivalence principle and the proper metric signature would imply also that the covariance principle must be restricted. However, general relativity as a physical theory is unaffected by the restriction due to the equivalence principle. *An important function of the equivalence principle test is to eliminate intrinsically unphysical Lorentz manifolds, any of which cannot be diffeomorphic to a physical space* (see § 6 and also [11]).

Since the proper metric signature of a Riemannian space is insufficient to ensure the satisfaction of the equivalence principle, further considerations must be made for its valid applicability. For instance, since the principle of equivalence implies relativistic causality, a physical space must satisfy relativistic causality. When gravity is present, the light speed is smaller than maximum speed c due to the gravitational effects of space constraction and time dilation. Therefore, *coordinate relativistic causality* (i.e., the light speed c is the maximum velocity of propagation for any event) can be used as a convenient criterion. For example, coordinate relativistic causality is satisfied by the exterior Schwarzschild solution [11]. The principle of causality (i.e., the causes of phenomena are identifiable) necessitates the asymptotic flatness of a metric due

to an isolated source. However, these necessary conditions cannot assure the theoretical validity of the equivalence principle in a metric space.

Perhaps, due to sensing some confusions related to understanding the equivalence principle, theorists such as Synge [24] advocated explicitly that the basis of general relativity should be the Einstein field equation alone rather than the equivalence principle. However, as pointed out by Klein [25], theoretically there is no satisfactory proof of rigorous validity of Einstein's field equation; and experimentally, as pointed out by Weisskopf [26], the validity of Einstein's equation has not yet been established beyond doubt. In fact, the invalidity of Einstein's equation for two-body problems was conjectured by Hogarth [27] in 1953; and Einstein himself had pointed out that his equation may not be valid for matter of very high density [3]. Moreover, it has been proven experimentally that Einstein's equation must be modified [13]. But, the equivalence principle remains indispensible because of its solid experimental foundation [13,28].

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ENDNOTES

1) A local Minkowskian space is a short hand to express that special relatvity is locally valid.

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