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THE QUESTION OF BOUNDEDNESS

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OF

A SPACE-TIME METRIC AND GRAVITATIONAL PLANE WAVES

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C. Y. Lo

N Applied and Pure Research Institute, N H

17 Newcastle Dr., Nashua, NH 03060, U.S.A.

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Abstract

The question of boundedness of a space-time metric was believed to be completely meaningless because space-time coordinates (due to a misinterpretation of the equivalence principle) were considered as arbitrary. On the other hand, the boundedness of a space-time metric is a necessary assumption of weak gravity and Einstein's radiation formula. To resolve this inconsistency, it is pointed out that when coordinate variables do not have length as the unit, to consider boundedness, the compensative factors should be removed to obtain a ***normalized*** metric. It is pointed out also that an arbitrary Gaussian system may not be able to serve as a space-time coordinate system which must satisfy physical principles. Based on these, it is proven that a normalized metric element must be bounded if excluding the region where the light speed is approaching zero. In agreement with this, the field equation for weak gravity also implies that a space-time metric in a Cartesian coordinate system must be bounded. Thus, for an exact solution, it is necessary to show that this requirement is satisfied for weak gravity. Since the gravitational (similar to electromagnetic) plane waves must also be justified as a (spatial) local idealization of a weak wave from an isolated finite source at a far distance, the unbounded "plane waves" of Bondi, Pirani, and Robinson are invalid as a local idealization in physics.

"According to the principle of equivalence, the metrical relations of the Euclidean geometry are valid relatively to a Cartesian system of reference of infinitely small dimensions, and in a suitable state of motion (free falling, and without rotation)." -- A. Einstein (The Meaning of Relativity, p. 90).

1. Introduction.

A major problem in general relatvity is that any Riemannian geometry metric with the proper metric signature would be accepted as a valid solution of Einstein's equation of 1916 [1]. This is, in part, due to the fact that the nature of the source term has been obscure since the beginning [2]. When a source term is given, the adequacy of this term for a physical situation is often not clear. For instance, the electromagnetic energy-stress tensor is inadequate for a dynamic problem [3] although it has been adequate for a static problem [4]. To deal with the problem whether a given source term is adequate, it is desirable to develope general physical requirements which are not specific for just a particular problem.

Moreover, it has been clarified that the notion of gauge in relativity, which is based on the diffeomorphism (a one-one, onto, infinitely differentiable C^∞ map between manifolds, and its inverse is also C^∞ [4]), is inadequate since a general covariance may not always be compatible with the equivalence principle [5,6]. For instance, the Galilean transformation is a diffeomorphism, but it results in disagreements with the Michelson-Morley experiment and a violation of the equivalence principle. Perhaps because of the inadequacy, this gauge notion was rejected by Eddington [7]. Einstein [2], as if to correct his earlier overstatement on covariance, has clearly limited the choice of space-time coordinate systems with the equivalence principle. Thus, not only a physical space should have a metric with the proper signature that allows to have local Minkowski spaces, but also ***must a local Minkowski space be obtainable by free falling*** [2].

Furthermore, some solutions (with the proper signature) cannot be diffeomorphic to physical spaces if the Einstein equation has an inadequate source tensor [3,5]. Thus, for a manifold, "defining" the light speed in terms of local Minkowski spaces is mathematically not well-defined and physically misleading. These lead to the realization that satisfying ***relativistic causality*** (i.e. a coordinate velocity of light has a maximum light speed c at vacuum) is a physical requirement which may not be satisfied by a solution [5,6].

However, the principle of equivalence is a passive physical requirement, and can be used to check the

validity of only a known solution. The principle of causality (i.e., the causes of phenomena are identifiable) [3,5], requires characteristics of a solution to be identifiable with physical causes (see also § 3). The principle of causality can also relate the symmetries between the causes and the resulting field [3,5], and thus is an active physical requirement which can be used before a solution is obtained. In this paper, another active physical requirement, the boundedness of a space-time metric will be elucidated (§§ 2-4).

In physics, every physical quantity is finite; and, for an isolated system, the magnitude of any physical quantity is bounded. In general relativity, Einstein's weak gravity assumption suggests that the space-time metric g_{ab}, being a physical quantity, is not an exception. In other words, for a finite isolated system, one has

$$|g_{ab}(x,y,z,t)| < \text{constant} \tag{1}$$

over space-time. In fact, eq. (1) is also a necessary implicit assumption of Einstein's radiation formula [8].

However, there are complications since some coordinate variables may not have the unit of length (§ 2). Also, in part due to Einstein's early error (§ 3) [9], many theorists failed to recognize the crucial differences between an arbitrary mathematical coordinate system and a valid physical space-time coordinate system [5,6]; and thus they believed that eq. (1) had no meaning in physics. This illustrates that "Too great an emphasis on geometry can only obscure the deep connection between gravitation and the rest of physics [8]."

As a result of these confusions (§§ 4 & 5), the "plane wave" solutions presented by Bondi, Pirani and Robinson [10] were accepted as a valid local idealization. Moreover, there is a book on colliding "plane waves" [11]. Another problem is that Damour and Schmidt [12] claimed the existence of a dynamic solution, although they have not proven the crucial characteristic, its boundedness for weak gravity [13].

2. The Nature of Variables and Normalization of Space-Time Metric.

Some theorists disregard eq. (1) because its validity would depend on the choice of variables. For instance, for a spherical coordinate system, the Minkowski metric is

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2), \tag{2}$$

instead of the metric, $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ in a Cartesian coordinate system. The metric elements such as r^2 and $\sin\theta^2 r^2$ are not bounded. But, they also are ***compensative factors*** which arise from the fact that both the coordinate variables θ and ϕ have a dimension other than length. To consider the physics of boundedness, these factors should be eliminated to have a normalized metric.

Now, consider the Schwarzschild metric [4] in a spherical coordinate system,

$$ds^2 = (1 - \frac{C}{r})dt^2 - (1 - \frac{C}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2), \tag{3a}$$

which is a vacuum solution for a massive star whose radius $R \geq 1.125\ C$. (According to the perfect fluid model for the massive energy-stress tensor source, an interior solution requires the radius of a star, $R > 9C/8$ [4].) For the Schwarzschild metric (3a), the ***normalized*** metric elements $\hat{g}_{ab}$ are:

$$\hat{g}_{tt} = (1 - \frac{C}{r}), \quad \hat{g}_{rr} = -(1 - \frac{C}{r})^{-1}, \quad \text{and} \quad \hat{g}_{\theta\theta} = \hat{g}_{\phi\phi} = -1 \tag{3b}$$

Therefore, the normalized metric elements are bounded for $r > 1.125\ C$.

The equivalence principle is readily related to the normalized metric since a light speed is related to the normalized metric elements [2]. Based on $ds^2 = 0$, for metric (3a) the light speed in the θ-direction is

$$\frac{rd\theta}{dt} = \pm\{-\hat{g}_{tt}/\hat{g}_{\theta\theta}\}^{1/2} = \pm (1 - \frac{C}{r})^{1/2}, \quad \frac{1}{3} < \left|\frac{rd\theta}{dt}\right| \leq 1, \tag{4a}$$

and

$$\frac{dr}{dt} = \pm\{-\hat{g}_{tt}/\hat{g}_{rr}\}^{1/2} = \pm (1 - \frac{C}{r}), \quad \frac{1}{9} < \left|\frac{dr}{dt}\right| \leq 1 \tag{4b}$$

for the r-direction. On the other hand, the metric element $g_{\theta\theta}$ is related to angular velocity, i.e.,

$$\frac{d\theta}{dt} = \pm\frac{1}{r}\{-g_{tt}/g_{\theta\theta}\}^{1/2} = \pm \frac{1}{r}(1 - \frac{C}{r})^{1/2}, \quad \text{and} \quad \text{limit} \left|\frac{d\theta}{dt}\right| = 0, \text{ as } r \to \infty \tag{4c}$$

Thus, that the metric element $g_{\theta\theta}$ is unbounded, reflects the fact that the angular velocity of light must go to zero as r increases, since the light speeds have a maximum. But, $\hat{g}_{\theta\theta}$ is bounded.

Boundedness of a normalized metric is independent of the specific space-time coordinate system used. For example, in a Cartesian coordinate system, Schwarzschild solution metric (3a) is transformed to [14]

$$ds^2 = (1 - \frac{C}{r})dt^2 - (dx^2 + dy^2 + dz^2) - \frac{C}{r}(1 - \frac{C}{r})^{-1}r^{-2}(xdx + ydy + zdz)^2. \qquad (3c)$$

It is clear that the elements of metric (3c) are bounded for $r > 1.125\ C$. It will be shown in § 3 that this is guaranteed by general physical considerations such as the equivalence principle.

It should be noted also that, for $r < C$, $rd\theta/dt$ is imaginary, and $|dr/dt| > 1$ would be possible. This means that eq. (4) is no longer valid since both g_{tt} and g_{rr} change sign; t becomes space-like and r becomes time-like. This also mean that metric (3) is neither static nor spherically symmetric. Moreover, Bonnor [15] pointed out, "This is the region of final approach to the black hole which is supposed to exist at $r = 0$; notice that this (physical) singularity is space-like." In other words, solution (3), if extended to $r < C$, would be incompatible with the equivalence principle which leads to eq. (4). Thus, the Schwarzschild solution actually supports Einstein's remark that his equation may not be valid for very compact objects [2].

In general, the metric can be written, instead of

$$ds^2 = g_{ab}\, dx^a\, dx^b\ , \qquad (5a)$$

but as

$$ds^2 = \hat{g}_{\mu\nu}\, d\hat{x}^\mu\, d\hat{x}^\nu\ , \qquad (5b)$$

where $d\hat{x}^\mu$ are compensated differentials. For example, $d\hat{\theta} = rd\theta$. Then, eq. (1) is modified to be generally

$$|\hat{g}_{\mu\nu}| < \text{constant}. \qquad (1)'$$

It will be shown that eq. (1)' is valid if one excludes the region where the light speeds were close to zero.

3. The Boundedness of Space–Time Metric and the Equivalence Principle.

In 1916, Einstein [9] argued that a Gaussian curvilinear coordinate system can serve as a space–time coordinate system in physics. This is, of course, incorrect because a space–time coordinate system must satisfy the equivalence principle and other physical principles [2,5]. As Einstein clarified in 1921 [2], the equivalence principle implies that some Gaussian systems cannot be serve as space–time coordinate systems. Thus, self–consistency needs a modification of the covariance [5,6]. Nevertheless, since Einstein has made little efforts beyond his book [2] to rectify this mistake, the confusion persists. Moreover, many relativists are really mathematician at heart, and are interested in solutions instead of their physical meaning [1]. Consequently, they did not realize the need of satisfying the equivalence principle as a physical requirement and are unwilling to give up a convenient mathematical tool of having the unrestrict gauge condition.

Among Einstein's peers, Eddington [7] is probably the only one who realized this mistake. He commented "space is not a lot of points close together; it is a lot of distances interlocked." He even criticized the limitation of Einstein's equivalence principle. Eddington's criticism was accepted by Einstein as shown by his 1954 article 'Relativity and Problem of Space' [16] in which he added the crucial phrase, "at least to the first approximation" on the indistinguishability between gravity and acceleration.

If any Gaussian system could be a space–time coordinate system, then it can be shown that the "space–time" metric were not bounded. For instance, consider the transformation (without changing the time unit),

$$t = C\{\exp(T/C) - \exp(-T/C)\}/2. \tag{6a}$$

Then

$$ds^2 = \frac{1}{4}\{\exp(T/C) + \exp(-T/C)\}^2 dT^2 - dx^2 - dy^2 - dz^2 \tag{6b}$$

repesents the Minkowski metric. But, the metric element $\{\exp(T/C) + \exp(-T/C)\}^2/4$ is not bounded, and (T,x,y,z) is not a space–time coordinate system. In other words, metric (6b) is not realizable in physics. If it were, the light speed in the x–direction, according to the light cone condition $ds^2 = 0$ [2], would be

$$\frac{dx}{dT} = \pm\frac{1}{2}\{\exp(T/C) + \exp(-T/C)\}, \quad \text{and} \quad \lim\left|\frac{dx}{dT}\right| = \infty, \quad \text{as} \quad T \to \infty \tag{6c}$$

Thus, in violation of relativistic causality, the light speed could be not finite. This illustrates that the boundedness of a normalized space-time metric in vacuum has its root on the equivalence principle [6]. According to general relativity, gravity or acceleration would lead to a time-time metric element smaller than 1 [9].

Now, consider another diffeomorphic transformation (without changing the distant unit),

$$x = C\{\exp(X/C) - \exp(-X/C)\}/2. \tag{7a}$$

Then

$$ds^2 = dt^2 - \frac{1}{4}\{\exp(X/C) + \exp(-X/C)\}^2 dX^2 - dy^2 - dz^2 \tag{7b}$$

repesents the Minkowski metric. But, the metric element $\{\exp(X/C) + \exp(-X/C)\}^2/4$ is not bounded. Consequently, (t,X,y,z) is not a valid space-time coordinate system in physics. If it were, the light speed in the X-direction, according to the light cone condition $ds^2 = 0$ [2], would be

$$\frac{dX}{dt} = \pm 2\{\exp(X/C) + \exp(-X/C)\}^{-1}, \quad \text{and} \quad \lim\left|\frac{dX}{dt}\right| = 0, \quad \text{as} \quad X \to \infty . \tag{7c}$$

Thus, without an identifiable physical cause, the light speed would approach zero (although the light speed is 1 in y- and z-directions). In general relativity, a very slow speed of light is due to strong gravity.

To see this seemingly very strong gravity as false, for simplicity, one may appeal the principle of causality which requires that the causes are identifiable [3,5]. Since metric (7b) for an isolated system, satisfies the field equation in vacuum, this implies that the seemingly strong gravity violates the principle of causality. This illustrates that the space-time metric must be restricted by physical principles.

Moreover, the gravitational time dilation implies that $\hat{g}_{tt} \leq 1$; and we observe gravitational red shifts but not blue shifts [9]. In general, for an orthogonal coordinate system, the light speed $L(s)$ in the s-direction is

$$|L(s)| = \{-\hat{g}_{tt}/\hat{g}_{ss}\}^{1/2} \leq 1. \tag{8a}$$

However, an upper light speed limit leads to only $\hat{g}_{tt} \leq -\hat{g}_{ss}$, and would still allow $|\hat{g}_{ss}|$ to be unbounded.

As shown by metric (3), if a star could be so compact that r = C belonged to the exterior of the star, one would have $\hat{g}_{tt} = 0$ and $\hat{g}_{rr} = \infty$ and the light speed is zero. However, if one limits the region to r > 1.125 C, then all the normalized metric elements are bounded, and the light speed has a lower limit.

Now, if one assumes that the light speed has a lower limit L_m (≤ 1), then one has

$$|\hat{g}_{tt}| \leq |\hat{g}_{ss}| \leq |\hat{g}_{tt}|(L_m)^{-2} \leq (L_m)^{-2}. \tag{8b}$$

Thus, the space metric elements are bounded above. They would also be bounded below, if $|\hat{g}_{tt}|$ also has a lower bound g_L . Since the light speeds has a lower bound, the existence of g_L is likely.

For a non-orthogonal space-time coordinate system, it can also be shown, with some lengthy algebraic calculations, that all normalized elements of a space-time metric are bounded above if the light has a lower speed limit. Thus, eq. (1)' is valid if the region which is possiby related to the so-called "black hole", are excluded (see also § 6). In particular, this means that a gravitational plane wave as an idealization, must be bounded. Noted that the boundedness of a space-time metric is independent of the Einstein equation. Therefore, such a ***boundedness is a physical requirement for the field equation***, not just for a solution.

4. Weak Gravity Considerations and Physical Requirements for the Plane Wave.

Relativity suggests the existence of gravitational waves because physical influence must be propagated with a finite speed [17]. Thus, the existence of the gravitational wave is a physical requirement for a valid field equation. The Einstein equation of 1916 seemed to satisfy this requirement since its linearized "approximation" clearly gives a wave solution. The Einstein equation of 1916 is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R = -K\, T(m)_{\mu\nu} , \tag{9}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci curvature tensor, $T(m)_{\mu\nu}$ is the energy-stress tensors for massive matter, and K ($= 8\pi\mathcal{K}c^{-2}$, where $\mathcal{K}$ is the Newtonian coupling constant) is the coupling constant.

However, Einstein discovered in 1936 that his equation does not admit a propagating wave solution and

therefore linearized gravity is not reliable. Recently, it has been shown that this linearization procedure is not generally valid in mathematics [18]. Thus, it would be necessary to justify the existence of a propagating wave solution independent of the Einstein equation of 1916.

Fortunately, for a massive source, the linear field equation as the first order approximation, can be justified [19] on the basis of the equivalence principle which requires modifying the Newtonian gravity, and earlier experiments such as the bending of light. The linear field equation for weak gravity is

$$\frac{1}{2}\partial_c\partial^c\,\bar{\gamma}_{ab} = -KT(m)_{ab}\,, \tag{10a}$$

where $\bar{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}\eta_{ab}\,\gamma$, $\gamma_{ab} = g_{ab} - \eta_{ab}$, $\gamma = \eta^{cd}\gamma_{cd}$, and η_{ab} is the flat metric. Eq. (10a) has a mathematical structure similar to that of Maxwell's equations. An inhomogeneous solution of eq. (10a) is

$$\bar{\gamma}_{ab}\,(x^i,t) = -\frac{K}{2\pi}\int\frac{1}{R}T_{ab}\left[y^i,\,(t-R)\right]d^3y, \quad \text{where } R^2 = \sum_{i=1}^{3}(x^i - y^i)^2\,. \tag{10b}$$

Solution (10b) would represent a wave if T_{ab} has a dynamical dependency on time t' $(= t - R)$ [8,20]. The implicit gauge condition is that the flat metric η_{ab} is the asymptotic limit at infinity.

Thus, the theoretical existence of gravitational waves is assured as a certainty to the first approximation as believed [4,7,8,14,20–23]. Also, although gravitational waves have not been directly observed [22], there are indirect evidences from observing the binary pulsars [24–26].

Moreover, from linear field eq. (10), some characteristics of an exact solution for the weak gravity due to dynamic massive matter can be obtained. They are as follows:

i) Solution (10b) manifests that the first order approximation of the space–time metric includes a propagating wave, and is an almost periodic function of time for a source in an almost periodic motion. Therefore, the principle of causality implies that the exact solution is also an almost periodic function.

ii) Moreover, if eq. (10a) gives a first order approximation, $\partial^a\,\bar{\gamma}_{ab}$ should be of second order [7,23] due to the conservation law, $\nabla^a\,T_{ab} = 0$ (which is independent of the notion of gauge).

iii) By definition, an exact space–time metric element (in a Cartesian coordinate system) for weak gravity

is a small deviation from the flat metric [2] and therefore must be bounded (i.e. $|g_{ab}| <$ constant). On the other hand, according solution (10b), the first order approximation of a γ_{ab} is also bounded (i.e., $|\gamma_{ab}| \leq$ constant). Thus, eq. (10) is consistent with the requirement of weak gravity. Moreover, for the case of including singular mass distributions, in the region too close to the singular source, γ_{ab} is no longer small and therefore eq. (10a) is not valid for this problem. Thus, one must remodel T_{ab} such that weak gravity can be applied without significantly changing the gravity of the other regions.

These characteristics are also valid to gravitational waves. Note that conditions i) and iii) are also satisfied by the electromagnetic wave and conditon ii) is similar to the Lorentz gauge. Thus, a gravitatioinal wave should have a fruitful analogy with electromagnetism [8] (see also Appendix A).

From linear field eq. (10), the first order approximation of a space-time metric is bounded. Because we are dealing with weak gravity, there is no possiblity that the speed of light is close to zero. Thus, as shown in §3, the exact space-time metric must be bounded. Due to theoretical self-consistency, this is also necessary since the first order approximation is dominant in the exact space-time metric. This dominance is used in the derivation of Einstein's radiation formula which is supported by binary pulsar experiments. Einstein's radiation formula is, in turn, the base to show [23] eq. (10) to be incompatible with the Einstein eq. (9).

5. The Questionable Validity of Existing "Plane Waves"

A crucial characteristic of the plane wave is its boundedness [8,20]. As pointed out by Bondi, Pirani and Bobinson [10], the gravitational plane wave is an idealization of a weak wave from a distant soure. They wrote "Our interest in plane waves derives not, of course, from the expectation that such waves might exist in nature, but from the presumption that at a great distances from a finite source of gravitational waves, these waves must appear to be approximately plane." However, their exact "plane wave" is

$$ds^2 = \exp(2\phi)(d\tau^2-d\xi^2)-u^2[\mathrm{ch}2\beta(d\eta^2+d\zeta^2)+\mathrm{sh}2\beta\cos 2\theta(d\eta^2-d\zeta^2)-2\mathrm{sh}2\beta\sin 2\theta d\eta d\zeta], \quad (11a)$$

where ϕ, β, and θ are functions of u (= $\tau-\xi$). It satisfies the differential equation (i.e. their eq. [2.8]),

$$2\phi' = u(\beta'^2 + \theta'^2 \, sh^2 \, 2\beta). \tag{11b}$$

Clearly, the variables have the unit of length. Although $\exp(2\phi) \leq 1$ is possible, metric (11a) is not bounded because this would require the impossibility of $|u^2| <$ constant. Note that u^2 grows anomaly large as time goes by. Such an unlimited increasing amplitude of a wave is in dissonance with that the source is isolated and finite. Moreover, according to metric (11a), the velocity of light has a very strange directional behavour. The light speed in η- or ζ-direction has the time limit zero although the light speed is 1 in the ξ-direction.

A plane wave is a idealization for (spatial) local characteristics (see Appendix A). For instance, in spite of the total energy of a plane wave is infinite, its local energy density should be finite. In analogy to the plane wave in electromagnetism, one expects that a gravitational plane wave would satisfy the three requirements. However, metric (11) is not even locally bounded. A wave whose amplitude grows with time, can hardly be regarded as due to an isolated source which radiates energy over time. Moreover, not only periodicity is violated, but also is $\partial^a \bar{\gamma}_{ab}$ not bounded. In short, metric (11) fails all three physical requirements (which cannot be criticized as too severe as the requirement of Rosen [27]).

It should be noted that, from metric (11), one cannot deduce a weak limit which is bounded as required by eq. (10). Thus, ***metric (11) for $G_{ab} = 0$ is incompatible with eq. (10).*** Also, metric (11) illustrates that the mathematical characteristic of planeness is insufficient for the physics of a plane wave.

Moreover, with metric (11), one may illustrates that an unbounded one-parameter family of solutions $g_{ab}(x^i,\lambda)$ for $R_{ab} = 0$ may have no meaning for a physical perturbation. Although metric (11a) is not bounded, a one-parameter family $g_{ab}(x^i,\lambda)$ can be obtained if one substitutes ϕ and θ in metric (11a) respectively with $\phi+\lambda$ and $\theta+\lambda$ since they satisfy the same differential equation (11b). However, not only $g_{ab}(x^i,0)$, but the expansion of λ order terms such as

$$h_{(1)\eta\eta} = -h_{(1)\zeta\zeta} = 2u^2 sh2\beta \, \sin 2\theta,$$

and

$$h_{(1)\eta\zeta} = -2u^2 sh2\beta \, \cos 2\theta, \tag{12}$$

where β and θ are function of u $(=\tau-\xi)$, are not bounded. Thus, no matter how small λ can be chosen, one cannot relate the family $g_{ab}(x^i,\lambda)$ to a physical perturbation. The existence of a smooth one-parameter family $g_{ab}(x^i,\lambda)$ for $R_{ab} = 0$ has meaning to the "reliability" of perturbation theory only if 1) globally $g_{ab}(x^i,0) \approx \eta_{ab}$, the flat metric, and 2) the family is bounded, i.e. $|g_{ab}(x^i,\lambda)| \leq$ constant.

Let us examine also the well-known plane-wave form proposed by Misner et al. [20] as the follows:

$$- ds^2 = L^2(e^{2\beta}\, dx^2 + e^{-2\beta}\, dy^2) + dz^2 - dt^2 , \tag{13}$$

where both L (background factor) and β (wave factor) are functions of $u(\equiv t - z)$, and they believe that a plane wave is bounded. Form (12) would be incompatible with relativistic causality [5] which requires:

$$L^{-2}e^{2\beta} \leq 1; \qquad \text{and} \qquad L^{-2}e^{-2\beta} \leq 1. \tag{14}$$

However, relation (13) may not be possible if L = 1 and $\beta \neq 0$ is a weak wave limit as stated by Misner et al. [20]. Note that form (12) cannot accommodate a circularly polarized wave [28]. Wald [4] also pointed out that there is insufficient evidence to support the decomposition in form (12) as natural.

Although form (12) does not have an appearance of obvious unboundedness, the problem remains. Plane-wave form (12) reduces the Ricci curvature to zero except

$$R_{uu} = 2[L'' + (\beta')^2 L]/L . \tag{15}$$

However, no exact wave solution has been obtained for $R_{uu} = 0$ [1]. The fact is that there is ***no*** bounded plane wave solution for Einstein equation $R_{ab} = 0$ (see § 6). Thus, the theoretical approach of Misner et al. [20] on the plane wave, is actually not self-consistent.

In both cases the transverse metric elements cannot be bounded. This demonstrates also that, for a dynamic problem, eq. (9) and eq. (10) are not compatible. Physically, this would mean that eq. (9) is not valid for a dynamic problem since gravitational waves will be created in a dynamic interaction [17].

6. Conclusions and Discussions

To see the invalidity of $G_{ab} = 0$ for waves, one can consider the Einstein equation of 1995 [23],

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R = -K\,[T(m)_{\mu\nu} - t(g)_{\mu\nu}], \tag{16a}$$

and

$$\nabla^{\mu}T(m)_{\mu\nu} = 0, \qquad \text{and} \qquad \nabla^{\mu}t(g)_{\mu\nu} = 0, \tag{16b}$$

where $t(g)_{\mu\nu}$ is the energy-stress tensor for gravity. Eq. (16b) implies that the equivalence principle is satisfied [23]. Since eq. (10) is an approximation of (16a), this equation has bounded wave solutions.

Eq. (16) supports Einstein's discovery [18] and Hogarth's conjecture [29] that there is no physical dynamic solution unless the gravitational energy-stress tensor is added to eq. (9) although some of its static solutions are supported by experiments. The anti-gravity coupling [30] is necessary due to the existence of gravitational waves [3,23]. Such a coupling implies the assumptions of singualrity theorems invalid [31].

Also, the possible cause of unboundedness (see §2) such as the notion of black holes and gravitational collapse to zero size, actually depend on many unverified assumptions [31]. Among these assumptions, that mass and energy are unconditionally equivalent and that the Einstein equation of 1916 is valid for dynamic problems, are incorrect [23,32]. Thus, the possibility of unboundedness has not been proven.

The question of boundedness is crucial for a dynamic solution of eq. (9). From eq. (16), it is clear that eq. (9), the Einstein equation of 1916, does not have the physical feature that allows a gravitational wave, like an electromagnetic wave, to carry energy-momentum. Thus, without this physical feature, there is no possibility that eq. (9) would admit a physical gravitational wave which is necessarily bounded.

In other words, the unboundedness of metric (11) is not accidential, but is due to the deficiency of eq. (9). From the linear field eq. (10) for weak gravity, boundedness is a necessary condition for the plane wave as well as Einstein's radiation formula [8,20]. It is based on the validity of Einstein's radiation formula (on the time average) that eq. (16) is derived [23]; otherwise a bounded dynamic solution would be impossible. Thus, because of these physical considerations, if general relativity is essentially correct, ***there is no bounded gravitational plane wave which satisfies*** $R_{ab} = 0$.

Nevertheless, based on the requirement of weak gravity, an implicit assumption for the Einstein equation of 1916 was that its exact dynamic solutions are bounded [33,34]. Apparently, many theorists were not aware of that ***a physical requirement may not necessarily be satisfied by a field equation*** [18].

Moreover, Einstein's radiation formula, which implies the existence of radiation, directly leads to the need of modifying eq. (9). Although Einstein's notion of gravitational energy-stress [14] is not localizable, physically it is impossible to transform a radiation away by changing the coordinate system. Thus, Einstein's notion can only be an approximation of the gravitational energy-stress tensor for some coordinate systems. It follows that eq. (9), which implies the exact validity of Einstein's conservation law [2], must be modified.

The boundedness of a space-time metric is a meaningful simple problem. In spite of indisputable strong evidences, eq. (1) was, nevertheless, disregarded by theorists who believe the so-called "orthodox view" which implies the validity of an arbitrary Gaussian system as a space-time coordinate system. (But, Einstein's book [2], which provides the corrected view (see § 3), was seldom referred to in the literature.) Note also that, this validity of an arbitrary Gaussian coordinate system provides the theoretical base for the absurd "conclusion" that the total energy flow due to radiation could be negative, positive or zero [35,36].

Furthermore, the so-called "orthodox view", which is advocated by the Hawking-Wheeler school of black holes [20,37], has at least four major problems: i) It does not agree with Einstein on important issues such as ***the equivalence principle***, the limitation of the Einstein equation of 1916 [2], and the issues related to black holes [38]. ii) It is not universally accepted as pointed out by Bonnor [15,39], Eddington [7], McCrea [40], and Pauli [14]. iii) It contains errors in mathematics as well as in physics [2,3,5,6, 15,23,30,32,35,36,38-41]. iv) It disagrees with the Michelson-Morley experiment and the binary pulsar experiments [3,6,23]. In fact, it has been proven explicitly that this "orthodox view" is a cause of inconsistency [5,6]. The analysis in this paper demonstates further how far it has gone wrong.

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Appendix A: The Plane Wave as a Spatial Local Idealization of a Wave from a Distant Source.

As pointed out and showed by Weinberg [8], a gravitational wave have a fruitful analogy with electromagnetism. Here, further details are provided. The fundamental reason is that the linear field equation (10a) and the Maxwell equations are d'Alembertian type of the form,

$$\partial_c \partial^c W(x^i,t) = \alpha S(x^i,t) \ . \tag{A1}$$

which governs the wave W with a finite source term S. A general inhomogeneous solution of (A1) is

$$W(x^i,t) = \frac{\alpha}{4\pi} \int \frac{1}{R} S[y^i, (t-R)]\, d^3y, \quad \text{where} \quad R^2 = \sum_{i=1}^{3} (x^i - y^i)^2 \ , \tag{A2}$$

and α is a consstant. When S is a periodic function of t, then W is a wave propagating outward.

When x^i is far away from the region of the source $S(y^i, t')$, $W(x^i,t)$ can be approximated as

$$W(x^i,t) \simeq \frac{\alpha}{4\pi r} \int S[y^i, (t-r)]\, d^3y = \frac{\alpha}{4\pi r} F(t-r), \quad \text{where} \quad r^2 = \sum_{i=1}^{3} (x^i)^2 \ . \tag{A3}$$

Consider

$$\frac{\partial}{\partial x^a} \left[\frac{1}{r} F(t-r)\right] = \frac{-x^a}{r^3} F(t-r) + \frac{-x^a}{r^2} F'(t-r). \tag{A4}$$

When S(t) is a periodic function of t, F and F' are of the same order. Then, if r is very large, the first term in right hand side of (A4) is negligible because of the extra factor 1/r.

Thus, for a large r_0 , W can be treated ***as if the denominator*** r_0 ***were a constant***, i.e.

$$W(x^i,t) \simeq \beta\, F(t-r), \quad \text{where} \quad \beta = \frac{\alpha}{4\pi r_0} \tag{A5}$$

can be thought of as if a constant. Note that, at radius r_0 , W and F satisfy the same equation

$$\partial_c \partial^c W(x^i,t) = 0 \quad \text{and} \quad \partial_c \partial^c F(x^i,t) \approx 0 \tag{A6}$$

if the terms with a factor of higher order $1/r^n$ are neglected. Both are propagating to the same r-direction.

When considering a narrow ray in the z-direction, according to (A4) its derivatives with respect of x, and y are very close to zero. Thus, within the narrow ray in the z-direction, one may consider that in effect

$$F(x^i,t) \approx F(t - z) \ . \tag{A7}$$

$\beta F(t-z)$ is called a plane wave and it satisfies eq. (A6), the wave equation in vacuum. The above proceedure is idealizing a wave with a distant source as a plane wave. However, it is crucial to recognize that this idealization is a spatial local idealization and is useful only for the investigation of spatial local characteristics. For instance, although the local energy density is finite, the total energy would be infinite.

Note also, while $\beta F(t - z)$ satisfies eq. (A6), not every plane solution of (A6), $f(u)$ $(u = t - z)$ is a plane wave. The reason is that $F(u)$ is a bounded periodic function of u because the source $S(x^i,t)$ which is the cause of gravity, is finite and periodic in t. For example, because $f(u) = u^2$ is neither bounded nor periodical $f(u)$ is not a plane wave. Similarly, $f(u) = \exp(u^2)$ is not a plane wave. These mistakes are good examples due to a superficial mathematical viewpoint which forgets the crucial physics behind the equations.

In electrodynamics, many problems are checked with a plane wave first and then physical calculations are done with a more realistic wave (A2). Not only the electromagentic plane wave is, but also, as stated by Bondi et al. [9], is the gravitational plane wave an idealization of a wave with a distant source.

Although gravity is governed by a non-linear equation which is not of d'Alembertian type, for weak gravity, its first order approximation, which is very dominating due to the smallness of the coupling constant K, is governed by a linear field equation (10a). If the source S is a periodic function of t, then due to the principle of causality, the exact solution of gravity should also be a periodic function of t. For a very large r_0, $1/r_0$ can be much smaller than K^n the nth order. Now, it should be clear that a gravitational plane wave, propagating to the z-direction, should also be a bounded and periodic function of u [8].

Therefore, metric (11), which is plane but not bounded (see § 5), cannot be considered as a plane wave which Bondi et al. [9] claimed as an idealization of a wave from a far distant finite source. Similarly, metric (12) also cannot be considered as a plane wave form.