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THE QUESTION OF DYNAMIC SOLUTIONS IN GENERAL RELATIVITY AND GRAVITATIONAL PLANE WAVES

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Abstract

There are contradictory claims related to whether the Einstein equation of 1916 has a dynamic solution. The exact "plane wave" solutions of Bondi et al., seemed to suggest that Einstein's 1936 discovery were incorrect. In 1990, Damour and Schmidt even claimed that they have proven the existence of bounded dynamic solutions. Nevertheless, in 1993 the non-existence of any bounded dynamic solution has been shown by Lo. To resolve this problem, the exact "plane wave" solutions and the proof of Damour and Schmidt are examined. It is found that the existing exact "plane-waves" are not really waves. They do not satisfy the physical requirements of boundedness and periodicity manifested by the linear field equation which is supported by observations on the binary pulsars. For the Einstein tensor $G_{ab} = 0$, there is no plane-wave solution which satisfies the physical requirements. Moreover, the existence of an exact circularly polarized plane-wave requires $G_{tt} > 0$. Based on known examples, it is confirmed that the claims of Damour and Schmidt on the reliability of perturbation theory in general relativity are incorrect. In particular, the existence of a bounded dynamic solution for the Einstien equation, was "proved" invalidly by assuming implicitly the boundedness of a wave solution in vacuum. However, dynamic solutions do exist for the Einstein equation of Lo's 1995 version.

THE QUESTION OF DYNAMIC SOLUTIONS IN GENERAL RELATIVITY AND GRAVITATIONAL PLANE WAVES

I. Introduction.

A major problem in general relativity is whether the Einstein equation of 1916 has a dynamic solution since there are contradictory claims. Although Lo [1] showed the non-existence of a dynamic solution in 1995, some still believed that Damour and Schmidt [2] had rigorously proven its existence in 1990. Apparently, misconceptions in physics are also responsible for the acceptance of this obscure mathematical error. Besides, it seems, there are reasons beyond mathematics for remaining in believing Damour and Schmidt.

In essence, there are four main reasons: i) The calculation of exact "plane waves" seems to support the existence of a dynamic solution. ii) Some current theories depend on the invalid assumption that a dynamic solution exists; otherwise it would be very difficult to justify, for instance, the notion of black wholes [3]. iii) In general relativity many calculations in the past depend on the validity of a perturbation method [4,5]. iv) Some relativists (who do not understand sufficient mathematical analysis to make an independent judgement) had accustomed to rely on Damour for the validity of mathematics in general relativity [6]. (Note that the compatibility between physical validity and mathematical rigor is a long standing problem in general relativity [7].) Therefore, to clarify this mistake, it seems also necessary to trace back the historical developments how related errors have managed to persist for more than 60 years.

Historically, the theory of gravitational waves was initiated by Einstein [8,9], and the existence of such waves had been assumed a certainty to the first approximation. However, in 1936, Einstein [10] discovered that his field equation in general relativity does not admit a propagating wave solution. Subsequently, Einstein and Rosen [11] investigated plane and cylindrical gravitational waves. Thus, Einstein and Rosen correctly recognized the investigation on plane waves is a *crucial* initial step in understanding gravitatational waves. Then, Rosen [12] came to the conclusion that there were no exact plane-wave metric filling all space-time.

However, Rosen's argument was not convincing because his result that there had to be a 2-space on which the determinant of the metric tensor vanished, may not be valid. Bondi, Pirani and Robinson [13], believed that Rosen's demand is too severe. For a general plane-wave satisfying the vaccum Einstein

equation, $G_{\mu\nu} = 0$, they believed in the less stringent conditions by Lichnerowicz [14] and Petrov [15]. Thus, Einstein's discovery seemed wrong and the Einstein equation of 1916 would admit a propagating wave.

However, from the viewpoint of physics, the question of plane-wave should be considered beyond the mathematical characteristic of planeness. Surprisingly, even the well-known physical requirement of boundedness on the amplitude of metric components, which has already been clearly pointed out by Einstein [16], was not utilized or simply ignored. As manifested by the weak field approximation [17], a propagating gravitational wave from an isolated system far away, is a periodic (almost periodic to be exact) function of time at a point of space. But, all existing exact "plane-wave" solutions do not reflect these characteristics [13, 17,18] although some authors explicitly declared that a gravitational plane-wave, similar to the electromagnetic plane wave, is also an idealization of a wave from a distant source. Thus, to address the problem of dynamic solution, the physical meaning of the exact plane wave should be clarified first (§§ 2 & 3).

In this paper, based on these physical requirements, it will be shown that there is no plane-wave solution for the equation $G_{\mu\nu} = 0$ (§ 6), i.e., Rosen [12] is essentially right. In term of physics, this failure is due to that $G_{\mu\nu} = 0$ (which has no energy-stress source) is invalid for a physical gravitational wave because it carries energy [13]. Moreover, the results of investigating the plane-waves unequvocally support the need of modifying the Einstein equation [18], and the modified equation of 1995 [1] (§§ 4 & 5).

Besides Einstein's remark, the need of modifying the Einstein equation can go back at least to Hogarth's [19] 1953 conjecture in which he stated clearly a gravitational energy-stress tensor in the source term is needed. However, due to the lack of experimental evidences, this conjecture has not resulted in a more concrete proposal. Note that Lo's proof for the modified Einstein equation is based on the binary pulsar experiments [20-22] which confirmed the linear field equation as the first order approximation [1].

Clearly, Lo's proof implies unequivocally that the 1990 claim of Damour and Schmidt [2] on the existence of bounded solutions for an isolated radiation source, is incorrect in mathematics. Moreover, based on examples, it is found directly that their claims are incorrect. In particular, their "proof" of boundedness was essentially by assuming boundedness implicitly on a vacuum solution. However, as pointed out by Weinberg [17], the initial data is restricted by the four components of the Einstein equation (see § 7). Thus, the initial data cannot be as arbitrary as suggested by Damour & Schmidt.

2. Physical Considerations and the Plane Wave as an Idealization

Relativity suggests the existence of gravitational waves because physical influence must be propagated with a finite speed [23]. Thus, the existence of the gravitational wave is actually a physical requirement for a valid field equation. The Einstein equation of 1916 [16] is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -K\, T(m)_{\mu\nu} , \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci curvature tensor, $T(m)_{\mu\nu}$ is the energy-stress tensors for massive matter, and K (= $8\pi\kappa c^{-2}$, where κ is the Newtonian coupling constant) is the coupling constant. However, gravitational waves have not been directly observed [24] although there are indirect observational evidences [20-22]. Theoretically, the existence of gravitational waves had been assumed a certainty to the first approximation [1,3,13,17,25-27]. The linear field equation is

$$\frac{1}{2} \partial_c \partial^c \bar{\gamma}_{ab} = -KT(m)_{ab} . \tag{2a}$$

where $\bar{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}\eta_{ab}\, \gamma$, $\gamma_{ab} = g_{ab} - \eta_{ab}$, $\gamma = \eta^{cd}\gamma_{cd}$, and η_{ab} is the flat metric. Eq. (2a) has a mathematical structure similar to that of Maxwell's equation. An inhomogeneous solution of eq. (2a) is

$$\bar{\gamma}_{ab}\,(x^i,t) = -\frac{K}{2\pi}\int \frac{1}{R} T_{ab}[y^i,\ (t-R)]\, d^3y, \quad \text{where } R^2 = \sum_{i=1}^{3} (x^i - y^i)^2 . \tag{2b}$$

Solution (2b) would represent a wave if T_{ab} has a dynamical dependency on time t' (= t − R). The implicit gauge condition is that the flat metric η_{ab} is the asymptotic limit at infinity.

However, Einstein himself discovered [10] in 1936 that linearized gravity is not reliable and that his eq. (1) does not admit a propagating wave solution. However, Bondi, Pirani and Bobinson [13], believed that a general plane-wave, which they considered to be valid [14,15], is

$$ds^2 = \exp(2\phi)(d\tau^2 - d\xi^2) - u^2[\mathrm{ch}2\beta(d\eta^2 + d\zeta^2) + \mathrm{sh}2\beta\, \cos 2\theta(d\eta^2 - d\zeta^2) - 2\mathrm{sh}2\beta\, \sin 2\theta d\eta d\zeta], \tag{3}$$

where ϕ, β, θ are functions of u (= $\tau-\xi$). It satisfies the differential equation (i.e. their eq. [2.8]),

$$2\phi' = u(\beta'^2 + \theta'^2 \, sh^2 \, 2\beta). \tag{4}$$

But, metric (3) is not bounded, because this would require the impossibility of $|u^2| <$ constant. Note that an unbounded f(u) grows anomaly large as time goes by, and is therefore questionable.

From the viewpoint of physics, however, a gravitational wave must satisfy at least, three requirements:

i) The amplitude of a gravitational wave must be bounded (i.e. $|g_{ab}| <$ constant) as weak gravity condition requires [16].

ii) Moreover, if eq. (2a) gives a first order approximation, $\partial^a \bar{\gamma}_{ab}$ should be of second order [1,28] due to the conservation law, $\nabla^a T_{ab} = 0$ (which is independent of the notion of gauge).

iii) As manifested by solution (2b), a gravitational wave is a propagating wave, and is an almost periodic function of time for a source in an almost periodic motion.

Note that conditions i) and iii) are also satisfied by the electromagnetic wave and conditon ii) is similar to the Lorentz gauge. Thus, a gravitatioinal wave should have a fruitful analogy with electromagnetism [17]. But, $R_{ab} = 0$ in all space-time would lead to a violation of these requirements (see §§ 3-7).

Bondi et al. claimed that "Our interest in plane waves derives not, of course, from the expectation that such waves might exist in nature, but from the presumption that at agreat distances from a finite source of gravitational waves, these waves must appear to be approximately plane." Thus, one would expect that their metric (3) satisfies these requirements. However, for metric (3), not only i) is violated, ii) is not satisfied since $\partial^a \bar{\gamma}_{ab}$ is also clearly not bounded. Moreover, metric (3) is not even an almost periodic function. Thus, metric (3) fails all three physical requirements (which cannot be criticized as too severe). Since metric (3) satisfies eq. (1), this is a clear evidence that eqs. (1) and (2) are not dynamically compatible [1].

Nevertheless, since metric (3) may not be the most general form of a plane-wave, we examine another general plane-wave form proposed by Misner, Thorne, and Wheeler [3] as the follows:

$$- ds^2 = L^2(e^{2\beta} dx^2 + e^{-2\beta} dy^2) + dz^2 - dt^2 , \tag{5}$$

where both L (background factor) and β (wave factor) are functions of $u(\equiv t - z)$. Form (5) would be incompatible with relativistic causality [29] which requires:

$$L^{-2}e^{2\beta} \leq 1; \qquad \text{and} \qquad L^{-2}e^{-2\beta} \leq 1. \tag{6}$$

However, since $L = 1$ and $\beta \neq 0$ is a weak wave limit according to Misner et al., relation (6) may not be possible. Also, form (5) cannot accommodate a circularly polarized wave [18]. As pointed out by Wald [30], there is insufficient evidence to support the decomposition in form (5) as natural.

Although form (5) manage to avoid an appearance of obvious unboundedness, the problem remains. Plane-wave form (5) reduces the Ricci curvature to zero except

$$R_{uu} = 2[L'' + (\beta')^2 L]/L . \tag{7}$$

However, no exact wave solution has been obtained for $R_{uu} = 0$ [7]. As mentioned above, the difficulty is not only in mathematics, but that no ***physical*** solution for Einstein equation $R_{ab} = 0$ exists (see §§ 4-6).

In fact, $R_{ab} = 0$ is incompatible with the notion of weak gravitational waves (see §§ 3-6). To see this, it would be enlightening to consider the weak field approximation eq. (2) further, since it is supported by the observations [20-22] on the radiation lost of binary pulsars.

3. The Weak-Field Approximation of Waves and Modification of the 1916 Einstein Equation

The inability to recognize $G_{\mu\nu} = 0$ as invalid for waves is due to that eq. (2) is ***mathematically*** mistaken as a first order approximation of eq. (1). In spite of Einstein's discovery of the incompatibility between eq. (1) and eq. (2) is not proven until 1993. Eq. (2) is actually an approximation of the modified equation,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R = -\kappa\,[T(m)_{\mu\nu} - t(g)_{\mu\nu}], \tag{8}$$

where $t(g)_{\mu\nu}$ is the energy-stress tensor for gravity. Eq. (8) is the Einstein equation of 1995 version. Then,

$$\nabla^{\mu}T(m)_{\mu\nu} = 0, \qquad \text{and} \qquad \nabla^{\mu}t(g)_{\mu\nu} = 0, \tag{9}$$

Eq. (9) implies that the equivalence principle would be satisfied [1]. In light of eq. (2) being an approximation of (8), let us consider the weak waves generated by massive matter.

According to eq. (2), at vaccum, a weak plane-wave satisfies the following two equations:

$$\partial_c \partial^c \bar{\gamma}_{ab} = 0 \; ; \tag{10a}$$

and

$$\partial^a \bar{\gamma}_{ab} = 0 \; . \tag{10b}$$

Note that the zero in (10) should be understood as representing a second order term [27] but not exactly zero; otherwise (10b) and (2a) imply no radiation [30,31]. However, the periodic nature of gravity and boundedness of a gravitational wave are clearly manifested only in solution (2b). Eq. (10) is the equation for the plane wave and implies that the pure gravitational plane-waves are transverse waves [17].

Also, one should not consider condition (10b) as a linearization of the harmonic gauge [3,17] which was not accepted by Eddington [28] and questioned by Einstein [16] and can be incompatible with the equivalence principle [1,29,32]. One may argue that (10b) would be satisfied by using the transformation,

$$x^{\mu} \rightarrow x'^{\mu} = x^{\mu} + \varphi^{\mu} \tag{11a}$$

which gives

$$\gamma'^{\mu\nu} = \gamma^{\mu\nu} - \partial^{\nu}\varphi^{\mu} - \partial^{\mu}\varphi^{\nu}. \tag{11b}$$

By solving

$$\partial_c \partial^c \varphi_b = \partial^a \bar{\gamma}_{ab} \; , \tag{11c}$$

it seems, a gauge function φ_a can be obtained. But, the mathematical similarity to the gauge in classical electrodynamics stops at this point because, as implicitly questioned by Einstein, a derivation of the gauge function $\partial_b \varphi_a$ may not be be bounded [29]. An unbounded metric would be invalid in physics [1,29].

The incompatibility between eq. (1) and eq. (2) can be proved by using Einstein's radiation formula. If one decomposes the Einstein tensor as a sum of the first order and the second order as follows [16]:

$$G_{ab} \equiv G^{[1]}{}_{ab} + G^{(2)}{}_{ab} \qquad \text{where} \qquad G^{(1)}{}_{ab} = \frac{1}{2}\partial_c\partial^c\,\bar{\gamma}_{ab} + H^{(1)}{}_{ab} \tag{12}$$

where

$$H^{(1)}{}_{ab} \equiv -\frac{1}{2}\partial^c[\partial_b\bar{\gamma}_{ac} + \partial_a\bar{\gamma}_{bc}] + \frac{1}{2}\eta_{ab}\partial^c\partial^d\bar{\gamma}_{cd}\,.$$

Eq. (10) implies $G^{(1)}{}_{ab} = 0$. One might argue [30] that $G_{ab} = 0$ although $G^{(2)}{}_{ab} \neq 0$ is that

$$G^{(1)}{}_{ab}(\gamma^{(2)}{}_{cd}) + G^{(2)}{}_{ab} = 0, \tag{13}$$

where $\gamma^{(2)}{}_{cd}$ is the second order approximation. But, eq. (13) is not valid if eq. (2) is the first order approximation; and consequently $G^{(2)}{}_{at} \sim O(1/r^2) \neq 0$ [17]. If g_{ab} is an almost periodic function of t', then it is alway possible to find a time-average of $G^{(1)}{}_{at}$ such that it has no term of $O(1/r^2)$ [1]. Thus, it is invalid to consider a possible dynamic solution of eq. (1) as suggested by Wald [30]. It follows that the plane-wave is actually an approximation of the Einstein eq. (8) of the 1995 version (see also §§ 4, 5 & 7).

Moreover, from the viewpoint of physics, Einstein's notion of gravitational energy-stress [27] actually implies the need of modifying eq. (1). Although Einstein's notion is not localizable, his radiation formula implies the existence of radiation in vacuum. However, physically it is impossible to transform a radiation away by changing the space-time coordinate system. Thus, Einstein's notion cannot be exact, but is an approximation of the gravitational energy-stress tensor for some coordinate systems. It follows that this impossibility means a necessary modification of eq. (1) which implies the exact validity of Einstein's conservation law [17]. This impossibility means also that eq. (1) is incompatible with radiation and therefore eq. (2).

4. Field Equation for the Exact Plane-Wave Idealization

To consider the most general exact plane-wave form, we must first derive the exact field equation for such an idealization. According to eq. (8), the field equation in vacuum should be

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = K \; t(g)_{\mu\nu}, \tag{8)'}$$

However, if one assume Einstein equation (1) were valid, then the field equation to be considered would be

$$R_{\mu\nu} = 0 \tag{14}$$

For simplicity, we consider a plane-wave propagating to the z-direction. As in the literature, the velocity of the gravitational wave is assumed to be c, the light speed in vacuum. This is justified by the facts that an accompany gravitational wave of the light must have the papagation speed as the light and that in a weak field, the light speed would be very close to c. Thus, the metric g_{ab} is a function of u (= ct −z), i.e.,

$$g_{lk} = g_{lk}(u) \; . \tag{15}$$

Let P^k be the momentum of a graviton. (For simplicity, the units is selected to have c =1.) Because the wave speed is 1, the mass of a graviton is zero; and one obtains [18] the conditions,

$$P^z = P^t, \;\; P^x = P^y = 0, \;\; \text{and} \;\; P^m \, g_{mk} = P_k = 0 \;\; , \tag{16a}$$

for k = x, y, and v (≡ t + z). Eq. (16a) is equivalent to

$$g_{xt} + g_{xz} = 0, \; g_{yt} + g_{yz} = 0, \; \text{and} \; g_{tt} + 2g_{tz} + g_{zz} = 0 \; , \tag{16b}$$

or

$$g^{xt} - g^{xz} = 0 \; , \; g^{yt} - g^{yz} = 0, \; \text{and} \; g^{tt} - 2g^{tz} + g^{zz} = 0. \tag{16c}$$

Eqs. (15) and (16) imply also the validity of the geodesic. Then, the tensor R_{ab} is reduced to

$$R_{tt} = - R_{tz} = R_{zz} \; , \tag{17a}$$

and the other components are zero. Then, $R_{ab} = 0$ is simplified to a differential equation of u,

$$R_{tt} = 0, \tag{18a}$$

where

$$R_{tt} = \{G'' - g_{xx}'g_{yy}' + (g_{xy}')^2 - G'(g'/2g)\}/2G, \tag{18b}$$

where

$$G \equiv g_{xx}\, g_{yy} - g_{xy}^2, \quad \text{and} \quad g = |g_{lk}| \tag{19}$$

is the determinant of the metric. The metric elements are connected by the following relation:

$$-g = G\, g_t^2\,, \quad \text{where} \quad g_t \equiv g_{tt} + g_{tz}. \tag{20}$$

Note that eq. (4) of Bondi et al. and eq. (7) of Misner et al. are special cases of eq. (18).

Equations (16)–(20) allow g_{xt}, g_{yt}, and g_{zt} to be set to zero (or equivalently $g_{uk} = 0$ for k = x, y, u). These orthogonal conditions are valid because there is no physical reason to suggest otherwise. In any case, these assigned values have little effect in subsequent calculations. In the above derivation, periodicity has not been used. Note that equation (16) allows the possibility of g_{tt} to have a wave component as in metric (3).

Now, there are four metric elements (g_{xx}, g_{xy}, g_{yy}, and g_{tt}) to be determined. However, there is only one differential equation (since eq. [20] is not really an equation if g is not specified by other means). Nevertheless, to show that there is no physical solution, eq. (18) is sufficient.

5. The Circularly Polarized Exact Plane–Waves

For an isolated system, Newtonian theory manifested the orbits are periodic functions. In general relativity, g_{ab} can be considered as an almost periodic function of t' (= t − R) since the radiaton is so small [17]. Thus, in a plane–wave idealization, g_{ab} can be considered as a periodic function as in the literature [3,17]. Moreover, for some cases, the periodicity of a plane wave can be proven as intrinsic.

A graviton of spin 2 is often identified with a circularly polarized wave [17]. Here, it will be shown

that a circularly polarized exact plane-wave does not satisfy eq. (14), but it would satisfy eq. (8)'. For this purpose, it is necesary to derive the circularly polarized wave from the expression of R_{tt} in eq. (18b) (but, ignores the relation $R_{tt} = 0$) instead of based on the usual linear equation (10). It is interesting to note that periodicity is also a result. For such a wave, the rotational invariants with respect to the z-axis are constants. These invariants are: G, $(g_{xx} + g_{yy})$, R_{tt}, $T(E)_{tt}$, g_{tz}, g, g_{tt}, and etc. Let us assume the invariant,

$$g_{xx} + g_{yy} = -2 - 2C, \text{ then } g_{xx} = -1 - C + B \text{ , and } g_{yy} = -1 - C - B. \tag{21}$$

Then, the expressions of R_{tt} and G in eq. (18) and eq. (19) imply,

$$B^2 + g_{xy}{}^2 = (1+C)^2 - G \text{ ,} \tag{22}$$

and

$$(B')^2 + (g_{xy}')^2 = 2GR_{tt} \geq 0 \tag{23}$$

are constants. It follows that eqs. (22) and (23) imply

$$B = B_\alpha \cos(\omega u + \alpha), \quad \text{and} \quad g_{xy} = \pm B_\alpha \sin(\omega u + \alpha) \text{ ,} \tag{24a}$$

where

$$\omega^2 = 2R_{tt}G/B_\alpha{}^2, \quad \text{and} \quad B_\alpha{}^2 = (1+C)^2 - G \geq 0. \tag{24b}$$

Thus, the circularly polarized metric is exactly

$$g_{xx} = -1 - C + B_\alpha \cos(\omega u + \alpha) \text{ ,} \quad g_{yy} = -1 - C - B_\alpha \cos(\omega u + \alpha).$$

and

$$g_{xy} = \pm B_\alpha \sin(\omega u + \alpha) \tag{24c}$$

Wave (24c) satisfies the linear eq. (10). However, eq. (24b) implies that $R_{tt} = 0$ only if $\omega = 0$ or $B_\alpha = 0$.

This means that eq. (14) cannot have a circularly polarized wave. Moreover, wave (24c) implies $R_{tt} > 0$. This requires an antigravity coupling, and thus wave (24c) is associated with eq. (8)' but not eq. (1).

Thus, in spite of the popular speculation of associating a graviton with eq. (14) [17], there is no possibility, within the current theory of eq. (1), to construct an acceptable metric representing the circularly polarized gravitational wave. Moreover, the other polarizations of plane waves also result in $R_{tt} \neq 0$.

6. A General Proof for the Non-existence of Plane-Waves for $R_{ab} = 0$

In general, it can be shown that there is no bounded wave solution for eq. (14) by using the required boundedness and periodic nature of the metric (see also §§ 2 & 3). Then, the curvature tensor component R_{tt}, on the time average, is necessarily non-negative for a plane wave. (This reflects the fact that, on the time average, $-G^{(2)}{}_{tz}$ is positive as manifested by Einstein's radiaton formula.) In the subsequent derivation, weak gravity is utilized. If a time average is taken over multiple periods, a time average of derivatives of a metric element must be zero. For simplicity, we assume also that the wave components are essentially monchromatic although one can see in the following derivations that such an assumption is not necessary.

It will be shown that, on the time average, $R_{tt} > 0$. For clarity, we define

$$-f \equiv \gamma_{xx} + \gamma_{yy} = -(f_1 + f_2) = -(f_1 + F_1 + F_2) \tag{25}$$

where f_1 is the time average of f, F_1 is of the first order and F_2 is of the second order. It follows that

$$G = 1 + f - f\gamma_{xx} - (\gamma_{xx})^2 - (\gamma_{xy})^2], \tag{26}$$

and

$$G' = f_2' - [f_2'\gamma_{xx} + f\gamma_{xx}' + 2\gamma_{xx}\gamma_{xx}' + 2\gamma_{xy}\gamma_{xy}']. \tag{27}$$

Note that all terms in (27), except possibly f and f_2', are at least of second order in the derivations. Simply algebraic calculation gives

$$G'' - g_{xx}'g_{yy}' + (g_{xy}')^2 = (F_2'' + F_1'') - (F_1'' + F_2'')\gamma_{xx} - f\gamma_{xx}'' - f_2'\gamma_{xx}'$$

$$- [2\gamma_{xx}\gamma_{xx}'' + (\gamma_{xx}')^2 + 2\gamma_{xy}\gamma_{xy}'' + (\gamma_{xy}')^2]. \qquad (28)$$

Now, it is clear that in R_{tt} the only term which can be of first order in the deviations is f_2''. Note that (28) includes all terms (from the curvature tensor) in the lowest order equation.

To approximate eq. (14), it would be useful to drive some mathemtical relations for later usage. A periodic function $A(\omega u)$ can be expressed by

$$A(\omega u) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega u + \alpha_n). \qquad (29a)$$

It follows that

$$A' = -\omega \sum_{n=1}^{\infty} nA_n \sin(n\omega u + \alpha_n),$$

and

$$(A')^2 = \omega^2 \sum_{n=1}^{\infty} A_n n^2/2 + \omega^2 F(A), \qquad (29b)$$

where

$$F(A) = \sum_{m\neq n}^{\infty} mn\, A_m A_n \sin(m\omega u + \alpha_m)\sin(n\omega u + \alpha_n) - \sum_{n=1}^{\infty} n^2 A_n^2 \cos 2(n\omega u + \alpha_n)/2.$$

The time average of $F(A)$ is zero. Since $A'' = -\omega^2 \sum_{n=1}^{\infty} n^2 A_n \cos(n\omega u + \alpha_n)$,

$$AA'' = -\omega^2 \sum_{n=1}^{\infty} n^2 A_n^2/2 + A''A_0 - \omega^2 F_1(A), \qquad (29c).$$

where

$$2F_1(A) = \sum_{m\neq n}^{\infty} [m^2+n^2] A_m A_n \cos(m\omega u + \alpha_m)\cos(n\omega u + \alpha_n) + \sum_{n=1}^{\infty} n^2 A_n^2 \cos 2(n\omega u + \alpha_n),$$

whose time average is also zero. Eqs. (29) implies.

$$(A')^2 + 2AA'' = -\omega^2 \sum_{n=1}^{\infty} n^2 A_n^2/2 + 2A''A_0 - \omega^2 [F(A) - 2F_1(A)], \qquad (29d).$$

Since all the constant terms in equation (29d) are negative, (28) is positive on the time average.

Now, let us consider the case where f_2' is of first order, then the first order equation is

$$F_1'' = 0 \tag{30a}$$

Since F_1 is a periodic function, the only possibility is $F_1 = 0$. This contradiction implies f_2' is of second order. The second order equation is

$$F_2'' = [(2\gamma_{xx} + f_1)\gamma_{xx}'' + (\gamma_{xx}')^2 + 2\gamma_{xy}\gamma_{xy}'' + (\gamma_{xy}')^2]. \tag{30b}$$

From eq. (26), one can see easily that the constant terms in eq. (30b) have the same sign. Consequently, eq. (30b) is not valid unless the wave components of γ_{xx} and γ_{xy} are zero. Since the above arguments are symmetric toward x and y, the wave component of γ_{yy} are also zero. Thus, the order of f_2'' is at least three.

In general, one may assume that the constant f_1 is of order n, and the least order of γ_{xx}' and γ_{xy}' is m if they are not zero. Then the lowest order equation is

$$f_2'' - f_1\gamma_{xx}'' = [(2\gamma_{xx}\gamma_{xx}'' + (\gamma_{xx}')^2 + 2\gamma_{xy}\gamma_{xy}'' + (\gamma_{xy}')^2]. \tag{30c}$$

If n is smaller than m, the order of the left hand side is at least of order 2m. But, this also not possible since the time average of the left hand side is zero, but the right hand-side is non-zero. If n is large than m, a similar problem would arise. Thus, it is not possible to have a wave components in γ_{xx}' , γ_{yy}' , or γ_{yy}'.

In conclusion, Rosen [12] is essentially correct, i.e., there is no physical plane-wave solution for $R_{ab} = 0$. Note that all existing "plane-wave" solutions are not bounded and that eq. (1) must be modified, otherwise it has no dynamic solution (see also § 7). It seems, before 1993 theorists were unaware of that such plane waves are not bounded, and therefore unphysical. For example, even Griffiths [33] wrote a book about colliding plane waves. A root of the problem is that linearized gravity was <u>incorrectly</u> accepted as generally valid [34], and therfore the linear field equation (2) were considered as derivable from eq. (1).

7. The Question of Dynamic Solutions in General Relativity

General relativity, being a theory which abandoned naive visualizations in favour of a conceptual analysis of the correspondence between observational data and the mathematical quantities in a theoretical formalism [27], can be a victim of over extrapolation without adequate analysis in physics. In particular, there are confusions on what is a physically valid solution because the equivalence principle was not well-understood [35]. In physics, deriving of a valid solution for a field equation is possible only if the physical requirements are clearly identified [1,18,29]. On the other hand, if the physical requirements are inadequate, an unphysical solution such as metric (3) would be incorrectly accepted as a "plane-wave" solution (see § 2).

In 1936, Einstein discovered that his eq. (1) does not admit a propagating wave solution. Its incompatibility with eq. (2) has been shown in § 3 and it has been proven that there is no plane wave solution for eq. (1). Based on these, Lo [1] has derived the 1995 version of Einstein equation (8), which clearly has a dynamic solution. However, one might still argue that equation (1) does mathematically admit a propagating wave solution [36]. Was Einstein's discovery wrong, and therefore Lo's derivation invalid? To clarify this question, one must understand that Einstein is a physicist, and thus would concern only physical solutions. To do justice to Einstein, one must check whether any of such mathematical solutions is valid in physics.

According to Einstein [16], a metric must satisfy the equivalence principle. For a physical solution, a weak source would produce a weak gravity. Furthermore, for a massive source, according to observations of the binary pulsars, linear field eq. (2) is the first order approximation [1]. Moreover, existing perturbation theories and weak gravity necessarily assume that a solution is bounded [4].

However, as manifested by metric (3), eq. (1) has no bounded physical solution for a dynamic situation. This can be proven by contradiction. If there were a bounded solution for an isolated system, according to the corresponding principle, eq. (1) should have a weak gravity if the source is not too strong. Let us assume that $\gamma^{(1)}{}_{ab} = \gamma_{ab} - \gamma^{(2)}{}_{ab}$ and define $\bar{\gamma}^{(i)}{}_{ab} = \gamma^{(i)}{}_{ab} - \eta_{ab}(\gamma^{(i)}{}_{cd}\eta^{cd})$, where i = 1, 2. If

$$\frac{1}{2}\partial_c\partial^c\,\bar{\gamma}^{(1)}{}_{ab} = -\,KT_{ab}\,, \tag{31}$$

then $\bar{\gamma}^{(2)}{}_{ab}$ is of the second order ***as required by the radiation formula***. Then, from eq. (1), one has

$$\frac{1}{2}\partial_c\partial^c\,\bar{\gamma}^{(2)}{}_{ab} + H^{(1)}{}_{ab} = -G^{(2)}{}_{ab}\,. \tag{32}$$

For a static problem, eq. (32) can be satisfied since both sides of eq. (32) would be of $O(K^2/r^4)$. But, for an isolated dynamic system, T_{ab} and γ_{ab} are ***almost periodic*** functions of time since the effect of gravitational radiation is extremely small [17]. Then, it can be shown that eq. (32) cannot be satisfied. It follows periodicity that $G^{(2)}{}_{at}$ is of $O(K^2/r^2)$ (see also § 3). But, a non-zero time-average of the left hand side is of $O(K^2/r^3)$. Thus, the two sides of eq. (32) cannot be matched. The impossibility of satisfying eq. (32) implies that the initial assumption of boundedness is not valid.

Note that $G^{(1)}{}_{at}$ contains only first order time derivatives, and thus the equations, $G_{at} = -KT(m)_{at}$ (a = x,y,z,t) restrict the initial data. For the idealization of an exact plane wave, it is obvious that eq. (32) cannot be satisfied. On the time average, $G^{(1)}{}_{tz} = 0$, $-G^{(2)}{}_{tz} > 0$, and $-G_{tz} > 0$, according to eq. (18).

Thus, for a dynamic problem, the basic assumption of boundedness for solutions of eq. (1) in the perturbative methods [4] is <u>invalid</u>. From the above analysis, the only possibility of having a bounded solution could be that eq. (2) is not a valid approximation. But, such a solution, if existed, would be unphysical since it is at odd with radiation experiments as well as the physical principles. There is no evidence for such an unphysical solution. On the other hand, the non-existence of a bounded plane-wave solution for eq. (1), supports the non-existence of a bounded dynamic solution (see §§ 5 & 6). If one takes the viewpoint of Bondi, Pirani and Robinson (see § 2), the impossibility of having a bounded plane wave solution can be considered as another conclusive evidence that eq. (1) does not admit a physical solution of propagating waves.

To illustrate the non-existence of a bounded radiating physical solution, let us examine a recent solution of $R_{ab} = 0$. For example, the cylindrical symmetry solution of Au, Fang & To [35] is

$$ds^2 = N^2(c^2dt^2 - dz^2) - L^2d\rho^2 - M^2\rho^2d\phi^2 \tag{33}$$

where

$$N^2 = \frac{1}{\rho^4}\exp(-4\int\Phi d\rho)\exp(2n_1), \qquad L^2 = \frac{1}{\rho^8}(1 + \rho\Phi)^2\exp(-6\int\Phi d\rho),$$

and

$$M^2 = \exp(2\int\Phi d\rho) \qquad \text{where} \qquad n_1 = n_1(ct - z), \quad \text{and} \quad \Phi = \Phi(\rho)$$

are resepectively arbitrary functions of $(ct - z)$ and of ρ. The function $n_1(ct - z)$ makes N^2 a propagating wave. If solution (33) were a physical solution, M should be a bounded function of ρ, i.e.,

$$\exp(2\int\Phi d\rho) < C_1^2 \quad , \tag{34}$$

for some positive constant C_1. But, this also means that N^2 is <u>not</u> bounded for small ρ. Moreover, if light velocity is not larger than its vacuum velocity c, one should have N^2/L^2 and $N^2/M^2 \leq 1$. It thus follows that

$$(1 + \rho\Phi)^2 \geq \rho^4 \exp(2\int\Phi d\rho)\exp(2n_1), \quad \text{and} \quad \exp(6\int\Phi d\rho) \geq \exp(2n_1)\rho^{-4}. \tag{35}$$

Hence,

$$(1/\rho + \Phi)^2 \geq \rho^{2/3} \exp(8n_1/3) \tag{36}$$

and therefore

$$\Phi^2 > \sim O(\rho^{2/3}). \tag{37}$$

But conditon (37) is also inconsistant with condition (34). Thus solution (33) is not a physical solution.

Another example is that Damour and Schmidt [2] claimed the existence of bounded physical solutions for an isolated radiating source. Their claims are shown in the abstract of their paper as follows:

The relation between perturbation theory and exact solutions in general relativity is tackled by investigating the existence and properties of smooth one-parameter families of solutions. On the one hand, the coefficients of the Taylor expansion (in the parameter) of any given smooth family of solutions necessarily satisfy the hierarchy of perturbation equations. On the other hand, it is the converse question (does any solution of the perturbation equations come from Taylor expanding some family of exact solutions?) which is of importance for the mathematical justification of the use of perturbation theory. This converse question is called the one of the "reliability" of perturbation theory. Using, and completing, recent results on the characteristic initial value probelm, the local reliability of perturbation theory of general relativity in vacuum is proven very generally. This result is then generalized to the Einstein-Yang-Mills equations (and therefore, in particular, to Einstein-Maxwell ones). These local results are then partially extended to the

global ones by: (i) proving the existence of semiglobal vaccum space-time (respectively, Einstein-Yang-Mills solutions) which are stationary before some retarded time u_0, and radiative after u_0, and which admit a smooth conformal structure at future null infinity: and (ii) constructing smooth one-parameter families of such solutions whose Taylor expansions are of the "multipolar post-Minkowskian" type which has been recently used in perturbation analyses of radiative space-times.

With such drastic claims, one would expect them to provide a few examples to illustrate the results. However, there is no example of a solution even for a radiating wave. They, just like Damour [4] in 1987, seemed not aware of that the existing waves of $R_{ab} = 0$ are unbounded. Consequently, there is no discussion on the problem of unboundedness; and inadvertently they implicitly assume that the waves were bounded. Some theorists might even argue that a physical solution in vacuum need not be bounded. However, this is not an issue here since Damour and Schmidt claimed in (ii) that eq. (2) provides the first order approximation.

In view of that all existing wave solutions of $R_{ab} = 0$ are unbounded, from the viewpoint of physics, a radiating physical solution should be due to false arguments. Note that the mentioned remote possibility (of having a bounded solution for which eq. (2) is not a valid approximation) is not an issue here because of ii). It is known that there is no possibility, within orthodox general relativity, to construct a physically acceptable (i.e. bounded) metric representing the gravitational wave necessarily accompanying a plane electromagnetic wave [18,29]. Therefore, their generalization to Einstein-Maxwell equations also cannot be valid in physics. It is also known that, for a circularly polarized wave, the linearized field equation does not provide an approximation [34]. Moreover, there is no excuse for not justifying, in terms of physics, the questionable situation of a sudden switch on radiation if it is excusable for not providing examples solved in close-forms.

Note that the local reliability is actually trivial since a continuous function is always locally bounded and a Minkowskian metric is locally diffeomorphic to the flat metric η_{ab}. Moreover, the existence of a smooth one-parameter family $g_{ab}(x^i,\lambda)$ for $R_{ab} = 0$ has meaning to the "reliability" of perturbation theory only if 1) globally $g_{ab}(x^i,0) \approx \eta_{ab}$, the flat metric, and 2) the family is bounded, i.e. $|g_{ab}(x^i,\lambda)| \leq$ constant.

To illustrate this, let us consider metric (3) of Bondi et al. Since metric (3) is not bounded, it is impossible to have a meaningful first order approximation. Nevertheless, a one-parameter family $g_{ab}(x^i,\lambda)$ can be obtained if one substitutes ϕ and θ in metric (3) respectively with $\phi+\lambda$ and $\theta+\lambda$ since they satisfy

the same differential equation (4). However, not only $g_{ab}(x^i,0)$, but the expansion of λ order terms such as

$$h_{(1)\eta\eta} = -h_{(1)\zeta\zeta} = 2u^2 sh2\beta \sin 2\theta, \quad \text{and} \quad h_{(1)\eta\zeta} = -2u^2 sh2\beta \cos 2\theta, \tag{38}$$

where β and θ are function of u $(=\tau-\xi)$, are not bounded. In fact, any plane wave can easily become a family of solutions since eq. (18) is homogeneous in the metric elements. If $g_{ab}(x^i,0)$ is not bounded, no matter how small λ is chosen, one cannot joint $g_{ab}(x^i,\lambda)$ with some stationary metric to form a solution of $R_{ab} = 0$. In short, a one-parameter family of solutions of $R_{ab} = 0$ may have no physical meaning at all.

Thus, Damour and Schmidt's claim of validity due to the freedom of choosing a very small λ is misleading and requires a rigorous proof. In particular, the validity of their choice of data for the purpose of a union (as mentioned) is itself an implicit dubious assumption. To say the least, the proof of their claim i) is incomplete and therefore their claim ii) is misleading since local reliability has little meaning in perturbation.

One may argue, however, their assumed validity is reasonable in physics since this only means a small deviations about some stationary metric. But, one should not confuse over the physical requirements as the mathematical characteristics of a solution of an equation which is under investigation. If a field equation is inappropriate, it may not be able to accommodate the physical requirements no matter how reasonable. The investigation of plane-waves in this paper shows that even necessary physical requirements are not satisfied by eq. (1). A "plane wave" of eq. (1) must have an amplitude increasing with time. The existence of only unbounded waves would imply that equation (1) is not valid for a dynamic problem. Experiments verify, however, the existence of bounded waves and thus reject eq. (1). Note also that some time-dependent solutions of eq. (1) even have unexplainable singularities [37].

Another major problem is that there seems no physical situation corresponds to a switch-on radiation. To complete their proof, Damour and Schmidt must do the impossibility, to show unexpectedly a family of bounded wave solutions $g_{ab}(x^i,\lambda)$ for $R_{ab} = 0$ with $g_{ab}(x^i,0) = \eta_{ab}$ and to prove, in spite of using the harmonic gauge [16,28,30,31], such a metric satisfying the equivalence principle. The futility of such an effort has clearly been demonstrated in the analaysis of plane waves. From the viewpoint of physics, this futility is even clearer. The impossibility of having a bounded plane wave solution for $R_{ab} = 0$ manifests that

a physical gravitational wave implies $R_{ab} \neq 0$ (see also § 5). Physically, this means that a gravitational wave carries the gravitational energy-momentum as shown in eq. (8), the Einstein equation of 1995 version.

8. Discussions and Conclusions

In Einstein's presentation [16], linear field equation (2) was derived as a consequence of the Einstein's equation (1). But, as Einstein discovered in 1936, his derivation is not generally valid. Then, Rosen [12] conceived the notion that the exact plane wave would be crucial in understanding general relativity. However, due to the lack of experimental support, the importance of Rosen's notion and Hogarth's conjecture [19] of 1953 on the need of modifying eq. (1), was not recognized. The explanation of the perihelion of Mercury and the work on exact "plane-waves" [13] provided a seemingly unshakable appearance for eq. (1).

Moreover, the confusion on the exact plane waves [14,15] played a crucial role in disregarding Einstein's discovery; and the incompatibility between eq. (1) and (2) was not proven until 1995 [1]. Although the weak wave was analyzed, the intrinsic differences between the weak gravitational wave and the exact "plane-wave" [13] were not addressed since eq. (2) were believed to be an approximation of eq. (1).

The investigation of plane waves has played a crucial role on the need of modifying eq. (1). First, when the gravity of an electromagnetic plane wave was considered [18,32], according to the principle of equivalence and the principle of causality, there is no possibility of having a physical solution within the theoretical framework of eq. (1). Thus, the validity of eq. (1) became finally questionable and the need of an antigravity coupling term in the source was apparent. In the studies of exact plane waves, as declared by Bondi et al. [13], a crucial point is that an exact plane wave must be considered as an idealization of a weak wave from a distant isolated source. Thus, the characteristics manifested in the weak gravitational waves [16] should serve as guidiances in the investigation of exact gravitational plane waves. This necessary connection in terms of physics, as shown, clearly leads to the modified Einstein eq. (8).

The physical picture on the need of modifying eq. (1) due to radiation, can be seen more clearly through an exact plane wave because, as an idealization, it has discarded the non-essentials. Thus, it can be shown that there is no possibility to have a circularly polarized plane wave within the framework of eq. (1), and that there is no bounded plane waves for $R_{ab} = 0$. A crucial non-mathematical error of Damour and Schmidt

[2] seemed to be that they were unaware of the non-existence of a bounded wave for $R_{ab} = 0$.

Moreover, Einstein's radiation formula implies the existence of radiation which also leads to the need of modifying eq. (1). Although Einstein's notion of gravitational energy-stress [27] is not localizable, physically it is impossible to transform a radiation away by changing the coordinate system. Thus, Einstein's notion can only be an approximation of the gravitational energy-stress tensor for some coordinate systems. It follows that eq. (1), which implies the exact validity of Einstein's conservation law [16], must be modified.

Experimentally, it is the Hulse-Taylor observation on the binary pulsar PSR 1913+16 leads to the unequivocal support of eq. (2) as the first order approximation for a dynamical problem that eq. (1) must be modified. Nevertheless, this experiment (and other subsequent observations) is not a theoretcal necessity in justifying eq. (2) because the validity of eq. (2), as a first order approximation, could have been derived and justified in terms of physical principles which led to general relativity if one utilizes earlier experiments such as the bending of light or the time delay of light [38]. Being independent of eq. (1), eq. (2) has been called the Maxwell-Newton Approximation [1]. Also, the Maxwell-Newton Approximation would be further confirmed by the Gravity Probe-B gyroscopes [39] experiment in the near future.

In conclusion, there is no dynamic solution for the Einstein equation of 1916. Therefore, it is necessary to modify the 1916 version, and conclusions drawn from this equation must be reviewed to determine the need of revision. On the other hand, the Einstein equation of 1995 version does have dynamic solutions since it has the Maxwell-Newton Approximation. However, since the exact form of the gravitational energy-stress tensor $t(g)_{ab}$ remains to be known, general relativity remains an incomplete theory as Einstein stated [16]. Also, the pure gravitational plane wave is mathematically similar to the plane wave in electromagnetism [17]. In terms of physics, however, this similarity implies that the gravitational energy-stress tensor with the antigravity coupling must be added to the source of the Einstein of 1916 version.

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