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Bosonization of Fermi Liquids

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Abstract

We consider systems of non-relativistic, interacting electrons at finite density and zero temperature in $d = 2, 3, \ldots$ dimensions. Our main concern is to characterize those systems that, under the renormalization flow, are driven away from the Landau Fermi liquid (LFL) renormalization group fixed point. We are especially interested in understanding under what circumstances such a system is a Marginal Fermi liquid (MFL) when the dimension of space is $d \geq 2$.

The interacting electron system is analyzed by *combining* renormalization group (RG) methods with so called "Luther-Haldane" bosonization techniques. The RG calculations are organized as a double expansion in the inverse scale parameter, λ^{-1}, which is proportional to the width of the effective momentum space around the Fermi surface and in the running coupling constant, g_λ, which measures the strength of electron interactions at energy scales $\sim \frac{v_F k_F}{\lambda}$.

For systems with a strictly convex Fermi surface, superconductivity is the only symmetry breaking instability. Excluding such an instability, the system can be analyzed by means of bosonization. The RG and the underlying perturbation expansion in powers of λ^{-1} serve to characterize the *approximations* involved by bosonizing the system.

We argue that systems with short-range interactions flow to the LFL fixed point. Within the approximations involved by bosonization, the same holds for systems with long-range, longitudinal, density-density interactions.

For electron systems interacting via long-range, transverse, current-current interactions a deviation from LFL behaviour is possible : if the exponent α parametrizing the singularity of the interaction potential in momentum space by $\hat{V}(|\vec{p}|) \sim \frac{1}{|\vec{p}|^\alpha}$ is greater than or equal to $d-1$, the results of the bosonization calculation are consistent with a MFL.

1 Introduction

In this paper, we consider systems of non-relativistic electrons at finite density and zero temperature in $d = 2, 3, \ldots$ dimensions. The interactions between electrons are described by two-body potentials or by current-current interactions. The Cooper channel which drives the BCS instability is turned off; (e.g. by assuming that the Fermi sphere of the non-interacting system has a suitable geometry [1, 2]).

Our main concern in this paper is to characterize those systems that, under the renormalization flow, are driven away from the Landau Fermi liquid (LFL) renormalization group (RG) fixed point. More concretely, we are interested in understanding under what circumstances such a system is a Marginal Fermi liquid (MFL) when the dimension of space is $d \geq 2$. This problem comes up, for example, in the study of single-layer quantum Hall fluids at filling fractions $\nu = \frac{1}{2}, \frac{1}{4}, \ldots$ and, perhaps, in the theoretical description of materials related to anisotropic HT_c superconductors (see [3, 4, 5, 6] and references therein) .

A one-dimensional MFL (or Luttinger liquid) at zero temperature can be characterized by the property that the electron propagator falls off like $|\vec{x}|^{-(1+\eta)}$, at large distances $|\vec{x}|$, for an exponent $\eta > 0$ that depends on the electron-electron interaction and characterizes a new RG fixed point. When η vanishes the system is a LFL . In the limit of large distance scales and low frequencies (scaling limit) the properties of a LFL are identical to those of a free system of non-interacting electrons (up to a renormalization of the residue of the one-particle pole and of the Fermi velocity). In one dimension, MFL correspond to a line of RG fixed points in the space of effective Hamiltonians (or actions) containing the fixed point corresponding to the free system.

In $d \geq 2$ dimensions, we define a MFL to be a Fermi liquid with an electron propagator falling off more rapidly than the free electron propagator by at least a fractional inverse power of the distance between the arguments, but not exponentially in the distance. Contrary to the superconducting instability, the instabilities leading to a MFL are not accompanied by symmetry breaking, and there is no energy gap in the excitation spectrum of such systems. Yet, Landau's picture of non-interacting quasi-particles does not apply to the physics of Luttinger liquids. Historically, MFL were discovered in the context of one-dimensional systems of interacting electrons [7], where they appear naturally, for a large class of two-body interactions. The experimental observation that the normal phase of anisotropic

HT_c superconductors exhibits many non-conventional features, incompatible with LFL theory, leads to the question whether electron-electron interactions can drive a system in *two* or *more* dimensions away from LFL's. The same question arises in the context of single-layer QH fluids at filling fractions ν with even denominators.

Recently, there have been many investigations of this problem. Besides numerical studies, two analytic techniques proved useful in attempts to understand it: the RG method, involving an expansion in an inverse scale parameter λ^{-1} [9], which has led to a wealth of rigorous results [1, 2, 9, 10], and the bosonization techniques [11] proposed, in this context, by Haldane [12].

In this paper, we study the stability and instability of the LFL by using the same two methods. We propose to combine them to clarify the picture of Fermi liquids. We implement the bosonization technique in a manner elucidating its standing in terms of the fermionic perturbation expansion and rendering the calculation of the electron propagator at large distance scales quite transparent. Our techniques apply to a broad class of two-body interactions. (We presented a preliminary account of our results at the 1994 Les Houches summer school "Fluctuating Geometries in Statistical Mechanics and Field Theories", [17])

Next, we summarize the contents of the various sections of this paper.

In Sect.2, we review the RG method for non-relativistic electron systems at finite density and zero temperature in a $d+1$-dimensional, euclidean space-time, with $d > 1$. The underlying perturbation theory is organized as a double expansion in an inverse scale parameter λ^{-1} and in the (dimensionless) running coupling constant, g_λ, of the two-body interactions.

The inverse scale parameter λ^{-1} is proportional to the width of a shell Ω_λ around the Fermi surface. Having integrated out the electron modes with momenta lying outside the shell Ω_λ, the effective action, S_λ, for the remaining modes describes the physical properties of the electron system at distance scales $\sim \frac{\lambda}{k_F}$, or energy scales $\sim \frac{v_F k_F}{\lambda}$, k_F and v_F being the Fermi momentum and velocity, resp. . The running coupling constant g_λ measures the strength of the two-body interactions in S_λ.

Given S_λ, one further reduces the effective momentum space around the Fermi surface to $\Omega_{\lambda'}$, with $\lambda' = M\lambda$, $M > 1$, and calculates the corresponding effective action $S_{\lambda'}$. This calculation is organized as a perturbation expansion in the parameters g_λ and λ^{-1}, assuming that they are small. Rescaling the resulting system

by a factor M, we obtain a system that is similar to the initial one, and we can compare the relevant and marginal parameters characterizing the corresponding effective actions. The RG method consists in executing these transformations iteratively, with the aim of deriving the scaling limit of the system, as $\lambda \to \infty$.
The underlying calculations can be interpreted in a transparent way by decomposing the shell Ω_λ into $N \sim \lambda^{d-1}$ boxes $B_{\omega_i}(\lambda)$, $i = 1, ..., N$, with sides of length $\sim \frac{1}{\lambda}$. The d-dimensional unit vectors $\vec{\omega}_i$, $i = 1, ..., N$, point to the centers of the corresponding boxes. This decomposition implies a decomposition of the electron field $\Psi^\sharp$ (where $\Psi^\sharp$ stands for Ψ^* or Ψ) into N components $\psi^\sharp_{\omega_i}$, $i = 1, ..., N$, whose Fourier modes have support in the corresponding boxes. A component field $\psi^\sharp_{\omega_i}$ describes quasi $1+1$-dimensional, "relativistic" electrons moving along the direction $\vec{\omega}_i$ with velocity v_F. The decomposition of the electron field into such component fields reproduces the electron propagator to leading order in an expansion in $\frac{1}{\lambda}$. In the non-interacting system, the subsystems along the different directions are independent. Two-body interactions couple the subsystems. After rescaling, each interaction process is suppressed by a factor $\frac{1}{N}$. Hence, the interacting system can be analysed by using "large-N" expansion techniques, [8, 9, 14]. For systems with a strictly convex Fermi surface, superconductivity is shown to be the only symmetry-breaking instability that can develop in the system. We propose to study electron systems which do not undergo such a symmetry breakdown – i.e. whose Cooper channel is turned off – by means of "bosonization". The framework of the RG will serve to characterize the approximations involved by "bosonizing" the system.

In Sect.3, we introduce the bosonization technique by calculating the scaling limit of the effective gauge field action (the generating function of connected current Green functions) for a non-interacting electron system. The effective gauge field action is obtained by coupling an external gauge field to the electron system and by integrating out the fermionic degrees of freedom.
The calculation of the scaling limit of the effective gauge field action is reduced to the calculation of the gauge field action of a family of independent Schwinger models, because the d-dimensional, non-interacting electron system decomposes – to leading order in an expansion in the inverse scale parameter λ^{-1} – into independent subsystems of quasi $1+1$ dimensional, "relativistic" fermions, one along each direction $[\omega] = \{\vec{\omega}, -\vec{\omega}\}$.
Gauge invariance must hold for each subsystem – i.e., in each direction – separately, and implies local conservation laws for the associated quasi $1+1$ dimen-

sional current densities $j^A_{[\omega]}$, $A = 0, 1$. These conservation laws imply that each current density $j^A_{[\omega]}$ can be expressed as a derivative of a bosonic field $\varphi_{[\omega]}$. It turns out that, for the non-interacting system, these fields are massless and gaussian, propagating only along the direction $[\omega]$.
By taking the effective action S_λ determined in the context of an RG analysis as an input, we can apply the bosonization technique to interacting systems, too. For systems whose Cooper channel is turned off, the electron-electron interactions are described – to leading order in the inverse scale parameter λ^{-1} – by an expression quadratic in the currents $j^A_{[\omega]}$. By replacing the fermionic currents $j^A_{[\omega]}$ by the corresponding bosonized expressions, we obtain a gaussian bosonic theory. Because the theory is gaussian, it can be studied more easily than the original fermionic theory where the interaction term is quartic in the electron fields.
In the calculation of the effective gauge field action, this technique is shown to reproduce the leading order of a (fermionic) perturbation expansion in powers of λ^{-1}. The leading contributions coincide with the ones of an RPA approximation. We determine the scaling limit of the effective gauge field action for systems with short range interactions and discuss the extension of the bosonization method to systems with long range interactions in three examples : systems with longitudinal density-density or transversal current-current interactions, and "Tomographic Luttinger Liquids".

In Sect.4, we apply the bosonization technique to the calculation of the electron propagator at large distance and time scales. We start from an effective action S_λ at energy scales $\sim \frac{v_F k_F}{\lambda}$ and the corresponding decomposition of the electron fields $\Psi^\sharp$ into N quasi $1+1$ dimensional components $\psi^\sharp_\omega$. We replace the action S_λ by its bosonized version, introduced in Sect.3, and , for each ray $[\omega] = \{\vec{\omega}, -\vec{\omega}\}$, we express the pair $\psi^\sharp_\omega, \psi^\sharp_{-\omega}$ in terms of a bose field $\varphi_{[\omega]}$. This is accomplished by applying the well-known bosonization formalism for $1+1$ dimensional relativistic fermions . However, one has to cope with a subtlety arising from the dependence of the quasi $1+1$ dimensional electron fields $\psi^\sharp_\omega$ on the components of the momentum perpendicular to the direction $[\omega]$. Again, the approximations involved can be characterized in the context of a (formal) perturbation expansion in powers of the inverse scale parameter λ^{-1}. Special care is taken to discuss the implications of linearizing the Fermi surface inside the boxes $B_\omega(\lambda)$.
We verify that the electron propagator of a system with short-range interactions tends, for large arguments, to the standard Landau Liquid form. Because of

screening – which is reproduced by bosonization – the same result holds for systems with long-range density-density interactions.
For systems of electrons interacting via long-range, transverse current-current interactions, we find the possibility for a deviation from LFL behaviour, depending on the exponent α which characterizes the singularity of the interaction potential in momentum space $\hat{V}(\vec{p}) \sim \frac{1}{|\vec{p}|^\alpha}$. The critical value for α is $d-1$. For $0 < \alpha < d-1$ and $\lambda \to \infty$, the electron propagator has the standard LFL form, whereas, for $d-1 \leq \alpha \leq 2$ and $\lambda \to \infty$,we argue that it shows MFL behaviour. In the second case, in order to obtain a theory which is form-invariant under scale transformations, the parameters of the effective action S_λ, used as an input of bosonization, must be functions of the scale parameter λ. A resulting consistency condition determines the expected flow of the parameters of S_λ under RG transformations.
Our analysis yields a view somewhat complementary to those arrived at in [3, 4, 5, 6].

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2 Effective Action on Large Scales and Fermionic Perturbation Theory

The action of a system of non-interacting, non-relativistic electrons in $d+1$ space-time dimensions is given by

$$S^0(\Psi^*, \Psi; \mu) = \int dt\, d^d x \left[\Psi^*(\vec{x}, t)\,(i\partial_0 + \mu)\,\Psi(\vec{x}, t) - \frac{1}{2m}\sum_{\ell=1}^{d} (\partial_\ell \Psi(\vec{x}, t))^*\, \partial_\ell \Psi(\vec{x}, t) \right] , \quad (1)$$

where m denotes the bare electron mass and the chemical potential, μ, specifies the mean electron density n^0_μ. We choose units such that $\hbar = 1$.
In a functional integral quantization, the electrons are described by two indepen-

dent, two-component Grassmann fields $\Psi(\vec{x},t) = \begin{pmatrix} \psi_\uparrow(\vec{x},t) \\ \psi_\downarrow(\vec{x},t) \end{pmatrix}$ and $\Psi^*(\vec{x},t) = \begin{pmatrix} \psi^*_\uparrow(\vec{x},t) & \psi^*_\downarrow(\vec{x},t) \end{pmatrix}$. The arrows $\uparrow,\downarrow$ stand for "spin up" and "spin down".

An ionic background and interactions between electrons are described by additional terms in the action of the form

$$S^U \quad = \quad -\int dt\, d^dx\, U(x)\Psi^*(x)\Psi(x)$$

and

$$S^V \quad = \quad -\frac{1}{2}\int dt\, d^dx \int ds\, d^dy\, \Psi^*(x)\Psi(x)V(\vec{x}-\vec{y})\, \delta(t-s)\, \Psi^*(y)\Psi(y) \quad ,$$

resp., where U is a one-body potential and V a two-body potential. The total action is given by

$$S \;=\; S^0 \;+\; S^U \;+\; S^V \quad . \tag{2}$$

Expectation values of functionals, $\mathcal{F}(\Psi^*,\Psi)$, of the Ψ and Ψ^* fields are calculated by evaluating Berezin-Grassmann integrals :

$$\langle\, \mathcal{F}(\Psi^*,\Psi)\, \rangle_\mu \;=\; \frac{\int \mathcal{D}(\Psi^*,\Psi)\; e^{iS(\Psi^*,\Psi\,;\mu)}\; \mathcal{F}(\Psi^*,\Psi)}{\int \mathcal{D}(\Psi^*,\Psi)\; e^{iS(\Psi^*,\Psi;\mu)}} \quad . \tag{3}$$

We work in the Grand Canonical Ensemble, where the chemical potential μ is held fixed. This implies that, in general, the mean electron density n_μ of the interacting system does not coincide with the mean electron density n^0_μ of the free system.

In order to calculate expectation values of the form in eq.(3), the total action S is split into a quadratic part, $S^2 = S^0 + S^U$, and a quartic part S^V. One expands the exponential e^{iS^V} in a power series and calculates the expectation values of the resulting polynomials in Ψ^* and Ψ, with respect to the Gaussian Berezin integration determined by S^2, by using Wick's theorem.

In this paper we consider only systems invariant under translations and rotations of space, i.e., the background is described by a constant potential $U(x) \equiv U$ that can be absorbed in a redefinition of the chemical potential ("jellium model").

For technical convenience, we analyse the system in the euclidean region reached by analytic continuation in the time t to the halfplane $\Im m[t] > 0$ and setting $x_0 = it$ (Wick rotation). For a system at a finite temperature T, the x_0-variable in the euclidean action is integrated over the interval $[-\frac{\beta}{2}, \frac{\beta}{2}]$, where β is proportional to the inverse temperature T^{-1}, and anti-periodic boundary

conditions are imposed at $x_0 = \pm\frac{\beta}{2}$. In this paper, however, we only consider systems at zero temperature, ($\beta \to \infty$).
Then the covariance of the Gaussian integration – i.e., the unperturbed electron propagator – is given by

$$\begin{aligned} G^0_{\alpha,\beta}(x-y) &:= -\langle\psi_\alpha(x)\psi^*_\beta(y)\rangle^0_\mu = -\int \frac{\mathcal{D}(\Psi^*,\Psi)}{\Xi^0_\mu}\, e^{-S^0(\Psi^*,\Psi;\mu)}\, \psi_\alpha(x)\psi^*_\beta(y) \\ &= \delta_{\alpha,\beta}\int_{\mathbf{R}} đp_0 \int_{\mathbf{R}^d} đ^d p\, \frac{e^{-ip_0(x_0-y_0)+i\vec{p}(\vec{x}-\vec{y})}}{ip_0 - \varepsilon^0_\mu(|\vec{p}|)} \quad , \end{aligned} \tag{4}$$

where $\varepsilon^0_\mu(|\vec{p}|) = \frac{\vec{p}^2}{2m} - \mu$ is the energy of a free electron, and the indices α, β label the spin orientations. We use the short-hand notation $\int đp(\cdot) := \int \frac{dp}{2\pi}(\cdot)$.

In the ground state of the unperturbed system, all one-particle states with wave vectors $\vec{p}$ satisfying $|\vec{p}| \leq k_F = \frac{1}{2}\sqrt{2m\mu}$ are occupied, and the electron density is given by

$$n^0_\mu = 2\int đ^d p\, \theta(k_F - |\vec{p}|)\ . \tag{5}$$

The Fermi surface , $\mathcal{S}^{d-1}_{k_F}$, is defined as the surface of this sphere and the Fermi wave number, k_F , sets the fundamental momentum scale of the system.
For the main results and conclusions of this paper to hold, the spherical symmetry is not essential. Our analysis applies as long as the Fermi surface is strictly convex. The situation changes, however, radically for systems of electrons hopping on a square lattice, at half filling, where the Fermi surface contains flat, parallel faces giving rise to "nesting phenomena". In addition to the superconducting instability, other instabilities like charge- or spin- density-wave instabilities can occur, and their interplay can lead to rather complicated phenomena. Such systems can be analyzed by using methods similar to those described in this section, but their properties are not yet fully understood.

We are interested in universal large-scale and low-energy properties of electron systems, i.e., in the so called *scaling limit* of such systems. The scaling limit can be constructed by using renormalization group (RG) techniques which are based on successively integrating out the modes of the electron fields Ψ^*, Ψ corresponding to wave vectors far from the Fermi surface, with the aim of deriving an effective action for the modes close to the Fermi surface; (the energy vanishes for modes whose momenta lie on the Fermi surface) .
An alternative way to obtain some "non-perturbative", large-scale and low-energy

information about an electron system consists in using the so called "Luther-Haldane" (LH) bosonization technique [11, 12, 13] . However, for electron systems in more than one space dimension, LH bosonization is not an exact method, i.e., it does not exactly resum the perturbation expansion in the coupling constant, g, of the quartic electron-electron interaction . *The aim of this paper is to incorporate the bosonization technique into the systematic framework of the RG, in order to estimate the effects of the approximations involved by "bosonizing" an electron system in more than one space dimension.*

In general, RG calculations for interacting electron systems require assuming that the coupling constant g of the quartic electron-electron interaction is small. For one-dimensional systems, the calculations have been carried out a long time ago (for a review, see [7]). In more than one space dimension, the situation is more complicated. However, during the last few years, there has been substantial progress accomplished by introducing the inverse scale parameter , λ^{-1}, proportional to the width of the effective wave vector space around the Fermi surface, as a supplementary expansion parameter, [9]. In this way, the dominant contributions to the scaling limit are obtained in a natural manner .

Here we sketch a version of the RG method for electron systems in more than one space dimension based on a double expansion in λ^{-1} and the running coupling constant, g_λ, which measures the strength of electron interactions at energy scales $\sim \frac{v_F k_F}{\lambda}$; for original results, see [1, 2, 9, 10] , and for reviews, [14, 17] . (An alternative way would consist in artificially introducing S different species of electrons and to organize the perturbation expansion in powers of λ^{-1} and S^{-1} or S, for $S \to \infty$ or $S \to 0$, resp. ; see [3].)

We adopt a Wilson-type formulation of the RG and accomplish the mode reduction in momentum space. The electron fields $\Psi^*(x), \Psi(x)$ are expressed in terms of their Fourier modes

$$\hat{\Psi}(p) \;=\; \int d^{d+1}x \; e^{ipx} \, \Psi(x) \qquad\qquad \hat{\Psi}^*(p) \;=\; \int d^{d+1}x \; e^{-ipx} \, \Psi^*(x) \quad ,$$

with $px := p_0 x_0 - \vec{p}\vec{x}$.
We assume that, for a sufficiently small value of the dimensionless coupling constant g, there exists a large scale factor $\lambda_0 \gg 1$ (with $\lambda_0 \to \infty$, as $g \to 0$), such that, in a first step, the integration over the electron modes $\hat{\Psi}(p), \hat{\Psi}^*(p)$ with

momenta $\vec{p}$ outside the shell

$$\Omega_{\lambda_0} := \left\{ \vec{p} \in \mathbf{R}^d \, , \, \left| \vec{p} - k_F \frac{\vec{p}}{|\vec{p}|} \right| \leq \frac{k_F}{2\lambda_0} \right\}$$

of width $\frac{k_F}{\lambda_0}$ around the Fermi surface leads to an *effective action* that has essentially the same form as the original action S. More precisely, the effective action for the remaining modes with momenta inside the shell Ω_{λ_0} is given by

$$S^{\text{eff}}_{\lambda_0}\left(\hat{\Psi}^*(p), \hat{\Psi}(p)\right) = -\log \int_{\vec{p}' \in \mathbf{R}^d \setminus \Omega_{\lambda_0}} \mathcal{D}\left(\hat{\Psi}^*(p'), \hat{\Psi}(p')\right) e^{-\left[S^0(\hat{\Psi}^*, \hat{\Psi}; \mu) + g S^V(\hat{\Psi}^*, \hat{\Psi})\right]} + \quad cte. \quad . \tag{6}$$

We assume that

$$S^{\text{eff}}_{\lambda_0}\left(\hat{\Psi}^*(p), \hat{\Psi}(p)\right) \sim S\left(\hat{\Psi}^*(p), \hat{\Psi}(p)\right) \quad .$$

In other words, for sufficiently small coupling constants ($< \frac{1}{\lambda_0^2}$) , possible instabilities develop only at energy scales smaller than $v_F \frac{k_F}{\lambda_0}$; in [1], it has been proven rigorously that this is true for a two-dimensional system with short-range interactions.

Given $S^{\text{eff}}_{\lambda_0}$, we wish to determine the effective action, $S^{\text{eff}}_{\lambda_1}$, on a lower energy scale $\sim v_F \frac{k_F}{\lambda_1}$, where $\lambda_1 = M\lambda_0$, $M > 1$, by integrating out the modes in the shells $\Omega_{\lambda_0} \setminus \Omega_{\lambda_1}$. Functional integration leads to a perturbation expansion for $S^{\text{eff}}_{\lambda_1}$ with a fermion propagator determined by the quadratic part of $S^{\text{eff}}_{\lambda_0}$.

In order to illustrate characteristic features of the resulting perturbation theory, we determine the unperturbed electron propagator $G^0(x)$, defined in eq.(4), for large values $x = \lambda\xi$ of its arguments, where $\xi \sim \frac{1}{k_F}$ and $\lambda \to \infty$. One finds

$$G^0(\lambda\xi) = \left(\frac{k_F}{2\pi}\right)^{d-1} \int_{S_1^{d-1}} d^{d-1}\omega \, e^{ik_F\vec{\omega}\lambda\vec{\xi}} \, . \cdot \int_{\mathbf{R}} dp_0 \int_{-\frac{k_F}{2\lambda}}^{\frac{k_F}{2\lambda}} dp_{\|} \, \frac{e^{-i(p_0\lambda\xi_0 - p_{\|}\lambda\vec{\omega}\vec{\xi})}}{ip_0 - v_F p_{\|}} \left[1 + O(\frac{1}{\lambda})\right] \, , \tag{7}$$

cf. [15], where $\vec{\omega}$ denotes a $d-$dimensional unit vector ($\vec{\omega} \in S_1^{d-1}$), and the Fermi velocity $v_F = \frac{k_F}{m}$ is given by the linearization of the energy function $\varepsilon^0_\mu(k_F\vec{\omega} + \vec{p}) = v_F|\vec{p} \cdot \vec{\omega}| + O(\vec{p}^2)$ around the Fermi sphere . For large arguments ($\sim \lambda$) of the electron propagator, only momenta inside the shell Ω_λ are important.

One can subdivide the shell Ω_λ into $N = (\frac{\lambda}{k_F})^{d-1}\mathrm{Vol}(\mathcal{S}_{k_F}^{d-1})$ disjoint, congruent blocks $B_{\omega^{(i)}}(\lambda)$ with sides of length $\sim \frac{k_F}{\lambda}$, centered at the points $k_F\vec{\omega}^{(i)}$, $i = 1, ..., N$, on the Fermi surface, as indicated in Fig.1 .The endpoints of the unit vectors $\vec{\omega}^{(i)}$, $i = 1, ..., N$, form a regular grid on $\mathcal{S}_1^{d-1}$.

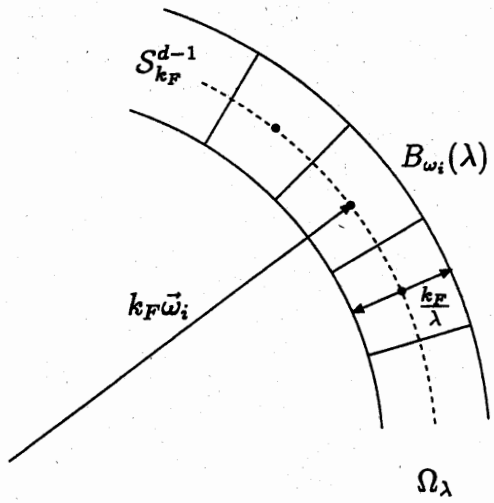


Fig.1

This discrete decomposition of the shell Ω_λ into N boxes $B_{\omega^{(i)}}(\lambda)$ yields a formula analoguous to eq.(7) for the large distance behaviour of the electron propagator :

$$G^0(\lambda\xi) = \sum_{i=1}^{N} e^{ik_F\vec{\omega}^{(i)}\lambda\vec{\xi}} \int_{\mathbf{R}} đp_0 \int_{\overline{B}_{\omega^{(i)}}(\lambda)} đ^d p \cdot \cdot \frac{e^{-i(p_0\lambda\xi_0 - \vec{p}\lambda\vec{\xi})}}{ip_0 - \varepsilon_\mu^0(k_F\vec{\omega}^{(i)} + \vec{p})} \left[1 + O(\frac{1}{\lambda})\right] \quad , \tag{8}$$

with $\overline{B}_{\omega^{(i)}}(\lambda) := \left\{\vec{q} - k_F\vec{\omega}^{(i)} \ , \ \vec{q} \in B_{\omega^{(i)}}(\lambda)\right\}$.
The energy function ε_μ^0 of the unperturbed system can be written as

$$\varepsilon_\mu^0(k_F\vec{\omega}^{(i)} + \vec{p}) = v_F p_{||} + \left[(\vec{p}_\perp)^2 + (\vec{p}_{||})^2\right] \quad , \tag{9}$$

where the momenta $\vec{p}$ are supposed to lie in the box $\overline{B}_{\omega^{(i)}}(\lambda)$, with $\vec{p}_{||} := p_{||}\,\vec{\omega}$, $p_{||} = (\vec{\omega}\cdot\vec{p})$, and $\vec{p}_\perp := \vec{p} - \vec{p}_{||}$. Compared to the first, linear term, the quadratic contributions are of higher order in $\frac{1}{\lambda}$, and one is tempted to neglect them. Neglecting the dependence on the perpendicular momenta, $\vec{p}_\perp$, amounts to replacing the piece of the Fermi surface, $\mathcal{S}_{k_F}^{d-1} \cap B_{\omega^{(i)}}(\lambda)$, in the box $B_{\omega^{(i)}}(\lambda)$

by a plane, cf. Fig 2 .

This approximation is harmless as long as $p_{\|} \gg (\frac{k_F}{\lambda})^2$. I.e., *given the decomposition of the shell Ω_λ into boxes $B_{\omega(i)}(\lambda)$, one can neglect the corrections to the linear part of the energy function ε^0_μ in the propagator, eq.(8), as long as one only integrates out modes with momenta $p_{\|} \gg (\frac{k_F}{\lambda})^2$.*

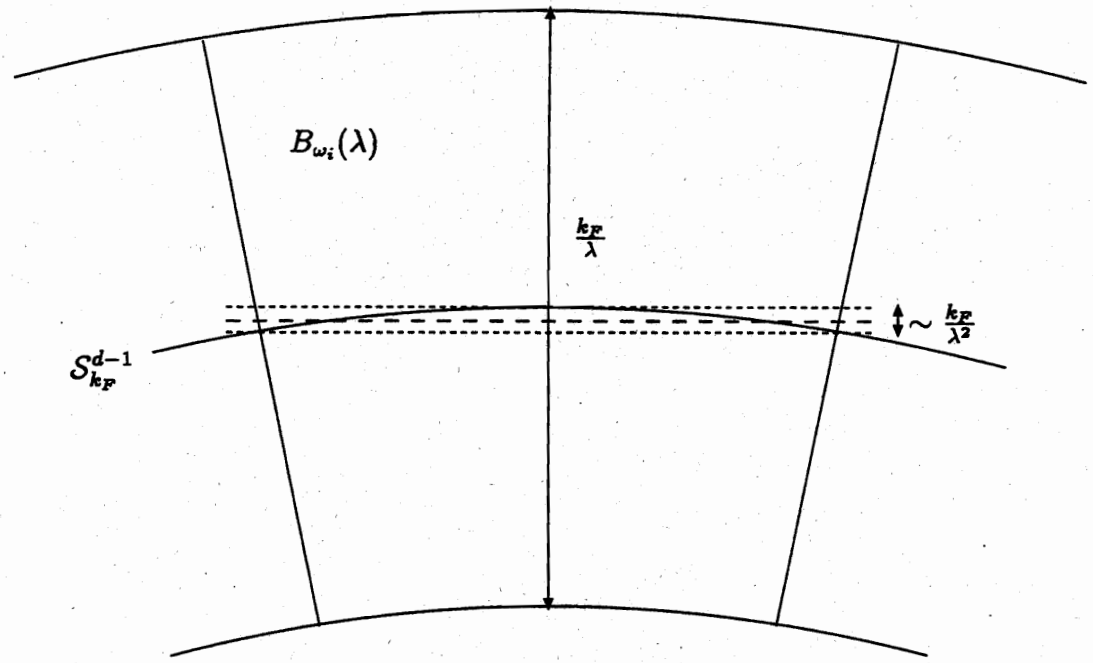


Fig.2

After having linearized the energy function ε^0_μ, eq.(8) can be reproduced by decomposing the electron fields Ψ^*, Ψ into N independent components $\psi^*_{\omega(i)}, \psi_{\omega(i)}$

$$\Psi^*(\lambda\xi) \approx \sum_{i=1}^{N} e^{-ik_F\vec{\omega}^{(i)}\lambda\vec{\xi}}\, \lambda^{-\frac{d}{2}}\, \psi^*_{\omega(i)}(\xi) \qquad \Psi(\lambda\xi) \approx \sum_{i=1}^{N} e^{ik_F\vec{\omega}^{(i)}\lambda\vec{\xi}}\, \lambda^{-\frac{d}{2}}\, \psi_{\omega(i)}(\xi) \quad , \tag{10}$$

with propagators

$$-\langle \psi_{\omega(i)}(\xi)\, \psi^*_{\omega(i)}(\eta)\rangle \;=\; \delta_{\omega(i),\omega(j)} \int_{\mathbf{R}} đk_0 \int_{\overline{B}_{\omega(i)}} đ^d k\, \frac{e^{-i(k_0(\xi_0-\eta_0)-\vec{k}(\vec{\xi}-\vec{\eta}))}}{ik_0 - v_F\vec{\omega}^{(i)}\vec{k}} \quad .$$

Here, "$\approx$" stands for "equal to leading order in $\frac{1}{\lambda}$ " and $\overline{B}_{\omega(i)} := \overline{B}_{\omega(i)}(1)$.

The Fourier modes of the component fields $\psi^\sharp_{\omega(i)}(\xi)$, i.e. $\hat{\psi}^\sharp_{\omega(i)}(k_0, \cdot)$, with $\Psi^\sharp =$

Ψ or Ψ^* , have support in the boxes $\overline{B}_{\omega(i)}$ with sides of length $\sim k_F$, and their propagators are given by

$$-\left\langle \hat{\psi}_{\omega(i)}(k_0,\vec{k})\,\hat{\psi}^*_{\omega(i)}(k'_0,\vec{k}\,')\right\rangle \;=$$

$$= \;\delta_{\omega(i),\omega(j)}\;(2\pi)^{d+1}\;\delta^{(d+1)}(k-k')\;\frac{1}{ik_0 - v_F\vec{\omega}^{(i)}\cdot\vec{k}}\;1_{\overline{B}_{\omega(i)}}(\vec{k}) \quad , \quad (11)$$

where

$$1_{\overline{B}_{\omega(i)}}(\vec{k}) \;:=\; \begin{cases} 1 \;, & \vec{k}\in\overline{B}_{\omega(i)} \\ 0 \;, & \text{else} \end{cases} \quad .$$

Note that the momenta k in eq.(11) are related to the momenta p in eq.(8) by the scale transformation $p=\frac{k}{\lambda}$. Whereas the electron fields $\Psi^\sharp(x=\lambda\xi)$ are functions of the physical coordinates (x), the component fields $\psi^\sharp_{\omega(i)}(\xi)$ are functions of the rescaled coordinates (ξ). In the rescaled system, the Fermi surface has the radius λk_F, and the boxes $\overline{B}_{\omega(i)}$ cover a shell Ω of thickness k_F around the Fermi surface. *Eq.(10) shows that, in order to describe the large scale physics of the unperturbed system, the electron fields Ψ^*,Ψ can be decomposed into $N\sim\lambda^{d-1}$ components $\psi^*_\omega,\psi_\omega$ which propagate only in the direction perpendicular to the Fermi surface.*

We now return to the interacting system. After having integrated out the modes with momenta in $\mathbf{R}^d\setminus\Omega_{\lambda_0}$, we can apply the decomposition (10) to the remaining electron modes in the effective action $S^{\text{eff}}_{\lambda_0}\left(\hat{\Psi}^*(p),\hat{\Psi}(p)\right)$, the error being of order $\frac{1}{\lambda_0}$. According to the assumptions stated above, the effective action $S^{\text{eff}}_0\equiv S^{\text{eff}}_{\lambda_0}$ at an energy scale $\sim v_F\frac{k_F}{\lambda_0}$ then has the form

$$S^{\text{eff}}_0 \;=\; S^2_0 \;+\; \delta S^2_0 \;+\; S^4_0 \;+\; \text{“ higher-order terms ”} \quad , \qquad (12)$$

with

$$S^2_0 \;\approx\; \sum_\omega\sum_{\sigma=\uparrow,\downarrow}\int_{I_\omega} đ^{d+1}k\;\frac{-1}{Z_0}\;\hat{\psi}^*_{\omega,\sigma}(k)\;\left(ik_0 - v_{F0}\vec{\omega}\vec{k}\right)\;\hat{\psi}_{\omega,\sigma}(k) \qquad (13)$$

$$\delta S^2_0 \;\approx\; \sum_\omega\sum_{\sigma=\uparrow,\downarrow}\int_{I_\omega} đ^{d+1}k\;\lambda_0\frac{\delta\mu_0}{Z_0}\;\hat{\psi}^*_{\omega,\sigma}(k)\;\hat{\psi}_{\omega,\sigma}(k) \qquad (14)$$

and

$$\begin{aligned} S^4_0 \;\approx\; & \frac{1}{2}\frac{k_F^{1-d}}{\lambda_0^{d-1}\,Z_0^2}\sum_{\omega_1,\ldots,\omega_4}\sum_{\sigma,\sigma'}\int_{I_{\omega_1}} đ^{d+1}k^{(1)}\ldots\int_{I_{\omega_4}} đ^{d+1}k^{(4)}\;\cdot \\ & \cdot\,\delta_{\omega_1+\omega_2,\omega_3+\omega_4}\;\delta^{(d+1)}\left(k^{(1)}+k^{(2)}-k^{(3)}-k^{(4)}\right)\;\cdot \\ & g_0^{\sigma,\sigma'}\left(\underline{\omega}\,;\tfrac{1}{\lambda_0}\underline{k}\right)\;\;\hat{\psi}^*_{\omega_4\sigma}(k^{(4)})\,\hat{\psi}^*_{\omega_3\sigma'}(k^{(3)})\,\hat{\psi}_{\omega_2\sigma'}(k^{(2)})\,\hat{\psi}_{\omega_1\sigma}(k^{(1)})\;, \end{aligned} \qquad (15)$$

where $\underline{\omega} := \{\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4\}$, $\underline{k} := \{k_1, k_2, k_3, k_4\}$, and I_ω stands for the integration domain $\mathbf{R} \times \overline{B}_\omega$.

For small values of $g\lambda_0^2$, the parameters $v_{F0}, Z_0, g_0^{\sigma,\sigma'}(\underline{\omega}; \frac{1}{\lambda_0}\underline{k})$ are renormalized only weakly with respect to the parameters v_F, $Z = 1$, $g\hat{V}$ in the original action. The set of dimensionless coupling functions $g_0^{\sigma,\sigma'}(\underline{\omega}; \frac{1}{\lambda_0}\underline{k})$ is related to the Fourier transform, $g\hat{V}(\frac{\vec{k}}{\lambda_0})$, of the original interaction potential by

$$k_F^{1-d} g_0^{\sigma,\sigma'}(\underline{\omega}; \tfrac{1}{\lambda_0}\underline{k}) \quad \approx \quad g_0\, \delta^{\sigma,\sigma'}\, \hat{V}\left(k_{F0}(\vec{\omega}_4 - \vec{\omega}_1) + \frac{1}{\lambda_0}(\vec{k}^{(4)} - \vec{k}^{(1)})\right) . \tag{16}$$

The Fourier transform, $\hat{V}$, of a *short-range* two-body potential V is smooth in momentum space, so that – to leading order in $\frac{1}{\lambda_0}$ – we can neglect the dependence on the small momenta. Thus, the coupling functions $g_0^{\sigma,\sigma'}(\underline{\omega}; \frac{1}{\lambda_0}\underline{k})$ in eq.(15) can be replaced by a set of coupling constants $g_0^{\sigma,\sigma'}(\underline{\omega})$.
For a *long-range* potential, whose Fourier transform $\hat{V}(p)$ is singular in momentum space, i.e., $\hat{V}(p) \sim \frac{1}{|p|^\alpha}$, with $\alpha > 0$, we set

$$k_F^{1-d} g_0^{\sigma,\sigma'}(\underline{\omega}; \tfrac{1}{\lambda_0}\underline{k}) \approx \begin{cases} g_0\, \delta^{\sigma,\sigma'}\, \hat{V}\left(k_{F0}(\vec{\omega}_4 - \vec{\omega}_1)\right) & , \quad \text{for } \vec{\omega}_4 \neq \vec{\omega}_1 , \\ g_0\, \delta^{\sigma,\sigma'}\, \hat{V}\left(\frac{1}{\lambda_0}(\vec{k}^{(4)} - \vec{k}^{(1)})\right) & , \quad \text{for } \vec{\omega}_4 = \vec{\omega}_1 . \end{cases} \tag{17}$$

We first restrict our analysis to short-range potentials; comments about long-range potentials will be made at the end of this section.

In general, a quadratic term δS_0^2 of the form given in eq.(14) is generated. It displaces the origin of the energy spectrum – i.e., the Fermi wave number – to $\lambda_0 k_{F0} \approx \lambda_0(k_F - \frac{\delta\mu_0}{v_{F0}})$. Under the condition that $\lambda_0 \frac{\delta\mu_0}{v_{F0}} \ll k_F$, we can absorb this term in a change of the parallel momentum , $k_{\|} \to k'_{\|} = k_{\|} + \lambda_0 \frac{\delta\mu_0}{v_{F0}}$, obtaining

$$\overline{S}_0^2 := S_0^2 + \delta S_0^2 = \sum_\omega \sum_{\sigma=\uparrow,\downarrow} \int_{I'_\omega} d^{d+1}k' \, \frac{-1}{Z_0}\, \hat{\psi}^*_{\omega,\sigma}(k') \left(ik'_0 - v_{F0}\vec{\omega}\vec{k}'\right) \hat{\psi}_{\omega,\sigma}(k') \quad . \tag{18}$$

Thus the propagators obtained from $\overline{S}_0^2$ are equal to the propagators determined by S_0^2 , except for a small displacement of their support, $\overline{B}_\omega \to \overline{B}'_\omega$.

By "higher-order terms", in eq.(12), we mean contributions corresponding to higher orders in the Taylor expansion of the coefficient functions of the quadratic and quartic terms in the momentum variables, or contributions involving more

than 4 electron fields. Engineering scaling suggests that both types of contributions are irrelevant, but we shall analyze these terms more carefully later in this section.

As usual, one divides the terms in the action into relevant, marginal and irrelevant ones, in accordance with their scaling dimension. The exponent , ν , characterizing the (leading) behaviour of an expression $\mathcal{F}\left(\hat{\Psi}^*(k), \hat{\Psi}(k), k\right)$ under scale transformations, i.e. ,

$$\mathcal{F}\left(\hat{\Psi}^*_{(\lambda)}(\tfrac{k}{\lambda}), \hat{\Psi}_{(\lambda)}(\tfrac{k}{\lambda})\,;\, \tfrac{k}{\lambda}\right) \quad \approx \quad \lambda^{\nu}\, \mathcal{F}\left(\hat{\Psi}^*(k), \hat{\Psi}(k)\,;\, k\right) \quad , \text{ for } \lambda \to \infty \,,$$

is called the scaling dimension of $\mathcal{F}$. The scaling dimension of the electron modes is $\frac{d}{2}+1$, i.e.,

$$\hat{\Psi}^{\sharp}_{(\lambda)}(\tfrac{k}{\lambda}) \;=\; \lambda^{\frac{d}{2}+1}\; \hat{\Psi}^{\sharp}(k) \quad . \tag{19}$$

This scaling dimension is fixed by the requirement that the scaling dimension of the quadratic action S_0^2 is zero.
Terms with scaling dimension greater than / equal to 0 are called relevant / marginal, and terms with negative scaling dimension are called irrelevant, as they are suppressed by inverse powers of λ , as $\lambda \to \infty$.
The two terms of the quadratic action S_0^2 , given by eq.(13), are marginal. Contributions to S_0^2 which arise from higher orders in the Taylor expansion of the energy function, ε , in the momentum variables are not displayed, as they are irrelevant. The quadratic term δS_0^2 , cf. eq.(14) , which causes a displacement of the Fermi wave number has scaling dimension 1 , i.e. , it is relevant. By holding the chemical potential μ fixed, the average electron density of the system – related to the Fermi wave number – changes during the RG iterations, and this requires a continual readaptation of the linearization point of the energy function.
The quartic interaction , eq.(15) , has scaling dimension $1-d$, i.e. , it is irrelevant in dimension $d \geq 2$. However, there are of order $N \sim \lambda_0^{d-1}$ different interaction terms, cf. eqs. (15) and (16). It can happen that – for special exterior momentum configurations – the sum over the different interaction terms compensates the scaling factor $\frac{1}{\lambda_0^{d-1}}$ (as discussed below) . Moreover, engineering scaling arguments are only valid as long as the dimensionless running coupling constants remain small during the RG iterations. If an instability is developing, in the sense that some couplings diverge as λ grows, our weak coupling analysis breaks down. The RG method will permit us to identify those processes that lead to instabilities.

Corrections due to higher orders in the Taylor expansion of the interaction potential around the points $k_{F0}(\vec{\omega}^{(4)} - \vec{\omega}^{(1)})$ are sub-leading, cf. eq.(16) . (Of course, this is only true for short-range interactions).
Local terms in the action involving 2ℓ electron fields ($\ell > 2$) have scaling dimension $-\ell d + (d+1)$; they are irrelevant and can be neglected. However, *non-local* terms of the type shown in Fig. 3a (straight lines stand for electron propagators G_ω and wiggly lines for local 4-vertices) can appear which have scaling dimension $(\ell - 1)(1 - d)$. These are tree-level contributions constructed exclusively out of $\ell - 1$ local 4-vertices; (the non-locality arises from the inner propagator lines). Although they have a negative scaling dimension, they can contribute in leading order to the renormalization of some special, lower order vertices. This happens in cases where momentum conservation allows one to sum over the $\omega-$labels of contracted pairs of incoming and outgoing propagator lines, leading to a contribution of $O(\lambda_0^{d-1})$ per contracted pair (cf. Fig. 3b) .

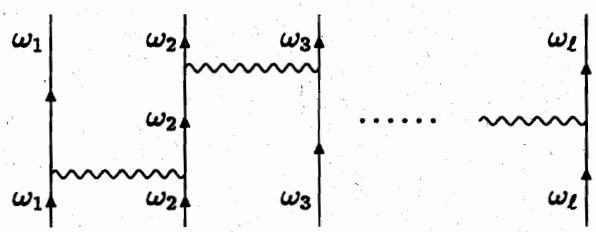


Fig.3a

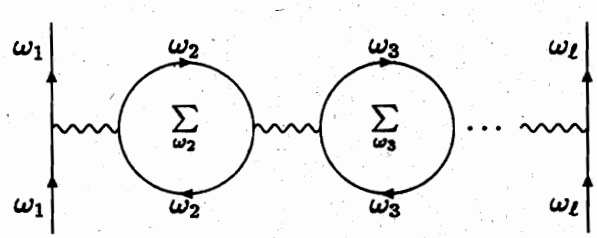


Fig.3b

The influence of such higher order vertices to the RG flow of the parameters $v_F, Z, \delta\mu, g(\underline{\omega})$ is analyzed in [16] ; they contribute in leading order in the inverse scale parameter λ, but do not cause any *qualitative* change of the flow. *For the*

sake of simplicity, we shall ignore them in the following, because we assume that g is small. (However, in the RG teatment of systems with long-range interactions, they play an important role in obtaining the correct form of screening) .

Given S_0^{eff} , we attempt to calculate the effective action S_1^{eff} at a larger scale $\lambda_1 = M\lambda_0 > \lambda_0$. Assuming that the dimensionless quartic coupling constants $g_0(\underline{\omega})$ are small, i.e. $|g_0(\underline{\omega})| \leq g$, we organise the perturbation theory in powers of g . The appearance of the small factor $\frac{1}{\lambda_0^{d-1}}$ in front of the quartic interaction term, eq.(15), suggests to introduce the inverse scale parameter $\frac{1}{\lambda_0}$ as a supplementary expansion parameter. This is also the parameter that controls the approximation of replacing the electron fields Ψ^*, Ψ by the component fields $\psi_\omega^*, \psi_\omega$, cf. eq.(10) . In order to obtain the effective action, S_n^{eff} , on successively larger scales $\lambda_n = M^n\lambda_0$, $n \to \infty$, we shall execute this procedure iteratively. This leads to the following formulation of the RG.

In the iteration step $n \to n+1$, we lower the energy scale in the effective action S_n^{eff} by a factor M which we choose to be an integer greater than 1 . The action S_n^{eff} is supposed to have the form specified in eqs. (12) - (15), with the scale parameter λ_0 replaced by $\lambda_n = M^n\lambda_0$, and the parameters $Z_0, v_{F0}, \delta\mu_0, g_0(\underline{\omega})$ replaced by $Z_n, v_{Fn}, \delta\mu_n, g_n(\underline{\omega})$. Then, the symbol "$\approx$" stands for "equal to leading order in $\frac{1}{\lambda_n}$" . There are $N_n = (\frac{\lambda_n}{k_F})^{d-1}\text{Vol}(\mathcal{S}_n)$ boxes $B_\omega^{(n)}$ which cover the shell Ω_n of standard width k_F around the spherical surface $\mathcal{S}_n$ of radius $\lambda_n k_{Fn-1} = \lambda_n(k_F - \sum_{i=0}^{n-1}\frac{\delta\mu_i}{v_{Fi}})$.
The iteration can be continued as long as the running coupling constants $g_n(\underline{\omega})$ remain small, i.e., $|g_n(\underline{\omega})| \leq g \ll 1$ and the displacement, $\lambda_n\frac{\delta\mu_n}{v_{Fn}}$, of the origin of the energy function from $\lambda_{n-1}k_{Fn-1}$ to $\lambda_n k_{Fn} := \lambda_n(k_{Fn-1} - \frac{\delta\mu_n}{v_{Fn}})$ is small compared to $\frac{k_F}{M}$, cf. Fig.4 . As in step 0, cf. eq.(18), this allows us to absorb the displacement of the chemical potential – described by δS_n^2 – in the term $\overline{S}_n^2$ by shifting the variables $k_{\|} \to k'_{\|} = k_{\|} + \lambda_n\frac{\delta\mu_n}{v_{Fn}}$. This yields a displacement of the Fermi wave number from $\lambda_n k_{Fn-1}$ to $\lambda_n k_{Fn}$ $(\approx \lambda_n k_{Fn-1})$.
We denote the shell of width $\frac{k_F}{M}$ around the spherical surface $\tilde{\mathcal{S}}_n$ of radius $\lambda_n k_{Fn}$ by $\tilde{\Omega}_n$. In the step $n \to n+1$, we eliminate modes with momenta $\vec{k}$ lying in $\Omega_n \setminus \tilde{\Omega}_n$. This yields an effective action

$$S_{n+1}^{\text{eff}}\left(\{\hat{\psi}_\omega^{\|}(k)\}\right) \approx$$

$$\approx -\log \int_{(k'_0,\vec{k}')\in\mathbf{R}\times\Omega_n\setminus\tilde{\Omega}_n} \mathcal{D}\{\hat{\psi}_\omega^{\|}(k')\}\, e^{-\left[S_n^{\text{eff}}(\{\hat{\psi}_\omega^{\|}(k')\})\right]} + cte. \quad , \qquad (20)$$

where $\{\psi_\omega^\sharp\}$ denotes the set of component fields with $\psi_\omega^\sharp = \psi_\omega^*$ or ψ_ω . In order to compare S_{n+1}^{eff} to S_n^{eff} , we subdivide each box $B_{\omega_i}^{(n)} \cap \tilde{\Omega}_n$ into M congruent boxes, $\tilde{B}_{\omega_{i,\ell}}^{(n)}$, $\ell = 1, ..., M$, with sides of length $\sim \frac{k_F}{M}$ (cf. Fig 4) .

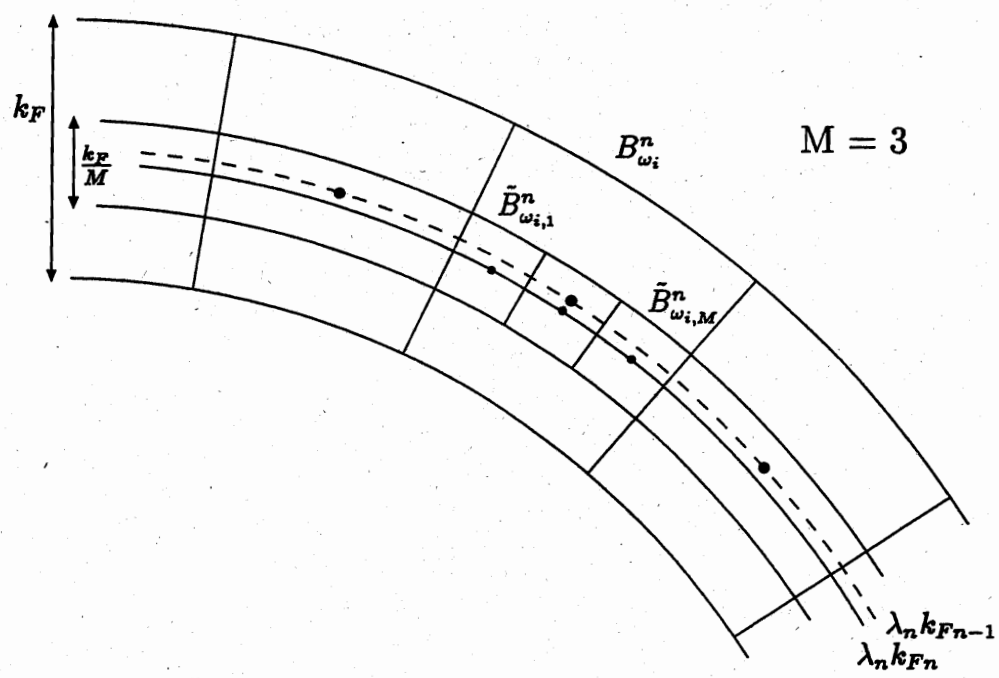


Fig.4

The resulting number of boxes is $N_{n+1} = M^{d-1} N_n$. This requires a refinement of the decomposition (10) of the electron field. *Note that, in this way, we implicitly take into account the curvature of the Fermi surface.* Using eq.(19), we then rescale the momentum variables, $\vec{k} \to \vec{k}' = M\vec{k}$, with $\vec{k}' \in B_{\omega_{i,\ell}}^{(n+1)}$, where the boxes $B_{\omega_{i,\ell}}^{(n+1)}$ have standard size.

We end up with an effective action S_{n+1}^{eff} of the same form as S_n^{eff}. The new effective parameters $Z_{n+1}, v_{Fn+1}, \delta\mu_{n+1}, g_{n+1}(\underline{\omega})$ can be expressed as functions of the set of parameters $\mathcal{P}_n := \{\delta\mu_n, v_{Fn}, g_n(\underline{\omega}), \lambda_n\}$:

$$\begin{aligned}
\lambda_{n+1} \frac{\delta\mu_{n+1}}{Z_{n+1}} &\approx \frac{M}{Z_n} \Sigma_\omega(k; \mathcal{P}_n)|_{k=0} \\
\frac{1}{Z_{n+1}} &\approx \frac{1}{Z_n} \left[1 + i \frac{\partial}{\partial k_0} \Sigma_\omega(k; \mathcal{P}_n)|_{k=0} \right] \\
\frac{v_{Fn+1}}{Z_{n+1}} &\approx \frac{1}{Z_n} \left[v_{Fn} + \frac{\partial}{\partial k_\parallel} \Sigma_\omega(k; \mathcal{P}_n)|_{k=0} \right]
\end{aligned} \tag{21}$$

and

$$k_F^{1-d}\frac{g_{n+1}(\underline{\omega})}{Z_{n+1}^2} \approx \frac{1}{Z_n^2}\,\Gamma(\underline{\omega},\underline{k};\mathcal{P}_n)|_{\underline{k}=0} \quad . \tag{22}$$

The functionals Σ_ω and Γ, which appear in these flow equations, turn out not to depend on the iteration step (to leading order in $\frac{1}{\lambda_n}$).

The functional Σ_ω is the self-energy and is obtained from the amputated one-particle irreducible (1PI) connected graphs renormalizing the propagator line $G_{\omega,n}$, and $\Gamma(\underline{\omega})$, the 4-vertex function, contains all 1PI connected graphs that renormalize the dimensionless coupling constant $g_n(\underline{\omega})$. We restrict our attention to determining the functionals Σ_ω and $\Gamma(\underline{\omega})$ to leading order in a double expansion in g and $\frac{1}{\lambda_n}$.

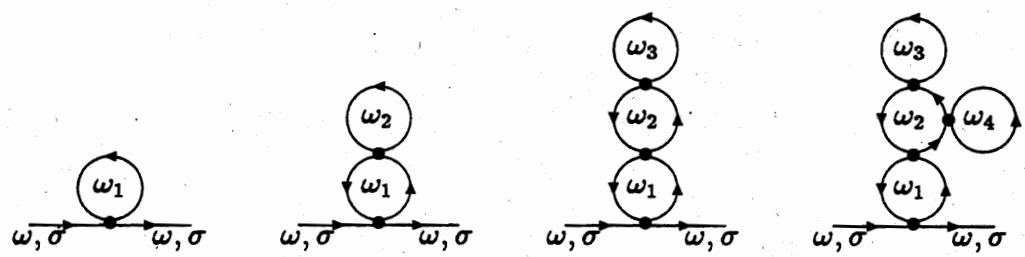
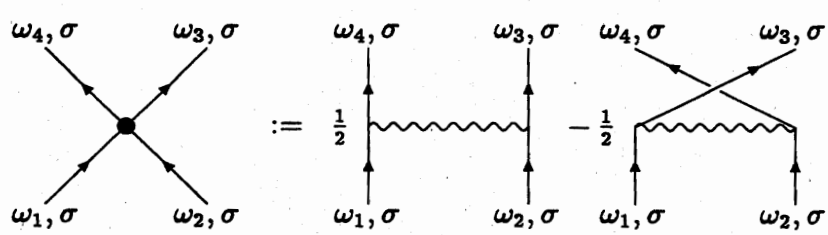


Fig.5

In Fig.5 , we display diagrams contributing to the self energy Σ_ω in leading order in $\frac{1}{\lambda_n}$ ("cactus diagrams") . To leading order in g , only the first one is retained. Straight lines stand for electron propagators $G_{\omega,n}$ given by

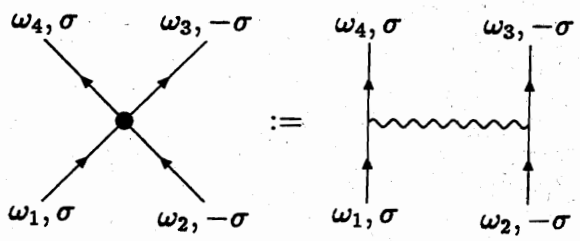
$$G_{\omega,n}(k) \quad = \quad \frac{1}{ik_0 - v_{Fn}k_{\|}} \cdot 1_{\overline{B}_\omega^{(n)}}(\vec{k}) \quad , \tag{23}$$

and dots stand for the following combinations of interaction vertices $-k_F^{1-d}\frac{g_n(\underline{\omega})}{\lambda_n^{d-1}}$, represented by wiggly lines :



, for parallel spins,

and



, for opposite spins,

where momentum conservation requires that $\vec{\omega}_1 + \vec{\omega}_2 = \vec{\omega}_3 + \vec{\omega}_4$.
In Fig.5, a summation, $\sum_{\omega}(\cdot)$, over $\vec{\omega} \in \mathcal{S}^{d-1}$ is associated with every electron loop and compensates the factor $\frac{1}{\lambda_n^{d-1}}$ per interaction vertex (for graphical reasons, the spin indices are omitted in the loops; we sum over all compatible spin configurations). One verifies easily that – for the case of short-range interactions – the contribution of the first diagram in Fig. 5 is a *constant* of order g . Using eq.(21), this implies that, to leading order, the parameters of the quadratic part of the action do not flow, i.e. ,

$$\begin{aligned} Z_{n+1} &\approx Z_n \approx Z_0 \\ v_{Fn+1} &\approx v_{Fn} \approx v_{F0} \\ \lambda_{n+1}\,\delta\mu_{n+1} &\approx O(g) \quad . \end{aligned} \tag{24}$$

In order to analyze the flow of the quartic coupling constants $g_n^{\sigma,\sigma'}(\underline{\omega})$, we classify them in terms of qualitatively different channels. Besides the spin indices, σ, σ' , they depend on the 4 discrete momenta $\underline{\omega} = \{\vec{\omega}_1, ..., \vec{\omega}_4\}$. At the nth iteration step, each of these unit vectors can take $N_n \sim \lambda_n^{d-1}$ different values $\vec{\omega}^{(i)}$, $i = 1, ..., N_n$. However, the four momenta $\underline{\omega}$ are not all independent, but must satisfy the momentum conservation $\vec{\omega}_1 + \vec{\omega}_2 = \vec{\omega}_3 + \vec{\omega}_4$, as required by translation invariance. Studying the geometry of momentum conservation, we can subdivide the set of coupling constants into three different channels. There is a qualitative difference between two and more than two dimensions.

In $d = 2$, cf. Fig. 6a, given the two incoming momenta $\vec{\omega}_1, \vec{\omega}_2$, with $\vec{\omega}_1 + \vec{\omega}_2 \neq \vec{0}$, there are exactly 2 possibilities to choose the outgoing momenta, either $\vec{\omega}_3 = \vec{\omega}_2$ and $\vec{\omega}_4 = \vec{\omega}_1$ or $\vec{\omega}_3 = \vec{\omega}_1$ and $\vec{\omega}_4 = \vec{\omega}_2$. In the exceptional case where $\vec{\omega}_1 + \vec{\omega}_2 = \vec{0}$, one is free to choose arbitrarily one of the $\lambda_n \sim N_n$ discrete values for $\vec{\omega}_4$; $\vec{\omega}_3$ is then determined as its antipode, $\vec{\omega}_3 = -\vec{\omega}_4$ (i.e. , in this case, not $\vec{\omega}_1$ and $\vec{\omega}_2$ are independent momenta, but $\vec{\omega}_1$ and $\vec{\omega}_4$) .

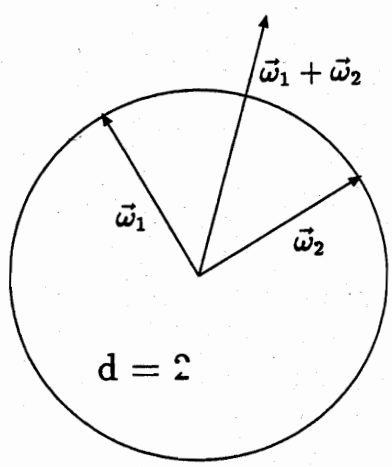


Fig.6a

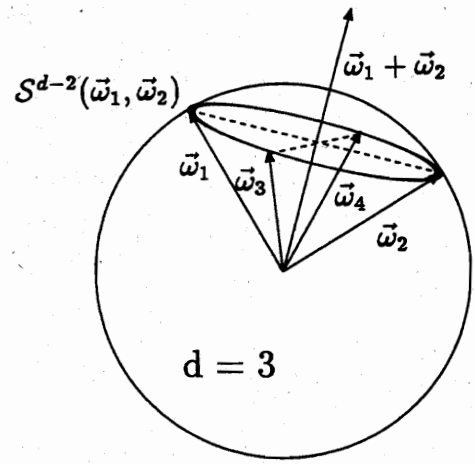


Fig.6b

In higher dimensions (cf. Fig. 6b for d=3), given the two incoming momenta $\vec{\omega}_1, \vec{\omega}_2$, with $\vec{\omega}_1 + \vec{\omega}_2 \neq \vec{0}$, one has $O(\lambda_n^{d-2})$ choices for the outgoing momenta. One of the outgoing momenta, e.g. , $\vec{\omega}_3$ can be chosen arbitrarily on the $d-2$ dimensional sphere $\mathcal{S}^{d-2}(\vec{\omega}_1, \vec{\omega}_2)$, $\vec{\omega}_4$ being determined as its antipode on $\mathcal{S}^{d-2}(\vec{\omega}_1, \vec{\omega}_2)$. Again, in the exceptional case, where $\vec{\omega}_1 + \vec{\omega}_2 = \vec{0}$, there are $O(\lambda_n^{d-1})$ choices, as in 2 dimensions (i.e., the number of choices of outgoing momenta is larger than for a generic configuration by a factor of $\sim \lambda_n$) .

Thus, in two dimensions, the couplings can be classified as follows :

$$\begin{aligned} g^d(\vec{\omega}_1 \cdot \vec{\omega}_2, \sigma \cdot \sigma') &:= g^{\sigma\sigma'}(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3 = \vec{\omega}_2, \vec{\omega}_4 = \vec{\omega}_1) \\ g^e(\vec{\omega}_1 \cdot \vec{\omega}_2, \sigma \cdot \sigma') &:= g^{\sigma\sigma'}(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3 = \vec{\omega}_1, \vec{\omega}_4 = \vec{\omega}_2) \\ g^c(\vec{\omega}_1 \cdot \vec{\omega}_4, \sigma \cdot \sigma') &:= g^{\sigma\sigma'}(\vec{\omega}_1, \vec{\omega}_2 = -\vec{\omega}_1, \vec{\omega}_3 = -\vec{\omega}_4, \vec{\omega}_4) \quad , \end{aligned} \tag{25}$$

cf. Fig.7 . Here, "d" stands for "direct", "e" for "exchange" and "c" for "Cooper"; we use the convention that $\sigma = \pm 1$ for "spin up" or "spin down", resp. .

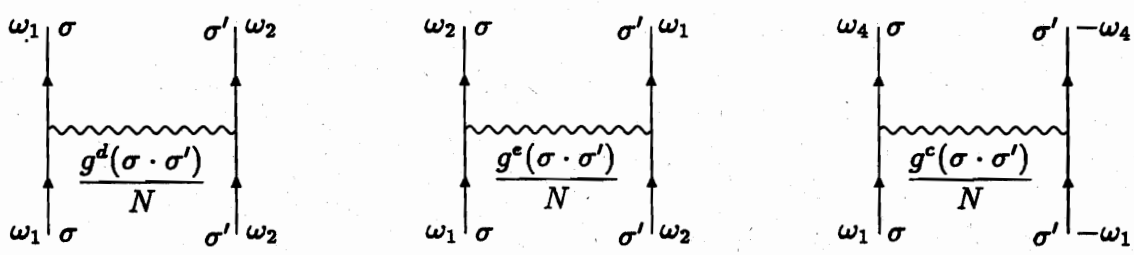


Fig.7

Note that because of rotational invariance the coupling constants, $g(\underline{\omega})$, really only depend on the scalar product, $\vec{\omega} \cdot \vec{\omega}'$, of two vectors on the unit sphere, i.e., on the angle, $\angle(\vec{\omega}, \vec{\omega}')$, between them. Analogously, they only depend on the relative orientations, $\sigma \cdot \sigma'$, (i.e. parallel or antiparallel) of the spin degrees of freedom.

For an electron system in more than 2 dimensions, momentum conservation allows more independent coupling constants than the ones listed in eq.(25). However, the analysis of their renormalization flow in leading order in $\frac{1}{\lambda_n}$ is similar to the one for two-dimensional systems. To leading order in $\frac{1}{\lambda_n}$, they neither flow nor do they influence the flow of the ones listed in eq.(25).

The contributions to the functional $\Gamma(\underline{\omega}, \mathcal{P}_n)$ of leading order in $\frac{1}{\lambda_n}$ can be determined by applying similar geometric considerations. However, now one has to study the geometry of momentum conservation for vectors lying *in thin shells around* the Fermi surface .

In Fig.8 , the contributions of leading order in $\frac{1}{\lambda_n}$ are displayed (except that self-energy contributions to the inner propagator lines are omitted). A summation, $\sum_{\omega}(\cdot)$, over $\vec{\omega} \in \mathcal{S}^{d-1}$ is associated with every electron loop (for graphical reasons, the spin indices are omitted in the loops; we sum over all compatible spin configurations).

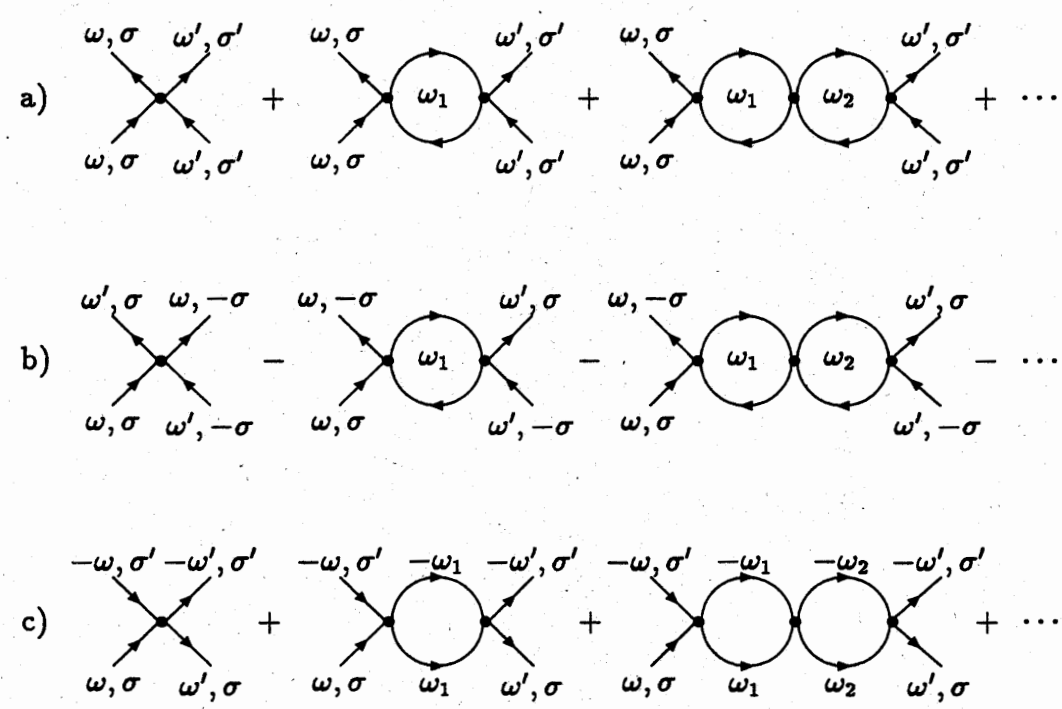


Fig.8

The renormalization of the coupling constants to leading order in g is determined – for each channel – by the first two diagrams. Diagrams a) renormalize the direct and exchange channel for electrons with parallel spins and the direct channel for electrons with opposite spins, diagrams b) renormalize the exchange channel for electrons with opposite spins and diagrams c) the Cooper channel. The contributions corresponding to diagrams a) and b) only lead to a weak renormalization of the direct and exchange channel and, solving eq.(22), one can show that – for sufficiently small initial conditions – the running coupling constants corresponding to these channels stay bounded. (This result remains true if one includes the leading order contributions of the higher vertices shown in Fig. 3 , cf [16]) .
The renormalization of the Cooper channel is – to leading order in $\frac{1}{\lambda_n}$ and omitting self energy renormalizations of inner propagator lines – independent of the direct and exchange channel :

$$g^C_{n+1}(\ell;\sigma\cdot\sigma') \approx \frac{g^C_n(\ell;\sigma\cdot\sigma')}{1+\beta g^C_n(\ell;\sigma\cdot\sigma')}\left[1+0(\frac{1}{\lambda_n})\right] \quad , \tag{26}$$

where β is a positive constant of order unity, and $\ell = 0,1,2,...$ refers to the angular decomposition of the Cooper channel : e.g., in $d=2$, the trigonometric decomposition

$$g^C_n(\vec{\omega}_1\cdot\vec{\omega}_2 = cos\vartheta) = \sum_{\ell=0}^{\infty} g^C_n(\ell)\, cos(\ell\vartheta)$$

is used (in $d=3$, the Cooper channel coupling function is expanded in a sum of Legendre functions, rather than trigonometric functions).
The solution of eq.(26) is

$$g^C_{n+1}(\ell;\sigma\cdot\sigma') \approx \frac{g^C_0(\ell;\sigma\cdot\sigma')}{1+(n+1)\beta g^C_0(\ell;\sigma\cdot\sigma')} \quad . \tag{27}$$

Suppose that one of the Cooper channel couplings, e.g. $g^C_0(\ell^*)$, is negative, $(\ell^* = 0,1,2,...)$. Then we encounter a singularity after $n^* = \frac{1}{\alpha|g^C_0(\ell^*)|}$ iteration steps, corresponding to an energy scale $v_F\frac{k_F}{\lambda_0}(\frac{1}{M})^{n^*}$. Thus, perturbation theory breaks down when $n \to n^*$. This is the celebrated superconducting instability (for a more detailed discussion, see [17]).
However, one should note that this instability only appears if the Fermi surface is symmetric under reflections at the origin, i.e., $\vec{k}\in\mathcal{S}^{d-1}_{k_F}$ implies that $-\vec{k}\in\mathcal{S}^{d-1}_{k_F}$.

If the reflected Fermi surface intersects the original Fermi surface transversally, the superconducting channel is not renormalized to leading order in $\frac{1}{\lambda_n}$.
If all initial values of the superconducting couplings are positive, i.e., $g_0^C(\ell) \geq 0$, $\ell = 0, 1, 2, \ldots$, then, following eq.(27), they all appear to flow to zero. However, in general, higher order corrections (in g and $\frac{1}{\lambda_n}$) to eq.(26) invalidate this conclusion; this is the "Kohn-Luttinger" - effect [18]. An analysis of this effect can be found in [1, 14].

In the following, we assume that the Cooper channel is turned off, as e.g. in systems of electrons coupled to magnetic impurities with broken parity- and time reversal invariance [1] . Then, for short-range interactions, all remaining coupling constants are marginal, i.e., $g_{n+1} \approx g_n$. (This result turns out to be stable against adding higher order corrections). In this case, as the coupling constants in the effective action S_n^{eff} are suppressed by a factor $\frac{1}{\lambda_n^{d-1}}$, our calculations indicate that the system flows to a Landau Liquid Fixed Point, as $n \to \infty$. In the following sections we confirm this expectation by calculating the electron propagator using *bosonization*. The effective action S_n^{eff} obtained by the RG analysis serves as an input for this calculation.

For long-range interactions, the situation is more complicated. For long-range density-density interactions of the form displayed in eq.(2), the effective action S_0^{eff} at a large, initial scale λ_0 has the form given in eqs. (12) - (15) . The effective interaction potential can be replaced by a set of coupling functions, as described in eq.(17). Due to the singularity of the interaction potential $\hat{V}(|\vec{p}|)$ at $|\vec{p}| = 0$, the interaction processes with $\vec{\omega}_4 \neq \vec{\omega}_1$ are less important than the direct scattering processes, where $\vec{\omega}_4 = \vec{\omega}_1$ and $\vec{\omega}_3 = \vec{\omega}_2$. In this channel, the singularity contributes a supplemantary factor λ_0^α to the quartic term (15) in the action which can partially or completely compensate the factor $\frac{1}{\lambda_0^{d-1}}$. However, resumming all diagrams of leading order in $\frac{1}{\lambda_0}$ leads to "screening" which renders the long-range interaction effectively short-ranged. Therefore similar results as for the short-range case are expected to hold. We shall confirm this expectation in the approximation obtained by bosonizing the system.
For long-range transverse current-current interactions, as they occur in Quantum Hall fluids at filling factors $\nu = \frac{1}{2}, \frac{1}{4}, \ldots$, the screening mechanism is ineffective. Calculating the electron propagator by the bosonization technique, we shall observe the possibility for a deviation from Landau Liquid behaviour, depending on the exponent α which characterizes the singularity of the interaction potential in

momentum space : for $\alpha \geq d-1$, we argue that the system is a MFL (similar predictions have previously been made in [3, 4, 5, 6]).

3 Effective Gauge Field Action and Bosonization

It is easiest to understand the meaning and accuracy of "bosonization" by calculating the scaling limit of the effective gauge field action, $\mathcal{W}(A)$, where A is an external electromagnetic vector potential. From the effective gauge field action one can determine the (connected) Green functions of currents by differentiating with respect to the gauge field A . Calculating the Green functions for the electron fields Ψ^*, Ψ is more complicated and is accomplished in the next section.

The external gauge field A_ρ, $\rho = 0, 1, ..., d$, is coupled to the electron system by replacing derivatives in the free action $S^0(\Psi^*, \Psi; \mu)$, eq.(1), by covariant ones $D_\rho(A) := \partial_\rho - ieA_\rho$ ("minimal coupling"). Here e is the elementary electric charge (we choose units such that $\hbar = c = 1$) . Eq. (1) is then replaced by

$$S^0(\Psi^*, \Psi; \mu, A) \;=\; S^0(\Psi^*, \Psi; \mu) \;+\; S^J(\Psi^*, \Psi; A) \quad , \tag{28}$$

where

$$S^J(\Psi^*, \Psi; A) \;=\; \int d^{d+1}x \sum_{\rho=0}^{d} A_\rho(x)\, j^\rho(\Psi^*, \Psi; A)(x) \quad ,$$

and the current density, j^ρ , is defined (in euclidean space-time) by

$$\begin{aligned} j^0(\Psi^*, \Psi)(x) \quad &= \quad -ie\Psi^*(x)\Psi(x) \\ j^\ell(\Psi^*, \Psi; A)(x) \;&=\; \tfrac{ie}{2m}\left[\Psi^*(x)D_\ell(A)\Psi(x) - (\Psi(x)D_\ell(A))^*\,\Psi(x)\right] \;, \end{aligned} \tag{29}$$

for $\ell = 1, ..., d$.

The effective gauge field action $\mathcal{W}(A)$ is obtained by integrating out the degrees of freedom of the electrons,

$$\mathcal{W}(A) \;:=\; -log\left\{(\Xi_\mu^V)^{-1}\int \mathcal{D}(\Psi^*, \Psi)\; e^{-\left[S^0(\Psi^*, \Psi; \mu, A) + S^V(\Psi^*, \Psi)\right]}\right\} \quad , \tag{30}$$

with $S^V(\Psi^*, \Psi)$ given by eq.(2) . It is the generating functional for the connected Green functions. At non-coinciding arguments, one has that

$$\prod_{i=1}^{n} \frac{\delta}{\delta A_{\rho_i}(x_i)}\, \mathcal{W}(A)\Big|_{A=0} \;=\; (-1)^n \left\langle \prod_{i=1}^{n} j^{\rho_i}(\psi^*, \psi; A; x_i)\right\rangle^{con}_{A=0} \;. \tag{31}$$

First, we consider a system of non-interacting electrons. The scaling limit, $\mathcal{W}^0_*(A)$, of the effective gauge field action $\mathcal{W}^0(A)$ has been calculated in [15]. Here, we just sketch the essential ideas and recall the main results.
We expand the effective action $\mathcal{W}^0(A)$ – eq.(30), with $S^V \equiv 0$ – in powers of the field A :

$$\mathcal{W}^0(A) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} dx_i^{d+1}\, C^{\rho_1,\ldots,\rho_n}(x_1,\ldots,x_n)\, A_{\rho_1}(x_1)\cdots A_{\rho_n}(x_n)\,. \tag{32}$$

The expansion coefficients, C , are given – at non-coinciding arguments – by the current Green functions, cf. eq.(31).
Next, we map the physical system in a space-time region $\Lambda^{(\lambda)}$ to a reference system in the region $\Lambda^{(1)} \equiv \Lambda$, where $\lambda > 1$ is a scale parameter, and μ is kept fixed. We shall be interested in the asymptotics when $\lambda \to \infty$. Under the rescaling map, points in $\Lambda^{(\lambda)}$ transform as

$$x = \lambda\xi \in \Lambda^{(\lambda)} \longmapsto \xi \in \Lambda \quad .$$

The gauge field $A^{(\lambda)}_\rho(x)$ – which probes the response of the electron system to small external electromagnetic fields – is chosen as follows :

$$A^{(\lambda)}_\rho(x) \;=\; \frac{1}{\lambda}\, a_\rho(\xi) \quad , \tag{33}$$

where $a_\rho(\xi)$ is an arbitrary, but fixed function on Λ. Thus, the gauge field scales like the momentum operator. In the following calculations we consider the formal thermodynamic limit $\Lambda \to \mathbf{R}^{d+1}$. To construct the scaling limit $\mathcal{W}^0_*(a)$ of $\mathcal{W}^0(A^{(\lambda)})$, we study the asymptotic form of the current Green functions, eq.(31), in the limit $\lambda \to \infty$, using eq.(8), and plug the result into eq.(32). We define the scaling limit $\mathcal{W}^0_*(a)$ as the *coefficient* of the most divergent term in an expansion of $\mathcal{W}^0(A^{(\lambda)})$ in powers of λ and λ^{-1} .
In the scaling limit, one encounters ultraviolet divergencies in the perturbation series of $\mathcal{W}^0_*(a)$. To fix a resulting ambiguity, we use standard Ward identities implied by gauge invariance, i.e.,

$$\frac{\partial}{\partial x_i^{\rho_i}} C^{\rho_1,\ldots,\rho_i,\ldots,\rho_n}(x_1,\ldots,x_i,\ldots,x_n) = 0 \quad .$$

The result of our analysis can be summarized as follows. In a calculation of electron Green functions at distance- and time scales of order λ, the free fermion

action $S^0(\Psi^*, \Psi; \mu)$ can be replaced by the following approximate action :

$$\begin{aligned} S^0(\Psi^*, \Psi) &\approx \sum_{\omega \in \mathcal{S}_1^{d-1}} \int_{I_\omega} đ^{d+1}k\, \hat{\psi}^*_\omega(k) \left(-ik_0 + v_F \vec{\omega}\vec{k}\right) \hat{\psi}_\omega(k) \\ &\approx \sum_{[\omega] \in \mathcal{S}^+} \int d^{d+1}\xi\, \overline{\psi}_{[\omega]}(\xi) \left(\gamma^0 \frac{\partial}{\partial \xi_0} + v_F \gamma^1 \vec{\omega} \frac{\partial}{\partial \vec{\xi}}\right) \psi_{[\omega]}(\xi) \\ &=: S^0\left(\{\psi^\sharp_\omega\}\right) \quad . \end{aligned} \tag{34}$$

The symbol "$\approx$" indicates that the approximate action reproduces the large distance- and time asymptotics of electron Green functions to leading order in $\frac{1}{\lambda}$. The sum, $\sum_{\omega \in \mathcal{S}_1^{d-1}}$, extends over the discrete set of unit vectors, $\vec{\omega}_i$, $i = 1, ..., N \sim \lambda^{d-1}$, whose endpoints lie on the surface, $\mathcal{S}_1^{d-1}$, of the $d-$dimensional unit sphere (cf. Section 2). The Fourier modes $\hat{\psi}^\sharp_\omega(k)$ have support on $I_\omega := \mathbf{R} \times \overline{B}_\omega$. In the second line of eq.(34), we introduce an (arbitrary) partitioning of $\mathcal{S}_1^{d-1}$ into a positive, $\mathcal{S}^+$, and a negative, $\mathcal{S}^-$, hemisphere and denote the ray through $\vec{\omega}$, for $\vec{\omega} \in \mathcal{S}^+$, by $[\omega] := \{\vec{\omega}, -\vec{\omega}\}$. This allows us to use the "relativistic" notation

$$\Psi_{[\omega]} := \begin{pmatrix} \psi_{-\omega} \\ \psi_\omega \end{pmatrix} \qquad \overline{\Psi}_{[\omega]} := \Psi^*_{[\omega]}\gamma^0 = \begin{pmatrix} \psi^*_\omega & \psi^*_{-\omega} \end{pmatrix} \quad , \tag{35}$$

with $\gamma^0 = \sigma_1$ and $\gamma^1 = \sigma_2$. In the following , we assume that electron spins are frozen in a fixed direction (we shall omit spin indices) .

In determining the effective gauge field action $\mathcal{W}^0_*$, the term $S^J(\Psi^*, \Psi; A)$, coupling the gauge field $A^{(\lambda)}_\rho(x) = \frac{1}{\lambda} a_\rho(\xi)$ to the electron fields, can be replaced by

$$\begin{aligned} S^J\left(\Psi^*, \Psi; A^{(\lambda)}\right) &\approx \int d^{d+1}\xi \sum_{\omega \in \mathcal{S}_1^{d-1}} \left[-ia_0(\xi) - v_F \vec{\omega}\vec{a}(\xi)\right] \cdot \\ &\qquad \cdot \sum_{\omega' \in \mathcal{S}_1^{d-1}} e^{i\lambda k_F(\vec{\omega}-\vec{\omega}')\vec{\xi}} : \psi^*_{\omega'}(\xi)\psi_\omega(\xi) : \\ &=: S^J\left(\{\psi^\sharp_\omega\}; a\right) \quad . \end{aligned} \tag{36}$$

Again, the symbol "$\approx$" indicates that, in a calculation of electron Green functions at distance- and time scales of order λ, this approximation yields the leading contribution in $\frac{1}{\lambda}$. As mentioned above, in the expansion of the effective gauge field action $\mathcal{W}^0_*$ in powers of A, we evaluate the expansion coefficients – i.e. the current Green functions – only at non-coinciding points. The omission of self contractions is indicated by the normal ordering of the product of electron fields in

eq.(36). The local terms of the expansion coefficients are determined by requiring gauge invariance.
Given a ray $[\omega]$, we define a current density

$$j_{[\omega]}(\xi) \;:=\; \sum_{\vec{\omega}'\in\mathcal{S}^+} \left\{ e^{ik_F(\vec{\omega}'-\vec{\omega})\lambda\vec{\xi}} \;:\psi^*_{-\omega'}(\xi)\psi_{-\omega}(\xi): \;+\; e^{ik_F(\vec{\omega}-\vec{\omega}')\lambda\vec{\xi}} \;:\psi^*_{-\omega}(\xi)\psi_{-\omega'}(\xi):\right\} \quad , \tag{37}$$

corresponding to electron motion in the direction of $-\vec{\omega}$, and a current density

$$\bar{j}_{[\omega]}(\xi) \;:=\; \sum_{\vec{\omega}'\in\mathcal{S}^+} \left\{ e^{-ik_F(\vec{\omega}'-\vec{\omega})\lambda\vec{\xi}} \;:\psi^*_{\omega'}(\xi)\psi_{\omega}(\xi): \;+\; e^{-ik_F(\vec{\omega}-\vec{\omega}')\lambda\vec{\xi}} \;:\psi^*_{\omega}(\xi)\psi_{\omega'}(\xi):\right\} \tag{38}$$

corresponding to electron motion in the direction of $\vec{\omega}$.
Using eq.(10), one can establish that the quasi $1+1$ dimensional current densities

$$\begin{aligned} j^0_{[\omega]}(\xi) \;&:=\; \tfrac{i}{2}\left(j_{[\omega]}(\xi)+\bar{j}_{[\omega]}(\xi)\right) \\ j^1_{[\omega]}(\xi) \;&:=\; \tfrac{1}{2}\left(j_{[\omega]}(\xi)-\bar{j}_{[\omega]}(\xi)\right) \quad , \end{aligned} \tag{39}$$

for $[\omega]\in[\mathcal{S}^{d-1}_1]$, are related to the $d+1$ dimensional current density, j^ρ, $\rho = 0,...,d$, defined in eq.(29), by

$$\begin{aligned} j^0(x) \;&:=\; j^0\left(\Psi^*,\Psi\right)(x) \;\approx\; \lambda^{-d}\sum_{\omega\in\mathcal{S}^+} j^0_{[\omega]}(\xi) \\ j^\ell(x) \;&:=\; j^\ell\left(\Psi^*,\Psi;A\right)\Big|_{A=0}(x) \;\approx\; \lambda^{-d}\sum_{\omega\in\mathcal{S}^+} v_F\,\omega^\ell j^1_{[\omega]}(\xi) \quad , \end{aligned} \tag{40}$$

where $\ell = 1,...,d$. Eq. (40) holds for current Green functions to leading order in $\frac{1}{\lambda}$, at non-coinciding arguments . For simplicity, the electron charge, $-e$, has been set to -1.
Eq. (36) can be written in the suggestive form

$$S^J\left(\Psi^*,\Psi;A^{(\lambda)}\right) \;\approx\; S^J\left(\{\psi^{\parallel}_\omega\};a\right) \;:=\; \sum_{\omega\in\mathcal{S}^+}\sum_{B=0,1}\int d^{d+1}\xi\; a^\omega_B(\xi)\; j^B_{[\omega]}(\xi) \quad , \tag{41}$$

with

$$a^\omega_0(\xi) := a_0(\xi) \quad , \quad a^\omega_1(\xi) := v_F\vec{\omega}\vec{a}(\xi) \quad .$$

One observes that – to leading order in the inverse scale parameter λ^{-1} – the original $d+1$ dimensional system decomposes into independent, quasi $1+1$ dimensional subsystems, one along each direction $[\omega]$ of $\mathbf{R}^d$. The $1+1$ dimensional

subsystems describe relativistic fermions moving along the direction $[\omega]$ *with velocity* $\pm v_F$. *Hence, the calculation of* $\mathcal{W}_*^0(a)$ *is reduced to the calculation of the effective gauge field action of a family of independent Schwinger models* .
One obtains

$$\begin{aligned}\mathcal{W}_*^0(a) &= \frac{1}{\lambda^{d-1}}\frac{1}{\pi}\left(\frac{k_F}{2\pi}\right)^{d-1}\frac{1}{v_F}\sum_{\omega\in\mathcal{S}^+}\int d^{d+1}\xi\sum_{B=0,1} a_B^{\omega,T}(\xi)a_B^{\omega,T}(\xi)\\ &= \frac{1}{2}\int_{\mathbf{R}\times\overline{B}} d^{d+1}k\sum_{\rho,\sigma=0}^{d}\hat{a}_\rho(-k)\,\Pi_*^{\rho\sigma}(k)\,\hat{a}_\sigma(k) \quad ,\end{aligned} \tag{42}$$

where we used

$$a_B^{\omega,T}(\xi) := \sum_{C=0,1}\left(\delta_{B,C}-\frac{\partial_B^\omega\partial_C^\omega}{(\partial_0^\omega)^2+(\partial_1^\omega)^2}\right)a_C^\omega(\xi)$$

and

$$\partial_0^\omega := \frac{\partial}{\partial\xi_0} \qquad \partial_1^\omega := v_F\vec{\omega}\frac{\partial}{\partial\vec{\xi}} \quad . \tag{43}$$

Note the non-trivial (and important) fact that $\mathcal{W}_*^0$ is only quadratic in the gauge field.
Invariance under gauge transformations, space rotations and parity - or time - reversal implies that the (euclidean) polarization tensor $\Pi_*^{\rho\sigma}$ has the general form

$$\begin{aligned}\Pi_*^{00}(k) &= \frac{\vec{k}^2}{k_0^2}\Pi_*^\ell(k) \qquad \Pi_*^{0i}(k) = \Pi_*^{i0}(k) = -\frac{k^i}{k_0}\Pi_*^\ell(k)\\ \Pi_*^{ij}(k) &= \Pi_*^t(k)\left(\delta^{ij}-\frac{k^ik^j}{\vec{k}^2}\right)+\Pi_*^\ell(k)\frac{k^ik^j}{\vec{k}^2} \qquad i,j=1,...,d\,.\end{aligned} \tag{44}$$

The calculation of the two independent functions Π_*^ℓ and Π_*^t, for non-interacting electrons, yields the result :

$$\begin{aligned}\Pi_*^\ell(k) &= \begin{cases}\frac{e^2}{\pi}k_Fv_F\left[\left(1+\sqrt{1+\left(\frac{v_Fk}{k_0}\right)^2}\right)\sqrt{1+\left(\frac{v_Fk}{k_0}\right)^2}\right]^{-1} & d=2\\ \frac{e^2}{\pi^2}\frac{(k_F)^2}{v_F}\frac{k_0^2}{\vec{k}^2}\left(1-\left|\frac{k_0}{v_Fk}\right|\ \mathrm{arctg}\ \left|\frac{v_Fk}{k_0}\right|\right) & d=3\end{cases}\\ \Pi_*^t(k) &= \begin{cases}\frac{e^2}{\pi}k_Fv_F\frac{|k_0|}{\sqrt{k_0^2+(v_Fk)^2}}-\Pi_*^\ell(k) = \frac{e^2}{\pi}k_Fv_F\left[1+\sqrt{1+\left(\frac{v_Fk}{k_0}\right)^2}\right]^{-1} & d=2\\ \frac{1}{2}\left(\frac{e^2}{\pi^2}(k_F)^2\left|\frac{k_0}{k}\right|\ \mathrm{arctg}\ \left|\frac{v_Fk}{k_0}\right|-\Pi_*^\ell(k)\right) & d=3\end{cases}\quad .\end{aligned} \tag{45}$$

Comparing this result to the small-k asymptotics of formulas found in standard textbooks, one finds agreement – up to the "diamagnetic term" of the transversal part which is proportional to $|\vec{k}|^2$ and yields lower order corrections in $\frac{1}{\lambda}$.

Eq.(42) is the result of resumming the leading contributions in the (formal) expansion of the effective gauge field action $\mathcal{W}^0(A)$ in powers of $\frac{1}{\lambda}$.
By power counting, the possible diagrams contributing to leading order in $\frac{1}{\lambda}$ to the effective gauge field action are the following ones :

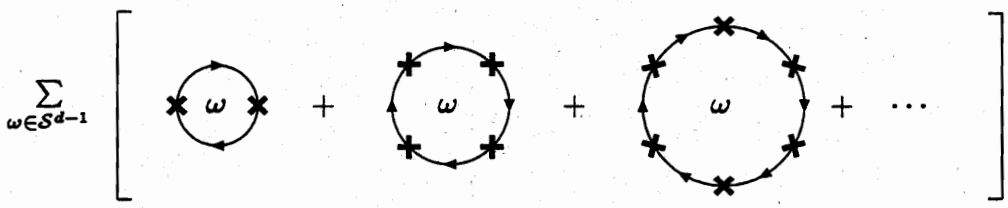


Fig.9

Straight lines stand for the electron propagators

$$G^0_\omega(k) \;=\; \frac{1}{ik_0 - v_F\vec{\omega}\vec{k}} \cdot \mathbf{1}_{\overline{B}_\omega}(\vec{k})$$

and crosses for the gauge field insertions a^ω_μ (note that, in contrast to the diagrammar introduced in the context of the RG, the electron propagators are understood to be integrated over the entire domain $I_\omega = \mathbf{R} \times \overline{B}_\omega$) .
However, eq.(42) states that, because of the special form of the propagators G^0_ω, only the first term in Fig.9 with two gauge field insertions contribute to $\mathcal{W}^0_*(a)$.
In addition, gauge invariance requires the introduction of local Schwinger terms which couple, along a ray $[\omega]$, right- and left-movers.

Remembering that, in the $1+1$ dimensional Schwinger model, the calculation of current Green functions in the fermionic theory can be reproduced by expressing the current densities in terms of a free, massless bose field (see, e.g., [15, 19], and references therein), we introduce following identifications :

$$j_{[\omega]}(\xi) \quad \longleftrightarrow \quad \frac{2}{\sqrt{\pi}}\left(\frac{k_F}{2\pi}\right)^{\frac{d-1}{2}} \partial^\omega \varphi_{[\omega]}(\xi)$$

$$\bar{\jmath}_{[\omega]}(\xi) \quad \longleftrightarrow \quad -\frac{2}{\sqrt{\pi}}\left(\frac{k_F}{2\pi}\right)^{\frac{d-1}{2}} \bar{\partial}^{\omega} \varphi_{[\omega]}(\xi) \quad , \tag{46}$$

for $\vec{\omega} \in \mathcal{S}^+$, with $[\omega] = \{\vec{\omega}, -\vec{\omega}\}$ denoting the corresponding ray, and

$$\partial^{\omega} := \frac{1}{2}\left(-i\frac{\partial}{\partial \xi_0} + v_F \vec{\omega}\frac{\partial}{\partial \vec{\xi}}\right) \qquad \bar{\partial}^{\omega} := \frac{1}{2}\left(i\frac{\partial}{\partial \xi_0} + v_F \vec{\omega}\frac{\partial}{\partial \vec{\xi}}\right) \quad .$$

Eq.(46) is equivalent to

$$j^B_{[\omega]}(\xi) \quad \longleftrightarrow \quad \frac{i}{\sqrt{\pi}}\left(\frac{k_F}{2\pi}\right)^{\frac{d-1}{2}} \varepsilon^{BC} \partial^{\omega}_C \varphi_{[\omega]}(\xi) \quad ,$$

for $B, C = 0, 1$ and $\vec{\omega} \in \mathcal{S}^+$.
The action of the set $\{\varphi_{[\omega]}(\xi)\}$ of bose fields is given by

$$S^0(\{\varphi_{[\omega]}\}) \;=\; -\frac{1}{2}\sum_{[\omega]} \int_{\mathbf{R}\times\overline{B}_\omega} \bar{d}^{d+1}k \; \hat{\varphi}_{[\omega]}(-k) \left[k_0^2 + (v_F \vec{\omega}\vec{k})^2\right] \hat{\varphi}_{[\omega]}(k) \quad , \tag{47}$$

where the Fourier modes, $\hat{\varphi}_{[\omega]}(k)$, of the bose field $\varphi_{[\omega]}(\xi)$ have support on $\mathbf{R}\times\overline{B}_\omega$. (For a given value k_0, the modes $\hat{\varphi}_{[\omega]}(k_0, \cdot)$ have a compact support. We choose this support to be $\overline{B}_\omega$, but, in principle, there is no need to choose a support identical to the one of the fermion modes $\hat{\psi}^{\sharp}_{\omega}(k_0, \cdot)$.)

One can verify that

$$\Xi_{\varphi}^{-1} \int \mathcal{D}\{\varphi_{[\omega]}\} \; e^{-S^0(\{\varphi_{[\omega]}\})} \left\{ \prod_{j=1}^{n} \left[\frac{i}{\sqrt{\pi}}\left(\frac{k_F}{2\pi}\right)^{\frac{d-1}{2}} \varepsilon^{B_j C_j} \; \partial^{\omega}_{C_j} \varphi_{[\omega_j]}(\xi_j)\right]\right\} \approx$$

$$\approx \; \Xi_{\psi}^{-1} \int \mathcal{D}\{\psi^{\sharp}_{\omega}\} \; e^{-S^0(\{\psi^{\sharp}_{\omega}\})} \left\{ \prod_{j=1}^{n} j^{B_j}_{[\omega_j]}(\xi_j) \right\} , \tag{48}$$

in the sense of distributions.

The representation (46) of the one-dimensional current densities, $j^B_{[\omega]}$, in terms of the bose fields, $\varphi_{[\omega]}$, is implied by the local conservation of the electron current, eq.(40) , along each ray $[\omega]$ seperately. These formulas are equivalent to Luther-Haldane bosonization [11, 12, 13].
The identification (46) reproduces only the leading term of the fermionic perturbation theory, i.e. :

$$\mathcal{W}^0_*(a) \;=\; \text{“}\lim_{\lambda\to\infty}\text{“} - log\left\{Z_{\varphi}^{-1} \int \mathcal{D}\{\varphi_{[\omega]}\} \; e^{-\left[S^0(\{\varphi_{[\omega]}\}) + S^J(\{\varphi_{[\omega]}\};a)\right]}\right\} \quad , \tag{49}$$

with

$$S^J(\{\varphi_{[\omega]}\};a) \;=\; \sum_{\omega\in\mathcal{S}^+}\int d^{d+1}\xi\; a^\omega_B(\xi)\,\varepsilon^{BC}\,\frac{i}{\sqrt{\pi}}(\frac{k_F}{2\pi})^{\frac{d-1}{2}}\partial^\omega_C\varphi_{[\omega]}(\xi) \quad .$$

The symbol "$\lim_{\lambda\to\infty}$" stands for determining the *coefficient* of the leading term of the expansion in λ and λ^{-1}.
The derivatives $\partial^\omega\varphi_{[\omega]}$ and $\overline{\partial}^\omega\varphi_{[\omega]}$ of the bosonic fields $\varphi_{[\omega]}$ describe the current density fluctuations, $j_{[\omega]}$ and $\overline{j}_{[\omega]}$, determining the scaling limit of the system. For a fixed direction $[\omega]$ they are composed of electron modes with momenta near the points $k_F\vec{\omega}$ and $-k_F\vec{\omega}$, resp., on the Fermi surface. In the scaling limit, they probe the Fermi surface only locally around the points $k_F\vec{\omega}$ and $-k_F\vec{\omega}$.

This implementation of bosonization is a special realization of a more general formalism, presented in [20].
Abelian gauge invariance implies the local conservation law

$$\partial_\rho j^\rho(x) \;=\; 0 \quad . \tag{50}$$

Instead of expressing the current density in terms of the elementary fields Ψ^*, Ψ and imposing the constraint (50), one can introduce new field variables which guarantee eq.(50) by construction. For $d+1$ dimensional currents, j^ρ , one needs, in general, antisymmetric gauge forms of rank $d-1$. However, considering only the scaling limit of the system, we can take advantage of a substantial simplification : the fermionic theory decomposes into a family of quasi $1+1$ dimensional subsystems ("dimensional reduction"), one along each direction $[\omega]$ of $\mathbf{R}$. Each subsystem describes quasi $1+1$ dimensional, "relativistic" electrons. Gauge inveriance has to be fullfilled for each subsystem – i.e. in each direction – seperately, and implies the conservation laws for the associated currents, $j^B_{[\omega]}$, $B=0,1$, i.e.

$$\frac{\partial}{\partial x_0}j^0_{[\omega]}(x) \;+\; v_F\vec{\omega}\frac{\partial}{\partial\vec{x}}j^1_{[\omega]}(x) \;=\; 0 \quad . \tag{51}$$

Expressing the currents $j^B_{[\omega]}$ in terms of the free, massless bose fields $\varphi_{[\omega]}$ guarantees that eq.(51) holds.

Next, we study the effective gauge field action for interacting systems.
In section 2, by using the RG group method, we successively eliminated the electron modes with momenta outside the shell $\Omega_{\lambda_n} := \left\{\,\vec{p}\in\mathbf{R}^d\;;\;\left|\vec{p}-k_F\frac{\vec{p}}{|\vec{p}|}\right| \;\le\; \frac{k_F}{2\lambda_n}\,\right\}$ of thickness $\frac{k_F}{\lambda_n} \ll k_F$ around the Fermi surface, in order to determine the effective action S_n for the remaining modes with momenta inside the shell Ω_{λ_n}. The

resulting effective action S_n describes the physics at energy scales smaller than $v_F \frac{k_F}{\lambda_n}$. Under the assumptions specified in Sect.2, one can reach very small energy scales $v_F \frac{k_F}{\lambda_n} \ll v_F k_F$ before the form of the corresponding effective action S_n differs considerably from the original action S.

In this section, we consider "spinless" electrons and suppose that, in addition, the Cooper channel is turned off. Then the terms S_n^2 and S_n^4, quadratic and quartic in the fields $\hat{\psi}_\omega^\sharp(k)$, are given by

$$S_n^2 \approx \sum_{\omega \in \mathcal{S}_1^{d-1}} \int_{I_\omega} đ^{d+1}k \, \hat{\psi}_\omega^*(k) \frac{-1}{Z_n} \left(ik_0 - v_{Fn} \vec{\omega} \vec{k} \right) \hat{\psi}_\omega(k) \tag{52}$$

and

$$\begin{aligned} S_n^4 \approx \frac{k_F^{1-d}}{2\lambda_n^{d-1}} \frac{1}{Z_n^2} \sum_{\omega_1, \omega_2 \in \mathcal{S}_1^{d-1}} \int_{I_{\omega_1}} đ^{d+1}k^{(1)} \ldots \int_{I_{\omega_1}} đ^{d+1}k^{(4)} \, . \\ \cdot \, \delta^{(d+1)} \left(k^{(1)} + k^{(2)} - k^{(3)} - k^{(4)} \right) \, g_n(\vec{\omega}_1 \vec{\omega}_2; \tfrac{k^{(1)} - k^{(4)}}{\lambda_n}) \, \cdot \\ \cdot \, \hat{\psi}_{\omega_1}^*(k^{(4)}) \, \hat{\psi}_{\omega_2}^*(k^{(3)}) \, \hat{\psi}_{\omega_2}(k^{(2)}) \, \hat{\psi}_{\omega_1}(k^{(1)}) \quad , \end{aligned} \tag{53}$$

with

$$g_n(\vec{\omega}_1 \vec{\omega}_2; \tfrac{k}{\lambda_n}) \quad = \quad \left\{ g_n^d(\vec{\omega}_1 \vec{\omega}_2; \tfrac{k}{\lambda_n}) \; - \; g_n^e(\vec{\omega}_1 \vec{\omega}_2; \tfrac{k}{\lambda_n}) \right\} \quad . \tag{54}$$

The "direct"- and "exchange"-coupling functions g^d and g^e have been defined in eq.(25). The symbol "$\approx$" stands for "equal to leading order in $\frac{1}{\lambda_n}$".

Contributions to S_n corresponding to higher orders in the Taylor expansion of the coefficient functions of the quadratic and quartic terms in the momentum variables, or contributions involving more than 4 electron fields are neglected. By engineering scaling they are irrelevant, cf. Sect.2 . For systems with a short range two-body interaction potential V, the coupling functions $g_n(\vec{\omega}_1 \vec{\omega}_2; \frac{k}{\lambda_n})$ can be replaced – to leading order in $\frac{1}{\lambda_n}$ – by the coupling constants $g_n(\vec{\omega}_1 \vec{\omega}_2)$. We first restrict our attention to such systems.

The term $S_n^J(\{\psi_\omega^\sharp\}; a)$ coupling the electron fields to the gauge field a has the form

$$S_n^J(\{\psi_\omega^\sharp\}; a) \approx \sum_{\omega \in \mathcal{S}^+} \sum_{B=0,1} \int d^{d+1}\xi \; a_B^\omega(\xi) \; j_{[\omega]}^B(\xi) \quad , \tag{55}$$

where $(a_0^\omega, a_1^\omega) := (a_0, v_{Fn} \vec{\omega} \cdot \vec{a})$, and the quasi $1+1$ dimensional currents $j_{[\omega]}^B(\xi)$ are defined as in the non-interacting system, cf. eqs. (37) - (40), but with

the products $\psi^*_{\omega'}(\xi)\psi_\omega(\xi)$ of the electron fields replaced by $\frac{1}{Z_n}\psi^*_{\omega'}(\xi)\psi_\omega(\xi)$ (in the non-interacting system $Z_n \equiv 1$). A Ward identity relates the renormalization of the vertex (55) to the renormalization of the electron propagator, preventing the appearence of a new, independent renormalization factor.
The quartic interaction term S^4_n given by eq.(53) can be expressed in terms of the quasi $1+1$ dimensional currents :

$$S^4_n\left(\{\psi^\sharp_\omega\}\right) = \frac{1}{8}\frac{k_F^{1-d}}{\lambda_n^{d-1}} \sum_{\omega_1,\omega_2\in\mathcal{S}^+} \int d^{d+1}\xi \int d^{d+1}\eta\, \delta^{d+1}_{k_F}(\xi-\eta)\cdot$$
$$\Big\{ g_n(\vec{\omega}_1\vec{\omega}_2)\left[j_{[\omega_1]}(\xi)j_{[\omega_2]}(\eta) + \bar{j}_{[\omega_1]}(\xi)\bar{j}_{[\omega_2]}(\eta)\right] -$$
$$- g_n(-\vec{\omega}_1\vec{\omega}_2)\left[j_{[\omega_1]}(\xi)\bar{j}_{[\omega_2]}(\eta) + \bar{j}_{[\omega_1]}(\xi)j_{[\omega_2]}(\eta)\right]\Big\}, \tag{56}$$

with

$$\delta^{d+1}_{k_F}(\xi-\eta) = \int_{\mathbf{R}} đk_0 \int_{|\vec{k}|\le\frac{k_F}{2}} đ^dk\; e^{ik(\xi-\eta)} \quad .$$

Replacing the quadratic term S^2_n of the action by the bosonic action $S^0(\{\varphi_{[\omega]}\})$ given in eq.(47) and inserting the bosonization identities (46) for the current densities in eqs. (55) and (56), we end up with a *gaussian* bosonic theory. The bosonized versions of eqs. (55) and (56) are given – in momentum space – by

$$S^J_n(\{\varphi_{[\omega]}\};a) = \sum_{\omega\in\mathcal{S}^+}\sum_{B,C=0,1}\int_{I_\omega} đ^{d+1}k\; \hat{a}^\omega_B(-k)\,\varepsilon^{BC}\,\frac{-1}{\sqrt{\pi}}\left(\frac{k_F}{2\pi}\right)^{\frac{d-1}{2}} k^\omega_C\varphi_{[\omega]}(k) \tag{57}$$

and

$$S^V_n\left(\{\varphi_{[\omega]}\}\right) = \frac{1}{\lambda_n^{d-1}}\left(\frac{1}{2\pi}\right)^d \sum_{\omega_1,\omega_2\in\mathcal{S}^+}\int_{I_{\omega_1}\cap I_{\omega_2}} đ^{d+1}k\,\cdot$$
$$\cdot\Big[g_n(-\vec{\omega}_1\vec{\omega}_2)\,\varphi_{[\omega_1]}(-k)\left(k^{\omega_1}\overline{k}^{\omega_2} + \overline{k}^{\omega_1}k^{\omega_2}\right)\varphi_{[\omega_2]}(k)$$
$$+ g_n(\vec{\omega}_1\vec{\omega}_2)\,\varphi_{[\omega_1]}(-k)\left(k^{\omega_1}k^{\omega_2} + \overline{k}^{\omega_1}\overline{k}^{\omega_2}\right)\varphi_{[\omega_2]}(k)\Big] \quad , \tag{58}$$

with

$$k^\omega = \frac{1}{2}(-ik_0 + v_F\vec{\omega}\vec{k}) \qquad \text{and} \qquad \overline{k}^\omega = \frac{1}{2}(ik_0 + v_F\vec{\omega}\vec{k}) \quad .$$

With the aim of calculating the scaling limit, $\mathcal{W}^V_*(a)$, of the effective gauge field action of the interacting electron system, we replace the fermionic action

S_n, given by eqs. (52) - (55), by their bosonic version, given by eqs. (47), (57) and (58), calculate the corresponding effective gauge field action, $\widetilde{\mathcal{W}}^V(a;\mathcal{P}_n)$, and determine the *coefficient*, $\widetilde{\mathcal{W}}^V_*(a;\mathcal{P}_n)$, of the leading contribution in an expansion of $\widetilde{\mathcal{W}}^V(a;\mathcal{P}_n)$ in powers of λ_n and λ_n^{-1}, as $\lambda_n \to \infty$. $\mathcal{P}_n$ stands for the set of parameters characterizing the fermionic action S_n. As the bosonic theory is gaussian, it is easier to study than the fermionic theory where the interaction term is quartic in the fields $\hat{\psi}^{\sharp}_{\omega}$.

But does the action $\widetilde{\mathcal{W}}^V_*(a;\mathcal{P}_n)$ derived by the bosonization procedure reproduce the correct result that one would obtain by iterating the RG transformations and by determining the resulting fixed point action ($n \to \infty$) ?
In order to answer this question, we have to clarify the approximations involved by bosonizing the electron system described by the action S_n. The approximations can be characterized in terms of the *formal* perturbation expansion of the fermionic theory in powers of λ_n^{-1}. *In fact, by bosonizing an electron system with action S_n, we resum all leading order contributions of the fermionic perturbation expansion of the effective gauge field action in powers of λ_n^{-1} – except self-energy renormalizations of inner propagator lines.* The diagrams reproduced after bosonization are the following ones :

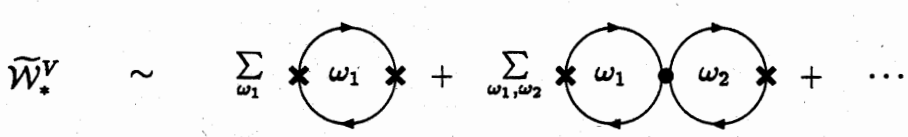


Fig.10

As above, straight lines stand for electron propagators and crosses for gauge field insertions. The interaction $\frac{k_F^{1-d}}{\lambda_n^{d-1}}\, g_n(\vec{\omega}_1\vec{\omega}_2)$ vertices are represented by dots. In the diagrams in Fig.10 , each factor $\frac{1}{\lambda_n^{d-1}}$ per interaction vertex is compensated by a bubble-summation, $\sum_{\omega}(\cdot)$.
These diagrams correspond to the so called RPA approximation .

In principle, to leading order in a (formal) expansion in $\frac{1}{\lambda_n}$, the propagator lines in the polarization bubbles in Fig.10 are renormalized by "cactus diagrams" of the form :

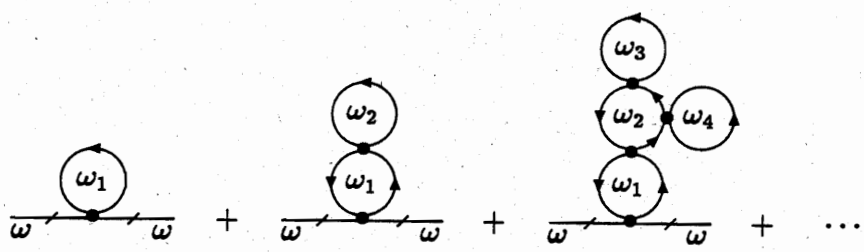


Fig.11

For systems with a short-range two-body potential V, their contribution is – to leading order in $\frac{1}{\lambda_n}$ – a *constant*, i.e., they only lead to a displacement of the chemical potential. As the polarization bubbles only depend on the difference of the momenta of the two inner propagator lines, they are not modified by such a displacement of the chemical potential.

Does this result imply that, by taking functional derivatives of $\widetilde{\mathcal{W}}_*^V(a;\mathcal{P}_n)$ with respect to the gauge field a and by evaluating the resulting current Green functions, cf. eq.(31), for large arguments $|x^{(i)} - x^{(j)}| = \lambda_n|\xi^{(i)} - \xi^{(j)}|$, with $\lambda_n \to \infty$, one obtains the leading contribution in $\frac{1}{\lambda}$ of the current Green functions of the interacting system ? One must remember that we use the fermionic action S_n as an input of bosonization. Because of the linearization of the pieces of the Fermi surface contained in the boxes $\overline{B}_\omega$, this action describes the properties of the system only correctly at momentum scales between $\frac{k_F}{\lambda_n}$ and $b\frac{k_F}{\lambda_n^2}$, as argued in Sect.2 .
It follows that the current Green functions derived from $\widetilde{\mathcal{W}}_*^V(a;\mathcal{P}_n)$ reproduce the leading order contribution in $\frac{1}{\lambda_n}$ of the current Green functions of the interacting system for arguments $|x^{(i)} - x^{(j)}| = \lambda_n|\xi^{(i)} - \xi^{(j)}|$ of order between λ_n and λ_n^2. In order to explore the current Green functions of the interacting system at larger distance- and time scales, we first have to to iterate the RG transformations further, with the aim of deriving effective actions S_m , $m > n$, describing the properties of the system at larger distance- and time scales. Such an effective action S_m can serve as a new input for the bosonization procedure.

In Sect.2, we have analyzed the flow of the set $\mathcal{P}_n$ of parameters characterizing the effective action S_n for electron systems with a short-range two-body potential V. All the parameters of the set $\mathcal{P}_n$ – except the scale factor λ_n – tend to finite values $\mathcal{P}_*$ in the limit $\lambda_n \to \infty$. This implies that

$$\mathcal{W}_*^V(a) \;=\; \text{“}\lim_{n\to\infty}\text{“}\; \widetilde{\mathcal{W}}^V(a;\mathcal{P}_n) \quad , \tag{59}$$

i.e. the scaling limit of the effective gauge field action is given by

$$\mathcal{W}_*^V(a) \;=\; \text{“}\lim_{n\to\infty}\text{“}\; -log\left\{\Xi_\varphi^{-1}\int \mathcal{D}\{\varphi_{[\omega]}\}\; e^{-\left(S_*^0(\{\varphi_{[\omega]}\})+S_*^J(\{\varphi_{[\omega]}\};a)\right)}\; e^{-S_*^V(\{\varphi_{[\omega]}\})}\right\} \quad , \tag{60}$$

where the action is defined by eqs. (47), (57) and (58). Again, "$\lim_{\lambda_n\to\infty}$" stands for determining the *coefficient* of the leading term of the Laurent expansion in λ_n and λ_n^{-1}.

Carrying out the *Gaussian* functional integral, we find for the leading term proportional to λ_n^{d-1} :

$$\mathcal{W}_*^V(a) \;=\; \frac{1}{2}\int d^{d+1}k \sum_{\rho,\sigma=0}^{d} \hat{a}_\rho(-k)\Pi_{V*}^{\rho\sigma}(k)\hat{a}_\sigma(k) \quad . \tag{61}$$

The polarization tensor $\Pi_{V*}^{\rho\sigma}(k)$ has the general form described in eq.(44), where $\Pi_{V*}^t(k)$ is given by $\Pi_*^t(k)$, and $\Pi_{V*}^\ell(k)$ can be expressed as a Neumann series in $g_*(\vec{\omega}\vec{\omega}')$. The explicit expression for the Neumann series is rather complicated in the general case (but straightforward to determine). We only write down the explicit expression for $\Pi_{V*}^\ell(k)$ in the (somewhat artificial) case when $\tilde{g}_*(\vec{\omega}_1\vec{\omega}_2) = g_*$. Then one obtains

$$\Pi_{g*}^\ell(k) \;=\; \Pi_*^\ell(k)\,\frac{1}{1+\frac{\vec{k}^2}{k_0^2}\frac{g_*}{k_F^{d-1}}\,\Pi_*^\ell(k)} \quad .$$

(The functions $\Pi_*^{\ell,t}$ are defined in eq.(46)) .

For systems with a long-range two-body potential V, the renormalization flow of the effective action S_n is not yet fully understood. Given the effective action S_n, we can calculate calculate the corresponding effective gauge field action $\widetilde{W}(a,\mathcal{P}_n)$ by bosonizing the system. This yields a result for the current Green functions in the domain of validity discussed above. In order to derive results at larger distance- and time-scales, one is obliged to make assumptions on the flow of the effective action S_n, as $n\to\infty$.

In the following, we discuss the calculation of $\widetilde{W}(a,\mathcal{P}_n)$ for three examples of systems with singular interactions :

i) **Long-range, density-density interactions**

We consider interactions of the type shown in eq.(2) with a long-range interaction potential gV, i.e., one whose Fourier transform, $g\hat{V}(|\vec{p}|) = \frac{g}{k_F^{d-1}}\left|\frac{k_F}{\vec{p}}\right|^{\alpha}$, becomes singular at $|\vec{p}| = 0$. The exponent α and the coupling constant g are supposed to be positive. We assume that the effective action S_n at an energy scale $v_{Fn}\frac{k_F}{\lambda_n}$ has the form specified in eqs. (52) - (55) and that all the parameters of the set $\mathcal{P}_n$ characterizing the action S_n – except the scale parameter λ_n – tend to finite values in the limit $n \to \infty$. In particular, the set of coupling functions $g_n(\underline{\omega}; \frac{1}{\lambda_n}\underline{k})$ is supposed to be related to the initial interaction potential by eq.(17), for arbitrary n. We can neglect the exchange channel with respect to the direct channel. In the direct channel – contrary to the case of short-range interactions – we have to retain the (singular) dependence on the small momenta, $\frac{1}{\lambda_n}\underline{k}$, as indicated in eq.(17). Hence, in eq.(53), the dominant coupling constants are given by

$$g_n(\vec{\omega}_1\vec{\omega}_2; \tfrac{1}{\lambda_n}\underline{k}) \;=\; k_F^{d-1} g_n \hat{V}(\tfrac{1}{\lambda_n}|\vec{k}|) \;=\; g_n \left|\frac{k_F\lambda_n}{\vec{k}}\right|^{\alpha} \quad . \tag{62}$$

By bosonizing the system, one obtains the result

$$\widetilde{\mathcal{W}}^D(a;\mathcal{P}_n) \;=\; \frac{1}{2}\lambda_n^{d-1}\int \bar{d}^{d+1}k \sum_{\rho,\sigma=0}^{d} \hat{a}_\rho(-k)\Pi_D^{\rho\sigma}(k;\mathcal{P}_n)\hat{a}_\sigma(k) \quad , \tag{63}$$

where the polarization tensor $\Pi_D^{\rho\sigma}(k;\mathcal{P}_n)$ has the same form as in eq.(44), with the two independent functions Π_D^{ℓ} and Π_D^{t} given by

$$\begin{aligned} \Pi_D^{\ell}(k;\mathcal{P}_n) \;&=\; \Pi_n^{\ell}(k)\,\frac{1}{1 + \frac{\vec{k}^2}{k_0^2} g_n \hat{V}(|\frac{\vec{k}}{\lambda_n}|)\Pi_n^{\ell}(k)} \\ \Pi_D^{t}(k;\mathcal{P}_n) \;&=\; \Pi_n^{t}(k) \end{aligned}$$

(The functions $\Pi_n^{\ell,t}$ are defined in eq.(46), with $v_F \to v_{Fn}$.)
As expected, this result coincides with the result of an RPA calculation

ii) **Tomographic Luttinger Liquid**

We introduce interactions which describe singular direct (or "forward") scattering processes, where the quasi-electron with momentum near $k_F\vec{\omega}$ interacts only with one with momentum near $-k_F\vec{\omega}$. This leads to the coupling constants

$$g_n(\vec{\omega}_1\vec{\omega}_2) \;=\; \delta_{\vec{\omega}_1,-\vec{\omega}_2}\,\lambda_n^{d-1}\,(2\pi)^{d-1}\,g_n \quad . \tag{64}$$

By inserting eq.(64) into eq.(60), one obtains :

$$\widetilde{\mathcal{W}}^{TL}(a;\mathcal{P}_n) = -log\Big(\int \frac{\mathcal{D}\{\varphi_{[\omega]}\}}{Z_\varphi}\, e^{-\frac{1}{2}\sum_{\omega\in\mathcal{S}^+} 4(1+\frac{g_n}{2\pi})\int d^{d+1}\xi\, \partial^\omega\varphi_{[\omega]}(\xi)\,\overline{\partial}^\omega\varphi_{[\omega]}(\xi)} \cdot e^{S_n^J(\{\varphi_{[\omega]}\};a)}\Big)$$

$$= \frac{1}{2}\lambda_n^{d-1}\int đ^{d+1}k \sum_{\rho,\sigma=0}^{d} \hat{a}_\rho(-k)\Pi_{TL}^{\rho\sigma}(k;\mathcal{P}_n)\hat{a}_\sigma(k) \quad .$$

The polarization tensor $\Pi_{TL}^{\rho\sigma}$ is renormalized by an overall factor $\left(1+\frac{g_n}{2\pi}\right)^{-1}$ with respect to the non-interacting system :

$$\Pi_{TL}^{\rho\sigma}(k;\mathcal{P}_n) = \left(1+\frac{g_n}{2\pi}\right)^{-1}\Pi_n^{\rho\sigma}(k) \quad . \tag{65}$$

iii) **Long-range, transverse, current-current interactions**

In the original system, interactions between transverse currents lead to an additional term in the action of the form

$$S^T(\Psi^*,\Psi) = -\frac{1}{2}\int d^{d+1}x\int d^{d+1}y\sum_{k=1}^{d} j_T^k(x)\, gV(|\vec{x}-\vec{y}|)\delta(x_0-y_0)\, j_T^k(y) \quad , \tag{66}$$

with

$$j_T^k(\Psi^*,\Psi;x) = \sum_{l=1}^{d}\left(\delta_{k,l}-\frac{\partial_k\partial_l}{\sum_{i=0}^{d}\partial_i^2}\right) j^l(\Psi^*,\Psi;x) \quad .$$

A physical realization of such a system is a QH system at filling factors $\nu = \frac{1}{2},\frac{1}{4},...$, which can be described as a system of free (composite) fermions interacting via long-range, transverse, current-current interactions as diplayed in eq.(66), cf. [3, 4, 6] .

One determines the effective interaction for the low-energy modes whose momenta lie in a thin shell Ω_n around the Fermi surface following the same procedure as for longitudinal interactions. Using eq.(41), one obtains

$$S_n^T(\{\psi_\omega^\sharp\}) = -\frac{g_n v_{Fn}^2}{2\lambda_n^{d-1}}\sum_{\omega_1,\omega_2\in\mathcal{S}^+}\int_{I_{\omega_1}\cap I_{\omega_2}} đ^{d+1}k\, \hat{V}(|\frac{\vec{k}}{\lambda_n}|)\cdot$$

$$\cdot\sum_{i,j=1}^{d}\omega_1^i\, j_{[\omega_1]}^1(-k)\,\mathcal{P}_{ij}^T(k)\,\omega_2^j\, j_{[\omega_2]}^1(k) \quad , \tag{67}$$

with

$$\mathcal{P}^T_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \qquad \text{and} \qquad \hat{V}(|\vec{p}|) = \frac{1}{k_F^{d-1}} \left| \frac{k_F}{\vec{p}} \right|^\alpha \quad .$$

For a QH system with unscreened Coulomb interactions, the exponent α is equal to 1 ; if one assumes the Coulomb interactions to be screened, the exponent α turns out to be 2.

We propose to calculate the effective gauge field action of this system by replacing the fermionic currents, $j^\mu_{[\omega]}(\overline{\psi}_{[\omega]}, \psi_{[\omega]})$, by their bosonic versions, $j^\mu_{[\omega]}(\varphi_{[\omega]})$, given in eq.(46). We have to make assumptions analoguous to the ones stated in (i) . One obtains

$$\widetilde{\mathcal{W}}^T(a; \mathcal{P}_n) \;=\; \frac{1}{2}\lambda_n^{d-1} \int \bar{d}^{d+1}k \sum_{\rho,\sigma=0}^{d} \hat{a}_\rho(-k)\, \mathbf{\Pi}^{\rho\sigma}_T(k; \mathcal{P}_n)\, \hat{a}_\sigma(k) \qquad , \tag{68}$$

with

$$\begin{aligned} \mathbf{\Pi}^\ell_T(k; \mathcal{P}_n) &= \mathbf{\Pi}^\ell_n(k) \\ \mathbf{\Pi}^t_T(k; \mathcal{P}_n) &= \mathbf{\Pi}^t_n(k) \frac{1}{1 + \mathbf{\Pi}^t_n(k) g_n \hat{V}(|\frac{\vec{k}}{\lambda_n}|)} \quad . \end{aligned}$$

Again, this reproduces the result of an RPA calculation.

4 Electron Propagator

In this section, we determine the bosonic expressions for the electron fields $\mathbf{\Psi}^*, \mathbf{\Psi}$, in order to calculate the electron propagator for interacting systems. We shall bosonize each one of the $N \sim \lambda^{d-1}$ component-fields $\psi^\sharp_\omega$ seperately ($\psi^\sharp$ stands for ψ^* or ψ). More precisely, for each ray $[\omega] = \{\vec{\omega}, -\vec{\omega}\}$, we express the pair $\psi^\sharp_\omega, \psi^\sharp_{-\omega}$ in terms of a bose field $\varphi_{[\omega]}$. This is accomplished by applying the well-known bosonization formalism for $1+1$ dimensional relativistic fermions summarized in appendix A . However, one has to cope with a subtlety arising from the dependence of the quasi $1+1$ dimensional electron fields $\psi^\sharp_\omega$ on the components of the momentum perpendicular to the direction $[\omega]$.

We start our discussion with the non-interacting system for which

$$-\left\langle \hat{\psi}_{\omega^{(i)}}(k)\, \hat{\psi}^*_{\omega^{(i)}}(k') \right\rangle^0 \;=\; \delta_{\omega^{(i)},\omega^{(j)}}\, (2\pi)^{d+1}\, \delta^{(d+1)}(k-k')\, \frac{1}{ik_0 - v_F \vec{\omega}^{(i)} \vec{k}}\, 1_{\overline{Q}_{\omega^{(i)}}}(\vec{k}) \quad . \tag{69}$$

In comparison to eq.(11), we replaced the boxes $\overline{B}_{\omega(i)}$ by cubes, $\overline{Q}_{\omega(i)}$, with sides of length k_F (the exact shape of the integration domain is irrelevant) . Because of the isotropy of the electron system, one observes a non-trivial dependence on the components of k only in the $0-$ and $\vec{\omega}-$direction. Propagation takes place in the radial direction, $\vec{\omega}$. This suggests a decomposition of the quasi particle fields $\psi^{\sharp}_{\omega}(\xi)$ into tensor products

$$\psi^{\sharp}_{\omega}(\xi) \; = \chi_{[\omega]}(\xi_{\perp}) \otimes \psi^{\sharp}_{\omega||}(\xi_0, \xi_{||}) \quad , \tag{70}$$

for $\vec{\omega} \in \mathcal{S}_1^{d-1}$.

The "radial" fields, $\psi^{\sharp}_{\omega||}$, describe $1+1$ dimensional, relativistic electrons, whereas the bosonic "angular" fields, $\chi_{[\omega]}(\xi_{\perp})$, just guarantee momentum conservation in the perpendicular direction.

The angular fields $\chi_{[\omega]}(\xi_{\perp})$ are Gaussian, of mean 0, and their propagator (covariance) is given , in momentum space, by

$$\left\langle \hat{\chi}_{[\omega]}(\vec{k}_{\perp})\, \hat{\chi}_{[\omega']}(\vec{k}'_{\perp}) \right\rangle_{\perp} \; = \; \delta_{[\omega],[\omega']}\, (2\pi)^{d-1}\, \delta^{(d-1)}(\vec{k}_{\perp} - \vec{k}'_{\perp}) \cdot 1_{\overline{Q}^{(d-1)}_{\omega}}(\vec{k}^{\perp}) \quad . \tag{71}$$

Their Fourier modes, $\hat{\chi}_{[\omega]}(\vec{k}_{\perp})$, have support in $\overline{Q}^{d-1}_{\omega}$.

The radial fields $\psi^{\sharp}_{\omega||}$ can be bosonized by applying the standard $1+1$ dimensional formalism . For each ray $[\omega]$, one uses the following identifications :

$$\begin{aligned}
\psi_{\omega 1}(\xi_0, \xi_{||}) \; &:= \; \psi_{-\omega||}(\xi_0, \xi_{||}) \; \longleftrightarrow \; \tfrac{1}{(2\pi)^{\frac{1}{4}}}\, D_{[\omega]}(\xi_0, \xi_{||}; 1) \; : e^{i\sqrt{\pi}\varphi^{||}_{[\omega]}(\xi_0, \xi_{||})} : \quad , \\
\psi^*_{\omega 1}(\xi_0, \xi_{||}) \; &:= \; \psi^*_{-\omega||}(\xi_0, \xi_{||}) \; \longleftrightarrow \; \tfrac{1}{(2\pi)^{\frac{1}{4}}}\, D_{[\omega]}(\xi_0, \xi_{||}; -1) \; : e^{-i\sqrt{\pi}\varphi^{||}_{[\omega]}(\xi_0, \xi_{||})} : \quad , \\
\psi_{\omega 2}(\xi_0, \xi_{||}) \; &:= \; \psi_{\omega||}(\xi_0, \xi_{||}) \; \longleftrightarrow \; \tfrac{1}{(2\pi)^{\frac{1}{4}}}\, D_{[\omega]}(\xi_0, \xi_{||}; 1) \; : e^{-i\sqrt{\pi}\varphi^{||}_{[\omega]}(\xi_0, \xi_{||})} : \quad , \\
\psi^*_{\omega 2}(\xi_0, \xi_{||}) \; &:= \; \psi^*_{\omega||}(\xi_0, \xi_{||}) \; \longleftrightarrow \; \tfrac{1}{(2\pi)^{\frac{1}{4}}}\, D_{[\omega]}(\xi_0, \xi_{||}; -1) \; : e^{i\sqrt{\pi}\varphi^{||}_{[\omega]}(\xi_0, \xi_{||})} : \quad ,
\end{aligned} \tag{72}$$

where $D_{[\omega]}$ is a disorder operator and the normal ordered exponential of $\varphi^{||}_{[\omega]}$ is a "vertex operator". The precise definitions of the expressions on the r.s. of eq.(72) appear in appendix A .

The fields $\varphi^{||}_{[\omega]}(\xi_0, \xi_{||})$ are $1+1$ dimensional free, massless Bose fields with an action given by

$$S^0_{[\omega]}(\varphi^{||}_{[\omega]}) \; = \; \frac{1}{2} \int_{\mathbf{R}\times[-\frac{k_F}{2}, \frac{k_F}{2}]} đk_0 đk_{||}\; \hat{\varphi}^{||}_{[\omega]}(-k_0, -k_{||}) \left(k_0^2 + (v_F k_{||})^2\right) \hat{\varphi}^{||}_{[\omega]}(k_0, k_{||}) \quad . \tag{73}$$

The Fourier modes $\hat{\varphi}^{||}_{[\omega]}(k_0, k_{||})$ have support on $\mathbf{R} \times [-\frac{k_F}{2}, \frac{k_F}{2}]$.
One can verify that from the bosonization formulas, eqs. (70)-(72), for the electron fields the ones for the current densities, eq.(46), follow. In appendix B, we show this in an example.

It is important to understand the relation of the radial bose fields $\varphi^{||}_{[\omega]}(\xi_0, \xi_{||})$ to the fields $\varphi_{[\omega]}(\xi)$ introduced in eqs.(46)-(47), which are related to the fermionic currents.
Following eq.(47), the action $S^0(\varphi_{[\omega]})$ of the bose field $\varphi_{[\omega]}$ is given, in momentum space, by

$$S^0(\varphi_{[\omega]}) \;=\; -\frac{1}{2}\int_{\mathbf{R}\times\overline{Q}_\omega} đ^{d+1}k\; \hat{\varphi}_{[\omega]}(-k)\; \left(k_0^2 + (v_F k_{||})^2\right)\; \hat{\varphi}_{[\omega]}(k) \quad , \qquad (74)$$

i.e., the field $\varphi_{[\omega]}$ propagates only along the direction $[\omega]$. It follows that the propagators of the $\varphi_{[\omega]}$-fields are related to the ones of the $\varphi^{||}_{[\omega]}$- fields by

$$\left\langle \varphi_{[\omega]}(\xi_0, \xi_{||}, \vec{\xi}_\perp)\; \varphi_{[\omega']}(\eta_0, \eta_{||}, \vec{\eta}_\perp) \right\rangle \;=$$
$$\delta_{[\omega],[\omega']}\; \delta^{d-1}_{k_F}(\vec{\xi}_\perp - \vec{\eta}_\perp)\; \left\langle \varphi^{||}_{[\omega]}(\xi_0, \xi_{||})\; \varphi^{||}_{[\omega']}(\eta_0, \eta_{||}) \right\rangle^{||} , \qquad (75)$$

with

$$\delta^{d-1}_{\omega,k_F}(\vec{\xi}_\perp) \;=\; \int_{\overline{Q}^{d-1}_\omega} đ^{d-1}k_\perp\; e^{i\vec{k}_\perp\vec{\xi}_\perp} \quad .$$

Remark : Comparing eq.(74) to eq.(73), one can decompose the field $\varphi_{[\omega]}(\xi)$ into a tensor product $\varphi_{[\omega]}(\xi) = \varphi^{||}_{[\omega]}(\xi_0, \xi_{||}) \otimes \chi_{[\omega]}(\vec{\xi}_\perp)$, where $\chi_{[\omega]}(\vec{\xi}_\perp)$ coincides with the bosonic Gaussian field introduced in eqs. (70) and (71).

Next, we study the effect of interactions of the form displayed in eq.(53) . As discussed in the preceeding sections, we use the effective actions S_n given in eqs. (52) and (53) as an input for the bosonization calculations. The action S_n is the result of an RG analysis where the electron modes with momenta $\vec{p}$ outside the shell Ω_n have been integrated out. It describes the properties of the electron system at energy scales smaller than $v_{Fn}\frac{k_F}{\lambda_n}$. Under the assumptions specified in Sect.2, one can reach very small energy scales $v_{Fn}\frac{k_F}{\lambda_n} \ll v_{Fn}k_F$ before the form of S_n deviates from the one given in eqs. (52) and (53). The (remaining) fermionic

degrees of freedom are described by the fields $\Psi^{\sharp}$. These fields $\Psi^{\sharp}(\lambda_n\xi)$ can be decomposed into $N_n \sim \lambda_n^{d-1}$ independent components $\psi^{\sharp}_{\omega}(\xi)$:

$$\Psi^*(\lambda_n\xi) \;=\; \sum_{\omega} e^{-ik_F\vec{\omega}\lambda_n\vec{\xi}}\,\lambda_n^{-\frac{d}{2}}\,\psi^*_{\omega}(\xi) \qquad \Psi(\lambda_n\xi) \;=\; \sum_{\omega} e^{ik_F\vec{\omega}\lambda_n\vec{\xi}}\,\lambda_n^{-\frac{d}{2}}\,\psi_{\omega}(\xi) \quad , \tag{76}$$

where the Fourier modes $\hat{\psi}^{\sharp}_{\omega}(k)$ have support in $\overline{Q}_{\omega}$ (cf. eq.(10)) . Momentum conservation guarantees that $\langle\psi_{\omega}\psi^*_{\omega'}\rangle^V \sim \delta_{\omega,\omega'}$, so that the interacting propagator, G , splits into the contributions of the component fields $\psi^{\sharp}_{\omega}$:

$$G(\lambda_n(\xi-\eta)) \approx \sum_{\omega} e^{ik_F\vec{\omega}\lambda_n(\vec{\xi}-\vec{\eta})}\,\lambda_n^{-d}\,G_{\omega,n}(\xi-\eta) \quad , \tag{77}$$

where $G_{\omega,n}(\xi-\eta)$ are the propagators of the component fields calculated by using the effective action S_n. The symbol "$\approx$" indicates that the equation holds to leading order in an expansion in $\frac{1}{\lambda_n}$.

As in eq.(8), this formula should be regarded as a discrete approximation to a continuous angular decomposition of the interacting propagator,

$$G(\lambda_n(\xi-\eta)) \approx \int_{\mathcal{S}_1^{d-1}} d^{d-1}\omega\; e^{ik_F\vec{\omega}\lambda_n(\vec{\xi}-\vec{\eta})}\,\lambda_n^{-1}\;G^{\|}_{\omega,n}\left(\xi_0-\eta_0,\vec{\omega}(\vec{\xi}-\vec{\eta})\right) \quad ,$$

for large arguments $\lambda_n(\xi-\eta)$, where the radial propagators $G^{\|}_{\omega,n}\left(\xi_0-\eta_0,\vec{\omega}(\vec{\xi}-\vec{\eta})\right)$ only depend on the 0-component and the component parallel to the direction $\vec{\omega}$. Hence, for the interacting system, the decomposition (70) of each component $\psi^{\sharp}_{\omega}$ into a tensor product of a radial field $\psi^{\sharp}_{\omega\|}(\xi_0,\xi_{\|})$ and an angular field $\chi_{[\omega]}(\vec{\xi}_{\perp})$ – with the propagator defined in eq.(71) – remains valid.

We calculate the propagator $G^{\|}_{\omega,n}(\xi_0,\xi_{\|})$ in the radial direction by bosonizing the system, i.e., we use the identifications (72) for the component fields and replace the fermionic action $S_n(\{\psi^{\sharp}_{\omega}\})$ by its bosonic version $S_n(\{\varphi_{[\omega]}\})$ given in eqs. (47) and (58). Then, the calculation of the propagator $G^{\|}_{\omega,n}(\xi_0,\xi_{\|})$ is reduced to evaluating an expactation value in the interacting bosonic ground state of a product of a disorder- and a vertex-operator in $\varphi^{\|}_{[\omega]}$ (see eq.(72)). In order to calculate this expectation value, we have to determine the action, $S_n(\varphi^{\|}_{[\omega]})$, of the radial bose field $\varphi^{\|}_{[\omega]}$ from the action $S_n(\{\varphi_{[\omega]}\})$ of the family $\{\varphi_{[\omega]}\,,\,[\omega]\in\mathcal{S}^+\}$ of $d+1$ dimensional bose fields.

Through the interaction (58), the field $\varphi_{[\omega]}$, for a given ray $[\omega]$, is coupled to all the other fields $\varphi_{[\omega']}$, for $[\omega']\in\mathcal{S}^1$. Integrating out the fields $\varphi_{[\omega']}$, for $[\omega']\neq[\omega]$, we obtain the ("effective") action $S_n(\varphi_{[\omega]})$ of the field $\varphi_{[\omega]}$ *which is still Gaussian :*

$$S_n(\varphi_{[\omega]}) \;=\; \frac{1}{2}\int_{\mathbf{R}\times\overline{Q}_{\omega}} \bar{d}^{d+1}k\;\hat{\varphi}_{[\omega]}(-k)\left[C^n_{[\omega]}(k)\right]^{-1}\hat{\varphi}_{[\omega]}(k) \quad , \tag{78}$$

where

$$C^n_{[\omega]}(k) \;=\; \langle \hat{\varphi}_{[\omega]}(-k)\hat{\varphi}_{[\omega]}(k)\rangle_n$$

is the propagator of the Fourier mode $\hat{\varphi}_{[\omega]}(k)$ in the interacting system. We determine the action $S_n(\varphi^{||}_{[\omega]})$ of the radial field $\varphi^{||}_{[\omega]}(\xi_0,\xi_{||})$ by averaging the inverse propagator $\left[C^n_{[\omega]}(k)\right]^{-1}$ in eq.(78) over the components $\vec{k}_\perp$ of $\vec{k}$ perpendicular to the direction $[\omega]$:

$$S_n(\varphi^{||}_{[\omega]}) \;=\; \frac{1}{2}\int_{\mathbf{R}} đk_0 \int_{[\frac{-k_F}{2},\frac{-k_F}{2}]} đk_{||}\; \hat{\varphi}^{||}_{[\omega]}(-k_0,-k_{||})\; h^n_{[\omega]}(k_0,k_{||})\; \hat{\varphi}^{||}_{[\omega]}(k_0,k_{||}) \quad , \tag{79}$$

with

$$h^n_{[\omega]}(k_0,k_{||}) \;:=\; k_F^{1-d}\int_{\overline{Q}^{d-1}_\omega} đ^{d-1}k_\perp\; \left[C^n_{[\omega]}(k_0,k_{||};\vec{k}_\perp)\right]^{-1} \quad .$$

Our procedure to calculate the radial, fermionic propagator $G^{||}_{\omega,n}$ is summarized in the following formula :

$$G^{||}_{\mp\omega,n}(\xi_0-\eta_0,\xi_{||}-\eta_{||}) \;=\; -\langle\, \psi_{\omega\alpha}(\xi_0,\xi_{||})\,\psi^*_{\omega\alpha}(\eta_0,\eta_{||})\,\rangle^{||}_n \;\approx$$
$$\frac{-Z_n}{\sqrt{2\pi}}\left\langle D_{[\omega]}(\xi_0,\xi_{||};1)D_{[\omega]}(\eta_0,\eta_{||};-1)\,:e^{i(-1)^{\alpha-1}\sqrt{\pi}\varphi^{||}_{[\omega]}(\xi_0,\xi_{||})}::e^{i(-1)^{\alpha}\sqrt{\pi}\varphi^{||}_{[\omega]}(\eta_0,\eta_{||})}:\right\rangle_{S_n(\varphi^{||}_{[\omega]})} , \tag{80}$$

where $\alpha=1$, for $-\omega$, and $\alpha=2$, for $+\omega$. The disorder field $D_{[\omega]}$ in the bosonized expression of the electron field guarantees the correct anticommutation relations, regardless of the nature of interactions.

Before diving into explicit calculations, we have to clarify the meaning of the symbol "$\approx$" in eq.(80), i.e., we have to clarify the approximations involved in calculating the electron propagator by means of bosonization.
As in Sect.3, where we have analysed the calculation of the effective gauge field action, we characterize our approximations in terms of the *formal* perturbation expansion of the fermionic theory in powers of λ_n^{-1}. By bosonizing the electron system with action S_n, we resum all leading order contributions of the (formal) fermionic perturbation expansion of the electron propagator in powers of λ_n^{-1} – except for self-energy renormalizations of inner propagator lines (as in Sect.3) .
The diagrams contributing to the propagator $G_{\omega,n}(k)$ that are reproduced by bosonization can be found in the following way (we restrict our discussion to 1PI self-energy contributions) :
First, draw all diagrams renormalizing the bare ω-propagator line that are composed only of ω-propagators and contain loops of at most two propagator lines,

i.e. "bubbles". Then add all diagrams that are generated by replacing an ω-bubble by an ω'-bubble formed by two ω'-propagator lines, for $\omega' \neq \omega$. In the following Fig.12, we display some examples :

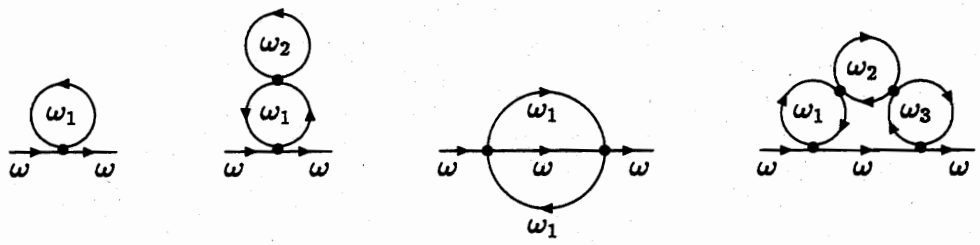
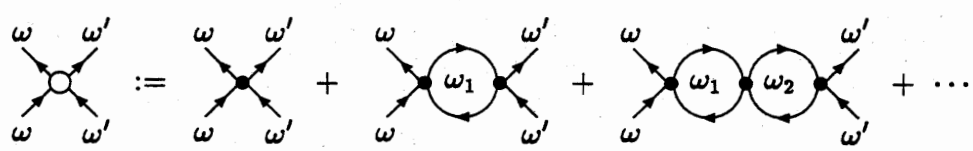


Fig.12

(The corresponding "diagrammar" has been defined in Sect.3 ; the summation, $\sum_{\omega_i}$, associated to each electron loop, is not displayed explicitly.)
A characteristic feature of bosonization is that it reproduces only diagrams containing loops of at most two propagator lines.

Introducing the effective interaction vertex



we can represent the diagrams reproduced as follows

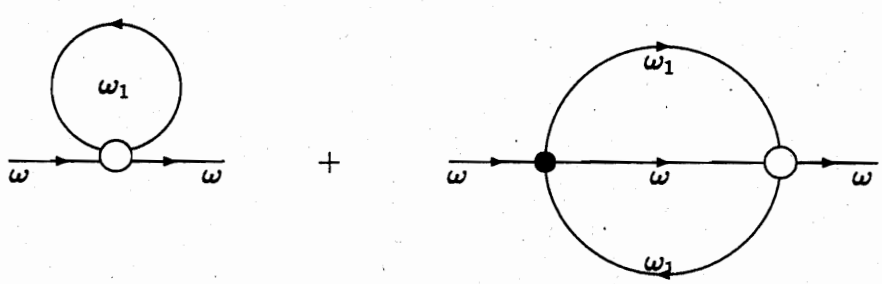


Fig.13

The first class of diagrams leads to a contribution of order 1, the second one to a contribution of order $\frac{1}{\lambda_n}$. One finds that all leading order diagrams are reproduced by bosonization, except for self-energy renormalizations of inner propagator lines. For short-range interactions, one can show by explicit calculations that the contribution of these self-energy renormalizations of inner propagator lines is *zero* – to the order considered in our calculation. For long range interactions, however, they could change the final result.

One should remember that we use the effective action S_n as an input of bosonization. Because of the linearization of the pieces of the Fermi surface contained in the boxes $\overline{B}_\omega$, this action describes the properties of the electron system correctly at (unrescaled) momentum scales between $\frac{k_F}{\lambda_n}$ and $\frac{k_F}{\lambda_n^2}$. Therefore, eq.(80) reproduces the electron propagator only correctly for arguments $|\xi - \eta|$ of order between $\frac{1}{k_F}$ and $\frac{\lambda_n}{k_F}$ (eq.(80) is written with respect to the rescaled system). In this range, formula (80) holds to leading order in a (formal) expansion in powers of $\frac{1}{\lambda_n}$.

To determine the electron propagator for larger arguments, we first have to calculate an effective (fermionic) action S_m, $m > n$, describing the properties of the system at larger distance- and time-scales. Such an effective action S_m can serve as a new input for the bosonization procedure.

We now return to the calculation of the bosonic propagator of the interacting system.

First, we consider systems with a short-range two-body interaction potential.

For the non-interacting system, eq.(74) implies

$$\langle\, \hat{\varphi}_{[\omega]}(k)\, \hat{\varphi}_{[\omega]}(k')\, \rangle^0 \;=\; (2\pi)^{d+1}\delta^{(d+1)}(k+k')\frac{1}{k_0^2 + (v_F\vec{\omega}\vec{k})^2}\; 1_{\overline{Q}_{\omega(i)}}(\vec{k}) \quad . \tag{81}$$

If one turns on interactions given by eq.(58), the propagator changes to become

$$\langle\, \hat{\varphi}_{[\omega]}(k)\, \hat{\varphi}_{[\omega]}(k')\, \rangle_n \;=$$

$$(2\pi)^{d+1}\delta^{(d+1)}(k+k')\frac{1}{k_0^2 + (v_{Fn}\vec{\omega}\vec{k})^2}\left[\sum_{\ell=0}^{\infty}\left(-\frac{1}{\lambda_n^{d-1}}\mathcal{T}^n(k)\right)^{\ell}\right]_{[\omega][\omega']} , \tag{82}$$

where

$$\mathcal{T}^n_{[\omega][\omega']}(k) \;=\; (\frac{1}{2\pi})^d\left[g_n(\vec{\omega}\cdot\vec{\omega}')\left(k_0^2 + (v_{Fn}\vec{\omega}\vec{k})(v_{Fn}\vec{\omega}'\vec{k})\right) + \right.$$

$$g_n(-\vec{\omega}\cdot\vec{\omega}')\left(-k_0^2+(v_{Fn}\vec{\omega}\vec{k})(v_{Fn}\vec{\omega}'\vec{k})\right)\Big]\frac{1}{k_0^2+(\vec{\omega}'\vec{k})^2} \quad .$$

For sufficiently small coupling constants, $|g_n(\vec{\omega}\cdot\vec{\omega}')| \leq g_c \ll 1$, the Neumann series converges, and one obtains

$$\langle\, \hat{\varphi}_{[\omega]}(k)\, \hat{\varphi}_{[\omega]}(k')\, \rangle_n \;=\; (2\pi)^{d+1}\delta^{(d+1)}(k+k')\frac{1}{k_0^2+(v_{Fn}\vec{\omega}\vec{k})^2}\left[1+\frac{1}{\lambda_n^{d-1}}f(k;g_n)\right] \quad , \tag{83}$$

where $f(k;g_n)$ is a bounded function in k. From Sect.2, we know that – for a system with short-range interactions – the coupling constants $g_n(\vec{\omega}\cdot\vec{\omega}')$ tend to finite limits, as $n\to\infty$. Thus, in the limit $\lambda_n\to\infty$, the effects of interactions disappear, and the system is driven to the non-interacting (Landau liquid) fixed point with a propagator given by (81). For short-range interactions, the dependence on $\vec{k}_\perp$ is irrelevant and suppressed by a factor $\frac{1}{\lambda_n^{d-1}}$.

Below, we shall see that, for sufficiently long-range transverse current-current interactions, the dependence on $\vec{k}_\perp$ is significant (i.e., singular, as $|\vec{k}_\perp|\to 0$, with $|k_0|, |k_{||}| < |\vec{k}_\perp||$). It is not suppressed by an inverse power of the scale factor λ_n. We have to average over the variable $\vec{k}_\perp$ in order to determine the effective dynamics of the bosonic degrees of freedom $\varphi^{||}_{[\omega]}$ in the direction along $[\omega]$. Using eq.(80), we then calculate the propagator of the radial, quasi-electron components $\psi^{\sharp}_{\omega||}$.

We start by studying two technically easier classes of systems with singular interactions. These systems correspond to examples (i)-(iii) of section 3 for which the polarization tensor $\Pi^{\rho\sigma}$ has been calculated explicitly.

i) Long-range, density-density interactions

The set of coupling constants is given by eq.(62), with $g_n\hat{V}(|\vec{p}|) = \frac{g_n}{k_F^{d-1}}|\frac{k_F}{\vec{p}}|^\alpha$, $\alpha>0$. One obtains

$$\langle\, \hat{\varphi}_{[\omega]}(k)\, \hat{\varphi}_{[\omega]}(k')\, \rangle_n \;=\; (2\pi)^{d+1}\delta^{(d+1)}(k+k')\frac{1}{k_0^2+(v_{Fn}\vec{\omega}\vec{k})^2}\cdot$$

$$\cdot\left\{1+\frac{2}{\lambda^{d-1}}(\frac{1}{2\pi})^d\frac{(v_{Fn}\vec{\omega}\vec{k})^2}{k_0^2+(v_{Fn}\vec{\omega}\vec{k})^2}\left[g_n^{-1}\hat{V}^{-1}(|\frac{\vec{k}}{\lambda_n}|)+\Pi_n^{00}(k)\right]^{-1}\right\} \quad , \tag{84}$$

where Π_n^{00} is defined in eq.(44), with $v_F \to v_{Fn}$.
Using that

$$\lim_{|\frac{\vec{k}}{k_0}|\to 0} \Pi_n^{00}(k) \longrightarrow c(\frac{\vec{k}}{k_0})^2 \qquad \text{and} \qquad \lim_{|\frac{k_0}{\vec{k}}|\to 0} \Pi_n^{00}(k) \longrightarrow c' \quad ,$$

one verifies that

$$\frac{(v_{Fn}\vec{\omega}\vec{k})^2}{k_0^2 + (v_{Fn}\vec{\omega}\vec{k})^2}\left[g_n^{-1}\hat{V}^{-1}(|\frac{\vec{k}}{\lambda_n}|) + \Pi_n^{00}(k)\right]^{-1} \sim$$

$$\sim \begin{cases} (\frac{\vec{\omega}\vec{k}}{k_0})^2\left[(\frac{g_n}{k_F^{d-1}})^{-1}|\frac{\vec{k}}{k_F\lambda_n}|^\alpha + c|\frac{\vec{k}}{k_0}|^2\right]^{-1} & , \text{ for}|\frac{\vec{k}}{k_0}| \to 0 \\ \left[(\frac{g_n}{k_F^{d-1}})^{-1}|\frac{\vec{k}}{k_F\lambda_n}|^\alpha + c'\right]^{-1} & , \text{ for}|\frac{k_0}{\vec{k}}| \to 0 \end{cases}$$

For $0 \le \alpha \le 2$, the rhs is a bounded function of k.

Under the assumption that all parameters $\mathcal{P}_n$ (except the scale parameter λ_n) characterizing the effective action S_n tend to finite values, as $n \to \infty$, it follows that, in the limit $\lambda_n \to \infty$, the effect of interactions on the bose propagator disappears, as for the system with short-range interactions.
Hence, in the limit $\lambda_n \to \infty$, the electron propagator G tends to the standard LFL form.

This result is due to the screening of the bare, long-range interaction potential $\hat{V}(|\vec{p}|)$. In eq.(84), the bare potential $\hat{V}(|\frac{\vec{k}}{\lambda_n}|)$ is replaced by an effective $RPA-$potential

$$g_n\hat{V}_n^{\text{eff}}(k_0,\vec{k}) \;:=\; \frac{g_n\hat{V}(|\frac{\vec{k}}{\lambda_n}|)}{1 \;+\; g_n\hat{V}(|\frac{\vec{k}}{\lambda_n}|)\Pi_n^{00}(k)} \;=\; \frac{1}{g_n^{-1}\hat{V}^{-1}(|\frac{\vec{k}}{\lambda_n}|) \;+\; \Pi_n^{00}(k)} \,.$$

In the static limit, $|\frac{k_0}{\vec{k}}| \to 0$, we obtain

$$\lim_{|\frac{k_0}{\vec{k}}|\to 0} g_n\hat{V}_n^{\text{eff}}(k_0,\vec{k}) \;=\; \frac{1}{(\frac{g_n}{k_F^{d-1}})^{-1}(\frac{k_F\lambda_n}{|\vec{k}|})^\alpha \;+\; c} \quad ,$$

i.e., in this limit, the effective potential is short-ranged.

ii) **Tomographic Luttinger Liquid**

With eq.(64), one obtains

$$\langle\, \hat{\varphi}_{[\omega]}(k)\, \hat{\varphi}_{[\omega]}(k')\, \rangle_n \;=\; (2\pi)^{d+1}\delta^{(d+1)}(k+k')\frac{1}{k_0^2+(v_{Fn}\vec{\omega}\vec{k})^2}\left[1+\frac{g_n}{2\pi}\right]^{-1} \quad .$$

As there is no dependence on the perpendicular momenta, one can immediately read off the effective dynamics for the modes $\hat{\varphi}^{||}_{[\omega]}$ propagating along $[\omega]$:

$$S_n^{TL}(\varphi^{||}_{[\omega]}) \;=\; \frac{1}{2}\int_{\mathbf{R}\times[-\frac{k_F}{2},\frac{k_F}{2}]} đk_0 đk_{||}\,\cdot$$
$$\cdot\,\hat{\varphi}^{||}_{[\omega]}(-k_0,-k_{||})\left[1+\frac{g_n}{2\pi}\right]\left(k_0^2+(v_{Fn}k_{||})^2\right)\hat{\varphi}^{||}_{[\omega]}(k_0,k_{||}) \quad .$$

The result for the quasi-electron propagator can be found with the help of formulae (A18) and (A19) of appendix A :

$$G_{\pm\omega,n}(\xi) \;=\; \delta^{(d-1)}_{\omega,k_F}(\vec{\xi}_\perp)\cdot\frac{\mp i}{2\pi}\,e^{\pm i\arg(\zeta)}\,\frac{Z_n}{|\zeta|^{1+\eta}} \quad , \tag{85}$$

where

$$\xi \;=\; (\zeta,\vec{\xi}_\perp) \qquad ,\text{with} \qquad \zeta \;:=\; (\xi_0,\frac{\xi_{||}}{v_{Fn}})\,,$$

and

$$\delta^{(d-1)}_{\omega,k_F}(\vec{\xi}_\perp) \;:=\; \int_{\bar{Q}^{d-1}} đ^{d-1}k_\perp\; e^{i\vec{k}_\perp\vec{\xi}_\perp} \quad .$$

The exponent η is given by

$$\eta \;=\; \frac{2(\frac{g_n}{\pi})^2}{1+2(\frac{g_n}{\pi})^2} \quad .$$

This interacting system describes a *Luttinger Liquid.*
The flow of the parameter Z_n in eq.(85) can be derived by assuming that, along each direction $\vec{\omega}$, the system is scale invariant :

Recall that

$$G(\lambda_n\xi) \;\sim\; \sum_\omega e^{ik_F\lambda_n\vec{\omega}\vec{\xi}}\,\frac{1}{\lambda_n^d}\,G_{\omega,n}(\xi) \;\sim\; \int d\omega\; e^{ik_F\lambda_n\vec{\omega}\vec{\xi}}\,\frac{1}{\lambda_n}\,G^{||}_{\omega,n}(\zeta)\,. \tag{86}$$

This equation and the assumption that the theory is scale invariant along each direction $\vec{\omega}$ implies the following matching-property for the Green functions $G^{\|}_{\omega,n}(\zeta)$:

$$G^{\|}_{\omega,n}(M\zeta) \quad \approx \quad \frac{1}{M} G^{\|}_{\omega,n+1}(\zeta) \quad . \tag{87}$$

By inserting eq.(85) :

$$\frac{Z_n}{|M\zeta|^{1+\eta}} \quad \sim \quad \frac{1}{M} \frac{Z_{n+1}}{|\zeta|^{1+\eta}} \quad ,$$

it follows that

$$\frac{Z_{n+1}}{Z_n} \sim \frac{1}{M^\eta} \qquad \text{or} \qquad Z_n \sim \frac{1}{(M^n)^\eta} \quad ,$$

i.e., for this system, the residue Z_n of the one-particle pole vanishes, as $n \to \infty$.

iii) Long-range, transverse current-current interactions

Such interactions have been introduced in eq.(66), with $g_n V(|\vec{p}|) = c_n |\frac{k_F}{\vec{p}}|^\alpha$, $c_n := \frac{g_n^T v_{Fn}^2}{k_F^{d-1}}$, and lead to the following bosonic propagator (for $[\omega] = [\omega']$) :

$$\langle\, \hat{\varphi}_{[\omega]}(k)\, \hat{\varphi}_{[\omega]}(k')\, \rangle_n \;=\; (2\pi)^{d+1} \delta^{(d+1)}(k+k') \frac{1}{k_0^2 + (v_{Fn}\vec{\omega}\vec{k})^2} \cdot$$

$$\cdot \left\{ 1 - \frac{2}{\lambda_n^{d-1}} \frac{1}{2\pi} (\frac{k_F}{2\pi})^{d-1} \frac{g_n \hat{V}(|\frac{\vec{k}}{\lambda_n}|)}{1 + g_n \hat{V}(|\frac{\vec{k}}{\lambda_n}|)\, \Pi_n^t(k)} \frac{k_0^2}{k_0^2 + (v_{Fn}\vec{\omega}\vec{k})^2} \left[\vec{\omega}^2 - \frac{(\vec{\omega}\vec{k})^2}{\vec{k}^2} \right] \right\} \quad , \tag{88}$$

with

$$\Pi_n^t(k) \;=\; \frac{2}{d-1} \frac{1}{2\pi} (\frac{k_F}{2\pi})^{d-1} \frac{1}{\lambda_n^{d-1}} \sum_{\omega \in \mathcal{S}^+} \frac{k_0^2}{k_0^2 + (v_{Fn}\vec{\omega}\vec{k})^2} \left[\vec{\omega}^2 - \frac{(\vec{\omega}\vec{k})^2}{\vec{k}^2} \right] \quad .$$

One finds that $\lim_{n\to\infty} \Pi_n^t(k) = \Pi_*^t(k)$, with $\Pi_*^t(k)$ defined in eq.(44).
Using that $\lim_{|\frac{k_0}{\vec{k}}| \to 0} \Pi_n^t(k) \sim |\frac{k_0}{\vec{k}}|$, one can see (cf. appendix C) that the contribution from the interactions to the propagator yields a singular $\vec{k}_\perp$-dependence, as $|\vec{k}_\perp| \to 0$ and $\big(|k_\||, |k_0|\big) \ll |\vec{k}_\perp|$. As discussed above, we then have to average over the perpendicular momenta $\vec{k}_\perp$, in order to determine the (effective) action, S_n^T , for the modes $\hat{\varphi}^{\|}_{[\omega]}$ propagating along $[\omega]$.

The somewhat tedious calculations are deferred to appendix C. Here, we only describe our results.
We obtain two different regimes, depending on the exponent α which characterizes the singularity of the interaction potential in momentum space by $\hat{V}(|\vec{p}|) \sim \frac{1}{|\vec{p}|^\alpha}$:

- For $0 \leq \alpha < d-1$, the bosonization calculation yields

$$G_{\omega,n}(\xi) \quad \sim \quad \delta^{(d-1)}_{\omega,k_F}(\vec{\xi}_\perp)\,(\frac{-i}{2\pi})\,e^{i\arg(i\xi_0+\frac{\xi_\parallel}{v_{Fn}})}\,\frac{Z_n}{\sqrt{(v_{Fn}\xi_0)^2+\xi_\parallel^2}} \quad . \tag{89}$$

The function $\delta^{(d-1)}_{\omega,k_F}(\vec{\xi}_\perp)$ has been defined in eq.(85). The influence of the interaction on the electron propagator is suppressed by a factor $\lambda_n^{-[(d-1)-\alpha]}$, as $\lambda_n \to \infty$.
From the matching condition (87), it follows that

$$Z_{n+1} \approx Z_n \qquad \text{and} \qquad v_{Fn+1} \approx v_{Fn} \quad .$$

The system tends to a LFL.

- For $d-1 < \alpha \leq 2$, we display the result in the region $|\xi_0| \ll \frac{|\xi_\parallel|}{v_{Fn}}$:

$$G_{\omega,n}(\xi) \quad \sim \quad \delta^{(d-1)}_{\omega,k_F}(\vec{\xi}_\perp)\,(\frac{-i}{2\pi})\,e^{i\arg(i\xi_0+\frac{\xi_\parallel}{v_{Fn}})}\,Z_n\,\cdot$$

$$\cdot \begin{cases} \exp\left(-u_{\rm I}k_F c_n^{\frac{d}{1+\alpha}}|\lambda_n\frac{\xi_\parallel}{v_{Fn}}|^{\frac{\alpha-(d-1)}{1+\alpha}}\right) & , \text{ for } |\xi_\parallel| \gg c_n v_{Fn}\lambda_n^\alpha \quad , \\ \exp\left(-u_{\rm II}\frac{k_F}{\lambda_n^{d-1}}\frac{|\xi_\parallel|}{v_{Fn}}\right) & , \text{ for } c_n v_{Fn}\lambda_n^\alpha \gg |\xi_\parallel| \gg \frac{v_{Fn}\lambda_n^{d-1}}{k_F} \quad , \\ \frac{1}{|\xi_\parallel|} & , \text{ for } \frac{v_{Fn}\lambda_n^{d-1}}{k_F} \gg |\xi_\parallel| \gg \frac{1}{k_F} \quad . \end{cases} \tag{90}$$

The constant c_n stands for $\frac{g_n v_{Fn}^2}{k_F^{d-1}}$, and $u_{\rm I}, u_{\rm II}$ are positive constants depending on the dimension d of space and the exponent α (cf. appendix C). We have neglected all terms which are of lower order in an expansion in λ_n^{-1} than the leading terms displayed. Two interpretations are consistent with this result :

1. The parameters $\mathcal{P}_n$ characterizing the effective action S_n are all of order 1 – except for the scale parameter λ_n – , i.e., they do not flow under RG transformations. Then, the system tends to a LFL, as

$\lambda_n \to \infty$: for finite arguments $|\xi_{||}|$, the propagator $G_{\omega,n}$ has the standard LFL form. It deviates from this form only for very large arguments $|\xi_{||}| \gg \frac{v_{Fn}\lambda_n^{d-1}}{k_F}$, where the results of a bosonization calculation are not reliable (remember that, because of the linearization of the pieces of the Fermi surface contained in the boxes $\overline{Q}_\omega$, the results of the present calculation are only reliable for arguments $|\xi_{||}|$ smaller than λ_n).

2. The parameters of the set $\mathcal{P}_n$ flow under RG transformations, i.e., they are functions of the scale parameter λ_n.
   The electron propagator deviates from the standard LFL form for finite arguments $|\xi_{||}|$, if the condition

$$c_n v_{Fn} \lambda_n^\alpha \sim O(\frac{1}{k_F})$$

   is satisfied. Further, the requirement that the system is form-invariant under scale transformations implies the matching condition (87). This condition can be satisfied if

$$c_n^{\frac{d}{1+\alpha}} v_{Fn}^{\frac{d-1-\alpha}{1+\alpha}} \sim const. \quad .$$

   Then it follows that

$$c_n \sim \lambda_n^{-\frac{\alpha(\alpha-d+1)}{1+\alpha}} \qquad \text{and} \qquad v_{Fn} \sim \lambda_n^{-\frac{\alpha d}{1+\alpha}} \quad .$$

   The flow of the parameter Z_n cannot be derived in the same way : for this, one would have to know subleading corrections to the argument of the exponential in eq.(90).
   Hence the system displays "Non Landau Liquid" behaviour, as suggested in [3].

The result of the bosonization calculation permits two consistent scenarios. In order to be able to decide which one is realized in the physical system one has to determine the flow of the parameters $\mathcal{P}_n$ under RG transformations. Preliminary calculations indicate that the second case is realized (cf. also [3, 4, 5, 6]).

- For the critical value $\alpha_c = d-1$ of the exponent α, we find for the electron propagator in the region $|\xi_0| \ll \frac{|\xi_\parallel|}{v_{Fn}}$:

$$G_{\omega,n}(\xi) \quad \sim \quad \delta^{(d-1)}_{\omega,k_F}(\vec{\xi}_\perp)\,(\frac{-i}{2\pi})\,e^{i\arg(i\xi_0+\frac{\xi_\parallel}{v_{Fn}})}\,Z_n\,.$$

$$\cdot \begin{cases} \exp\left(-v\,k_F c_n \ln^2|\xi_\parallel|\right) & , \text{ for } |\xi_\parallel| \gg c_n v_{Fn}\lambda_n \quad , \\ \frac{1}{|\xi_\parallel|} & , \text{ for } c_n v_{Fn}\lambda_n \gg |\xi_\parallel| \gg \frac{1}{k_F} \quad , \end{cases} \tag{91}$$

where v is a constant of order one.
As for $\alpha > d-1$, we can distinguish two possible scenarios, consistent with the results of the bosonization calculation :
If the parameters c_n, v_{Fn} are not functions of the scale parameter λ_n, the propagator has the standard LFL form, as $\lambda_n \to \infty$.
In order to obtain a non-LFL behaviour, the product $c_n v_{Fn}\lambda_n$ of parameters must be of order one. Then, the matching condition (87) requires that $c_{n+1} \approx c_n$. By the first condition, it follows that $v_{Fn} \sim \frac{1}{\lambda_n}$.
Again, in order to decide which one of the two scenarios is realized, the flow of the parameters under RG transformations must be investigated. The calculations in [3, 4, 5, 6] point in the second direction.

Appendix A : Bosonization of the Electron Propagator in $1+1$ Dimensions

In this appendix, we review the bosonization of a relavistic $1+1$ dimensional electron system by using functional integrals. Our aim is to summarize the procedure for calculating the electron propagator. The general ideas of the bosonization technique have been presented in [19, 20].

We want to investigate a $1+1$ dimensional electron system in euclidean space-time, whose action is given by

$$S^0(\overline{\Psi},\Psi) \;=\; \int d^2\xi\;\overline{\Psi}(\xi)\,\gamma^\mu\partial_\mu\,\Psi(\xi) \quad . \tag{A.1}$$

Quantization is accomplished by using functional integrals. The field Ψ denotes a two-component Grassmann field, and $\overline{\Psi} := \Psi^* \gamma^0$, where Ψ^* is an independent Grassmann field. Choosing the chiral representation of the γ-matrices, i.e. ,

$$\gamma^0 = \sigma_1 \qquad \gamma^1 = \sigma_2 \qquad \gamma^5 = -i\gamma^0\gamma^1 = \sigma_3 \qquad ,$$

the two components $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ of Ψ are the antiholomorphic and holomorphic modes, resp., which are the euclidean analogues of left- and right-movers, resp. . We shall perturb the system by a current-current interaction of the form

$$V(\overline{\Psi}, \Psi) \;=\; -\frac{g}{2} \int d^2\xi \; j^\mu(\xi) j^\mu(\xi) \quad , \tag{A.2}$$

where the one dimensional currents, j^μ, are defined by

$$j^\mu(\xi) \;=\; -i : \overline{\Psi}(\xi)\gamma^\mu \Psi(\xi) : \quad \mu = 0, 1 \quad .$$

This is the so called (massless) Thirrring model, or, in the context of solid state physics, the Tomonaga-Luttinger model.
Before studying the interacting system, we establish the bosonization formulas for the free system. Identifying the fermionic current density j^μ with the bosonic expressions

$$j^\mu(\overline{\Psi}, \Psi; \xi) \;\leftrightarrow\; j^\mu(\varphi; \xi) \;=\; \frac{i}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \varphi(\xi) \quad , \tag{A.3}$$

one can reproduce the Green functions for the currents

$$\begin{aligned} \langle j^{\mu_1}(\xi_1) \cdots j^{\mu_n}(\xi_n) \rangle_F^0 \;&=\; \int \frac{\mathcal{D}(\overline{\Psi}, \Psi)}{Z_F} \, e^{-S^0(\overline{\Psi}, \Psi)} \, j^{\mu_1}(\xi_1) \cdots j^{\mu_n}(\xi_n) \\ &=\; \langle j^{\mu_1}(\varphi, \xi_1) \cdots j^{\mu_n}(\varphi, \xi_n) \rangle_B^0 \;=\; \int \frac{\mathcal{D}(\varphi)}{Z_B} \, e^{-\tilde{S}^0(d\varphi)} \, j^{\mu_1}(\varphi, \xi_1) \cdots j^{\mu_n}(\varphi, \xi_n) \quad , \end{aligned} \tag{A.4}$$

where the action $\tilde{S}(d\varphi)$ [1] of the free, massless bose field φ is given by

$$\tilde{S}^0(d\varphi) \;=\; \frac{1}{2} \int d^2\xi \; \partial_\mu \varphi(\xi) \partial_\mu \varphi(\xi) \;=\; \frac{1}{2} \int d\varphi(\xi) \wedge * \, d\varphi(\xi) \quad . \tag{A.5}$$

Here, we have introduced the one form $d\varphi \;=\; \partial_\mu \varphi \; d\xi^\mu$, and $*$ denotes the Hodge star operation. We use standard notations of the calculus with differential

[1] In this appendix, we write the action $S(\varphi)$ as a functional, $\tilde{S}(d\varphi)$, of derivatives of the bosonic field.

forms. In [20] one finds a brief summary of the main definitions. (Note that, in [20], we use slightly modified conventions; especially $\phi \to \varphi := \frac{\phi}{2\sqrt{\pi}}$ and $j'^{\mu} \to j^{\mu} = i j'^{\mu}$.)

In order to express the Green functions of the electric fields $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ in the bosonic theory, one has to introduce disorder fields $D(\underline{y}, \underline{q})$.
For non-zero integers, $\underline{q} = \{q^{(1)}, ..., q^{(n)}\}$, satisfying $\sum_{i=1}^{n} q^{(i)} = 0$, we define the one form

$$\Pi_{\underline{\eta},\underline{q}}(\xi) \;:=\; \sum_{i=1}^{n} \Pi_{\eta^{(i)},q^{(i)}}(\xi) \tag{A.6}$$

with

$$\begin{aligned} \Pi_{\eta^{(i)},q^{(i)}}(\xi) \;&=\; \Pi_{\nu}^{\eta^{(i)},q^{(i)}}(\xi)\, d\xi^{\nu} \;:=\; -\sqrt{\pi}\, q^{(i)} \left[* d\, \Delta^{-1}\, \delta^{(2)}_{\eta^{(i)}} \right](\xi) \\ &=\; \frac{1}{2\sqrt{\pi}} q^{(i)} \frac{\xi^{\mu} - \eta^{(i)\mu}}{|\xi - \eta^{(i)}|}\, \varepsilon_{\mu\nu} d\xi^{\nu} \qquad\qquad \mu, \nu = 0, 1 \quad . \end{aligned}$$

Here, we use that

$$\partial_{\mu}\partial_{\mu}\Delta^{-1}(\xi) \;=\; -\delta^{(2)}(\xi) \qquad \text{, i.e. ,} \qquad \Delta^{-1}(\xi) \;=\; -\frac{1}{2\pi} \ln|\xi| \quad . \tag{A.7}$$

In physics terminology, $\Pi_{\eta,q}$ is the vector potential of a magnetic vortex of charge q at the point $\eta = (\eta_0, \eta_1)$. The field strength is given by

$$d\Pi_{\eta,q}(\xi) \;=\; \sqrt{\pi}\, q * \delta(\xi - \eta) \quad ,$$

If D is a two-dimensional domain then

$$\frac{1}{\sqrt{\pi}} \int_D d\Pi_{\eta,q}(\xi) \;=\; \begin{cases} q & , \text{ if } \eta \in D \\ 0 & , \text{ else} \end{cases} \quad .$$

Locally on $\mathbf{R}^2 \setminus \{\eta\}$, the one form $\Pi_{\eta,q}$ can be written as the derivative of a function $\alpha_{\eta,q}(\xi)$, i.e. ,

$$\Pi_{\eta,q}(\xi) \;=\; d\,\alpha_{\eta,q}(\xi) \quad . \tag{A.8}$$

By identifying the euclidean space-time with the complex plane, $\xi = (\xi_0, \xi_1) \mapsto \overline{\xi} = i\xi_0 + \xi_1$, the function $\alpha_{\eta,q}(\xi)$ can be represented as

$$\alpha_{\eta,q}(\xi) \;=\; -\frac{q}{2\sqrt{\pi}} \arg(\overline{\xi} - \overline{\eta}) \quad . \tag{A.9}$$

Note that this function is not globally defined on $\mathbf{C}$; we identify the branch-cut with the negative, real axis.

The expectation value of a disorder field $D(\underline{\eta},\underline{q}) = \prod_{i=1}^{n} D(\eta^{(i)}, q^{(i)})$ is given by

$$\langle\, D(\underline{\eta},\underline{q})\,\rangle_B \;=\; \left\{ \frac{\int \mathcal{D}\varphi\; e^{-\tilde{S}(d\varphi+\Pi_{\underline{\eta},\underline{q}})}}{\int \mathcal{D}\varphi\; e^{-\tilde{S}(d\varphi)}} \right\}_{\rm ren} \quad . \tag{A.10}$$

On the r.h.s. of eq.(A.10), a multiplicative renormalization is necessary, in order to eliminate an infinite "self-energy".

Correlation functions involving disorder fields $D(\underline{\eta},\underline{q})$ and functionals, $\mathcal{F}(d\varphi,\varphi)$, depending on the bosonic field φ and its derivative $d\varphi$ are defined by

$$\langle\, D(\underline{\eta},\underline{q})\,\mathcal{F}(d\varphi,\varphi)\,\rangle_B \;=\; \left\{ \frac{\int \mathcal{D}\varphi\; \mathcal{F}(d\varphi+\Pi_{\underline{\eta},\underline{q}}\,,\,\varphi+\alpha_{\underline{\eta},\underline{q}})\; e^{-\tilde{S}(d\varphi+\Pi_{\underline{\eta},\underline{q}})}}{\int \mathcal{D}\varphi\; e^{-\tilde{S}(d\varphi)}} \right\}_{\rm ren} \quad . \tag{A.11}$$

For the electron fields, the following identifications hold :

$$\begin{aligned}
\psi_1(\xi) \;&\longleftrightarrow\; \tfrac{1}{(2\pi)^{\frac{1}{4}}}\, D(\xi,1)\; :e^{i\sqrt{\pi}\varphi(\xi)}: \;&=:\; \psi_1(\varphi;\xi)\,,\\
\psi_2(\xi) \;&\longleftrightarrow\; \tfrac{1}{(2\pi)^{\frac{1}{4}}}\, D(\xi,1)\; :e^{-i\sqrt{\pi}\varphi(\xi)}: \;&=:\; \psi_2(\varphi;\xi)\,,\\
\psi_1^*(\xi) \;&\longleftrightarrow\; \tfrac{1}{(2\pi)^{\frac{1}{4}}}\, D(\xi,-1)\; :e^{-i\sqrt{\pi}\varphi(\xi)}: \;&=:\; \psi_1^*(\varphi;\xi)\,,\\
\psi_2^*(\xi) \;&\longleftrightarrow\; \tfrac{1}{(2\pi)^{\frac{1}{4}}}\, D(\xi,-1)\; :e^{i\sqrt{\pi}\varphi(\xi)}: \;&=:\; \psi_2^*(\varphi;\xi)\,,
\end{aligned} \tag{A.12}$$

with the convention that, in a product composed of several ψ'_αs , we write the disorder fields to the left of all functionals depending on φ .

The normal order of the exponentials is defined by

$$:e^{i\int d^2\xi\,\varphi(\xi)f(\xi)}: \;=\; e^{i\int d^2\xi\,\varphi(\xi)f(\xi)}\; e^{\frac{1}{2}\int d^2\xi\, f(\xi)\left(\Delta+m_0^2\right)^{-1} f(\xi)} \quad , \tag{A.13}$$

where $\Delta = -\partial_\mu\partial_\mu$, f denotes a test function, and m_0 is a positive constant.

It is a staightforward calculation (cf.[19, 20]) to verify that

$$\langle \prod_{i=1}^{n} \psi^\sharp_{\beta_i}(\xi_i)\,\rangle_F^0 \;=\; \langle \prod_{i=1}^{n} \psi^\sharp_{\beta_i}(\varphi;\xi_i)\,\rangle_B^0 \quad .$$

Remark: We use the prescription that the function $\langle \prod_{i=1}^n \psi^\sharp_{\beta_i}(\xi_i) \rangle$ is the analytic continuation of the function $\langle \hat{T} \prod_{i=1}^n \psi^\sharp_{\beta_i}(\xi_i) \rangle$, where $\hat{T}$ denotes the time-ordering operator for Grassmann fields, i.e.,

$$\hat{T}\left[\psi^\sharp(\xi_1)\cdots\psi^\sharp(\xi_n) \right] := \mathrm{sgn}(\sigma)\, \psi^\sharp(\xi_{\sigma(1)})\cdots\psi^\sharp(\xi_{\sigma(n)}) \quad ,$$

where σ is the permutation such that $\xi^0_{\sigma(1)} > \xi^0_{\sigma(2)} > \cdots > \xi^0_{\sigma(n)}$.

We shall make use of the following basic identities (cf. [19]) :

$$\int \frac{\mathcal{D}\varphi}{Z}\, e^{-\frac{1}{2}\int d\xi\, \partial_\mu\varphi\partial_\mu\varphi} \prod_{j=1}^n : e^{i\varepsilon_j\varphi(\xi_j)} : =$$

$$= \begin{cases} 0 & , \text{ if } \sum_{j=1}^n \varepsilon_j \neq 0 \,, \\ \prod_{i=1}^n c(m_0)^{\frac{\varepsilon_j^2}{4\pi}}\, e^{-\frac{1}{2\pi}\sum_{1\le i\le n} \varepsilon_i\varepsilon_j \ln(\frac{1}{|\xi_i-\xi_j|})} & , \text{ if } \sum_{j=1}^n \varepsilon_j = 0 \,, \end{cases}$$

where the constant m_0 can be choosen such that $c(m_0) = \frac{1}{2\pi}$,

$$-\frac{1}{2}\int d^2\zeta\, \Pi^\mu_{\underline{\xi},\underline{\eta}}(\zeta)\Pi^\mu_{\underline{\xi},\underline{\eta}}(\zeta) =$$

$$= -\frac{\pi}{2}\left[\sum_{i=1}^n \Delta^{-1}(\xi_i,\xi_i) + \sum_{i=1}^n \Delta^{-1}(\eta_i,\eta_i) \right]$$

$$-\frac{1}{2}\left[\sum_{1\le i<j\le n} \ln|\xi_i-\eta_j| - \sum_{1\le i<j\le n} \ln|\xi_i-\xi_j| - \sum_{1\le i<j\le n} \ln|\eta_i-\eta_j| \right] ,$$

and

$$\int d^2\zeta\, \Pi^\mu_{\underline{\xi},\underline{\eta}}(\zeta)\, \partial_\mu\varphi(\zeta) = 0 \quad ,$$

where the vortices located at the points $\underline{\xi} := (\xi_1,\cdots,\xi_n)$ carry the charge $q=-1$ and the ones located at the points $\underline{\eta} := (\eta_1,\cdots,\eta_n)$ carry the charge $q=1$.

As an application, we calculate the Green function $\langle\psi_2^*(\xi)\psi_2(\eta)\rangle^0$ for the free system. Let $\xi_0 > \eta_0$. Then

$$\begin{aligned} \langle\, \psi_2^*(\xi)\psi_2(\eta)\, \rangle^0_F &= \langle\, \psi_2^*(\varphi;\xi)\psi_2(\varphi;\eta)\, \rangle^0_B \\ &= \frac{1}{\sqrt{2\pi}}\langle\, D(\xi,-1)D(\eta,1) : e^{i\sqrt{\pi}\varphi(\xi)} :: e^{-i\sqrt{\pi}\varphi(\eta)} :\, \rangle^0_B \\ &= \frac{1}{\sqrt{2\pi}}\Big[\, e^{i\sqrt{\pi}\left(\alpha_{\xi,\eta}(\xi)-\alpha_{\xi,\eta}(\eta)\right)} \ . \end{aligned}$$

$$\cdot \int \frac{\mathcal{D}\varphi}{Z}\, e^{-\tilde{S}^0\left(d\varphi+\Pi_{\xi,\eta},d\varphi+\Pi_{\xi,\eta}\right)} \; : e^{i\sqrt{\pi}\varphi(\xi)} :: e^{-i\sqrt{\pi}\varphi(\eta)} : \Big]_{\rm ren}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{i}{2}[\arg(\overline{\eta}-\overline{\xi})+\arg(\overline{\xi}-\overline{\eta})]}\, e^{-\frac{1}{2}\ln|\xi-\eta|} \; .$$

$$\cdot \int \frac{\mathcal{D}\varphi}{Z}\, e^{-\tilde{S}^0(d\varphi,d\varphi)} \; : e^{i\sqrt{\pi}\varphi(\xi)} :: e^{-i\sqrt{\pi}\varphi(\eta)} :$$

$$= \frac{1}{2\pi}\, \frac{i}{i(\xi_0-\eta_0)+(\xi_1-\eta_1)} \quad . \tag{A.14}$$

One observes that

$$\langle\, \psi_2(\eta)\psi_2^*(\xi)\,\rangle \;=\; -\,\langle\, \psi_2^*(\xi)\psi_2(\eta)\,\rangle \quad .$$

Similarly one obtains

$$\langle\, \psi_1^*(\xi)\psi_1(\eta)\,\rangle \;=\; \frac{1}{2\pi}\, \frac{-i}{-i(\xi_0-\eta_0)+(\xi_1-\eta_1)} \tag{A.15}$$

and

$$\langle\, \psi_1(\xi)\psi_2(\eta)\,\rangle \;=\; \langle\, \psi_1(\xi)\psi_2^*(\eta)\,\rangle \;=\; \langle\, \psi_2(\xi)\psi_1^*(\eta)\,\rangle \;=\; \langle\, \psi_1^*(\xi)\psi_2^*(\eta)\,\rangle \;=\; 0 \quad .$$

These are the correct formulas for the free system (compare, for example appendix D of [15], where the same conventions are used).

One can show that the same bosonization identities hold for the interacting system, with the interaction V given by eq.(A.3) (cf. [20]). The only difference is that expectation values of bosonic operators are taken with respect to the interacting system, i.e.

$$\langle\, (\,\cdot\,)\,\rangle_B^V \;:=\; \int \frac{\mathcal{D}\varphi}{Z_V}\, e^{-[\,\frac{1}{2}\int d\xi\, \partial_\mu\varphi(\xi)\partial_\mu\varphi(\xi)\,+\,V(d\varphi)\,]}\,(\,\cdot\,) \quad , \tag{A.16}$$

where the functional $V(\varphi)$ is obtained from the functional $V(\overline{\psi};\psi)$ by using the identities (A.3), i.e. ,

$$V(d\varphi) \;=\; \frac{g}{2\pi}\int d^2\xi\; \partial_\mu\varphi(\xi)\partial_\mu\varphi(\xi) \quad . \tag{A.17}$$

We repeat the calculation of the Green function $\langle\psi_2^*(\xi)\psi_2(\eta)\rangle^V$ for the interacting system :

$$\begin{aligned}\langle\, \psi_2^*(\xi)\psi_2(\eta)\,\rangle_F^V \;&=\; \langle\, \psi_2^*(\varphi;\xi)\psi_2(\varphi;\eta)\,\rangle_B^V \\ &=\; \frac{1}{\sqrt{2\pi}}\langle\, D(\xi,-1)D(\eta,1) : e^{i\sqrt{\pi}\varphi(\xi)} :: e^{-i\sqrt{\pi}\varphi(\eta)} :\,\rangle_B^V\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \Bigg[\int \frac{\mathcal{D}\varphi}{Z_V} \, e^{-(1+\frac{g}{\pi})\tilde{S}^0\left(d\varphi+\Pi_{\xi,\eta},d\varphi+\Pi_{\xi,\eta}\right)} \, . \\
&\qquad \cdot \; : e^{i\sqrt{\pi}\left(\varphi(\xi)+\alpha_{\xi,\eta}(\xi)\right)} :: e^{-i\sqrt{\pi}\left(\varphi(\eta)+\alpha_{\xi,\eta}(\eta)\right)} : \Bigg]_{\text{ren}} \\
&= \frac{i}{\sqrt{2\pi}} \, e^{-i\arg(\bar{\xi}-\bar{\eta})} \, e^{-\frac{1}{2}(1+\frac{g}{\pi})\ln|\xi-\eta|} \, . \\
&\qquad \cdot \int \frac{\mathcal{D}\varphi}{Z_V} \, e^{-(1+\frac{g}{\pi})\tilde{S}^0(d\varphi,d\varphi)} \; : e^{i\sqrt{\pi}\varphi(\xi)} :: e^{-i\sqrt{\pi}\varphi(\eta)} : \\
&= \frac{i}{2\pi} \, e^{-i\arg(\bar{\xi}-\bar{\eta})} \, \frac{1}{|\xi-\eta|^{1+\beta}} \quad , \qquad \text{(A.18)}
\end{aligned}
$$

where

$$
\beta = \frac{2\left(\frac{2g}{\pi}\right)^2}{1+2\left(\frac{2g}{\pi}\right)^2} \quad .
$$

Similarly,

$$
\langle \, \psi_1^*(\xi)\psi_1(\eta) \, \rangle^V = \frac{-i}{2\pi} \, e^{i\arg(\bar{\xi}-\bar{\eta})} \, \frac{1}{|\xi-\eta|^{1+\beta}} \quad . \qquad \text{(A.19)}
$$

The exponent β describes the decay of the elecron propagator for large arguments. The dependence on the coupling constant is a characteristic feature of Luttinger Liquids.

Appendix B : Consistency of bosonization formulas

In this appendix, we show on the example of $\langle \, \bar{\jmath}_{[\omega_1]}(\xi) \; \bar{\jmath}_{[\omega_2]}(\eta) \, \rangle^0$ that the bosonization formulas (70) – (73) for the electron fields $\psi^\sharp_\omega$ imply the bosonization formulas (46) – (48) for the electron currents $j_{[\omega]}$ and $\bar{\jmath}_{[\omega]}$. The generalization that this statement holds for arbitrary products of current densities evaluated in the non-interacting ground state is staightforward.

Remark :
The calculation of expectation values of products of current densities in the *interacting* ground state is organized as a perturbation expansion in powers of the set of coupling constants $g_n(\vec{\omega}\cdot\vec{\omega}')$ and the inverse scale parameter λ_n^{-1}. For systems with interactions of the form given in eq.(53) (i.e. interactions which can

be expressed in terms of the current densities $j_{[\omega]}$ and $\bar{j}_{[\omega]}$, cf. eq.(56)) , such an expansion reduces the calculation to the task of evaluating products of current densities $j_{[\omega]}$ and $\bar{j}_{[\omega]}$ in the non-interacting ground-state – where our statement applies.

By inserting eq.(38), one obtains for the expression $\langle\, \bar{j}_{[\omega_1]}(\xi)\, \bar{j}_{[\omega_2]}(\eta)\, \rangle^0$:

$$
\begin{aligned}
\langle\, \bar{j}_{[\omega_1]}(\xi)\, \bar{j}_{[\omega_2]}(\eta)\, \rangle^0 \;=\; & \\
\sum_{\omega_1',\omega_2' \in \mathcal{S}^+} \Big\{ & e^{ik_F\lambda(\vec{\omega}_1-\vec{\omega}_1')\vec{\xi}}\, e^{ik_F\lambda(\vec{\omega}_2-\vec{\omega}_2')\vec{\eta}}\, \langle\, :\psi^*_{\omega_1'}(\xi)\psi_{\omega_1}(\xi): :\psi^*_{\omega_2'}(\eta)\psi_{\omega_2}(\eta):\, \rangle^0 \\
& + e^{ik_F\lambda(\vec{\omega}_1-\vec{\omega}_1')\vec{\xi}}\, e^{ik_F\lambda(\vec{\omega}_2'-\vec{\omega}_2)\vec{\eta}}\, \langle\, :\psi^*_{\omega_1'}(\xi)\psi_{\omega_1}(\xi): :\psi^*_{\omega_2}(\eta)\psi_{\omega_2'}(\eta):\, \rangle^0 \\
& + e^{ik_F\lambda(\vec{\omega}_1'-\vec{\omega}_1)\vec{\xi}}\, e^{ik_F\lambda(\vec{\omega}_2-\vec{\omega}_2')\vec{\eta}}\, \langle\, :\psi^*_{\omega_1}(\xi)\psi_{\omega_1'}(\xi): :\psi^*_{\omega_2'}(\eta)\psi_{\omega_2}(\eta):\, \rangle^0 \\
& + e^{ik_F\lambda(\vec{\omega}_1'-\vec{\omega}_1)\vec{\xi}}\, e^{ik_F\lambda(\vec{\omega}_2'-\vec{\omega}_2)\vec{\eta}}\, \langle\, :\psi^*_{\omega_1}(\xi)\psi_{\omega_1'}(\xi): :\psi^*_{\omega_2}(\eta)\psi_{\omega_2'}(\eta):\, \rangle^0 \Big\} \\
= & \\
\Big\{ & e^{ik_F\lambda(\vec{\omega}_1-\vec{\omega}_2)(\vec{\xi}-\vec{\eta})}\, \langle\, \psi^*_{\omega_2}(\xi)\psi_{\omega_2}(\eta)\, \rangle^0\, \langle\, \psi_{\omega_1}(\xi)\psi^*_{\omega_1}(\eta)\, \rangle^0 \\
& + \delta_{\omega_1,\omega_2} \sum_{\omega_1' \in \mathcal{S}^+} e^{ik_F\lambda(\vec{\omega}_1-\vec{\omega}_1')(\vec{\xi}-\vec{\eta})}\, \langle\, \psi^*_{\omega_1'}(\xi)\psi_{\omega_1'}(\eta)\, \rangle^0\, \langle\, \psi_{\omega_1}(\xi)\psi^*_{\omega_1}(\eta)\, \rangle^0 \\
& + \delta_{\omega_1,\omega_2} \sum_{\omega_1' \in \mathcal{S}^+} e^{ik_F\lambda(\vec{\omega}_1'-\vec{\omega}_1)(\vec{\xi}-\vec{\eta})}\, \langle\, \psi^*_{\omega_1}(\xi)\psi_{\omega_1}(\eta)\, \rangle^0\, \langle\, \psi_{\omega_1'}(\xi)\psi^*_{\omega_1'}(\eta)\, \rangle^0 \\
& + e^{ik_F\lambda(\vec{\omega}_2-\vec{\omega}_1)(\vec{\xi}-\vec{\eta})}\, \langle\, \psi^*_{\omega_1}(\xi)\psi_{\omega_1}(\eta)\, \rangle^0\, \langle\, \psi_{\omega_2}(\xi)\psi^*_{\omega_2}(\eta)\, \rangle^0 \Big\}
\end{aligned}
$$

In order to simplify these expressions we can apply a "discretized" version of lemma (3.21) in [15] :

Lemma :

For a decomposition of the surface $\mathcal{S}_1^{d-1}$ of the d-dimensional unit sphere (with $d > 1$) into congruent, quadratic patches with sides of length $\frac{1}{\lambda}$ we define the set $\mathcal{M}$ of d-dimensional unit vectors $\vec{\omega}_i$, $i = 1, ..., N = \frac{Vol(\mathcal{S}_1^{d-1})}{\lambda^{d-1}}$ pointing to the centers of these patches. Given a vector $\vec{\sigma}_j \in \mathcal{M}$ and some test function $f(\vec{\omega}_i; \vec{\xi})$, one finds, in the limit $\lambda \to \infty$, the following asymptotic formula :

$$\int_{\mathbf{R}^d} d^d\xi \sum_{\vec{\omega}_i \in \mathcal{M}} f(\vec{\omega}_i;\vec{\xi}) \left[\delta^{d-1}_{\omega_i,k_F}\left(\vec{\xi}_\perp(\vec{\omega}_i)\right) \; e^{ik_F\lambda\vec{\xi}(\vec{\omega}_i-\vec{\sigma}_j)} \right]$$

$$= \int_{\mathbf{R}^d} d^d\xi \sum_{\vec{\omega}_i \in \mathcal{M}} f(\vec{\omega}_i;\vec{\xi}) \left[\delta_{\omega_i,\sigma_j} \, \delta^{d-1}_{\sigma_j,\Lambda}\left(\vec{\xi}_\perp(\vec{\sigma}_j)\right) \quad + \quad O(\frac{1}{\lambda}) \right] , \quad \text{(B.1)}$$

where

$$\vec{\xi}_\perp(\vec{\sigma}) \; := \; \vec{\xi} - (\vec{\xi}\cdot\vec{\sigma})\,\vec{\sigma}$$

and

$$\delta^{d-1}_{\sigma_j,\Lambda}\left(\vec{\xi}_\perp(\vec{\sigma}_j)\right) \; := \int_{[\frac{-\Lambda}{2},\frac{\Lambda}{2}]^{d-1}} \bar{d}^{d-1}k_\perp \; e^{i\vec{k}_\perp\vec{\xi}_\perp(\vec{\sigma}_j)} \quad .$$

Here, $\Lambda \ll \lambda k_F$ denotes an arbitrary momentum which can be chosen to be equal to k_F.

The proof of this lemma is analoguous to the proof of the continuous version of the lemma presented in appendix C of [15] .
By applying this lemma to the expressions appearing above and by inserting the bosonization formulas (70) – (73) for the electron fields $\psi^{\sharp}_{\omega}$, we obtain the following formula :

$$\langle\, \bar{\jmath}_{[\omega_1]}(\xi)\, \bar{\jmath}_{[\omega_2]}(\eta)\, \rangle^0 \;\approx\; -4\,(\frac{k_F}{2\pi})^{d-1}\, \delta_{\omega_1,\omega_2}\, \delta^{d-1}_{\omega_1,k_F}\left(\vec{\xi}_\perp(\vec{\omega}_1)\right) \cdot$$

$$\langle\, \psi^*_{\omega_1||}(\xi_0,\xi_{||};\varphi^{||}_{[\omega_1]})\psi_{\omega_1||}(\eta_0,\eta_{||};\varphi^{||}_{[\omega_1]})\, \rangle_{S^0(\varphi^{||}_{[\omega_1]})}\, \langle\, \psi_{\omega_1||}(\xi_0,\xi_{||};\varphi^{||}_{[\omega_1]})\psi^*_{\omega_1||}(\eta_0,\eta_{||};\varphi^{||}_{[\omega_1]})\, \rangle_{S^0(\varphi^{||}_{[\omega_1]})} \,, \quad \text{(B.2)}$$

where $\psi^{\sharp}_{\omega||}(\xi_0,\xi_{||};\varphi^{||}_{[\omega]})$ is a short-hand notation for the bosonized expressions of the radial electron fields $\psi^{\sharp}_{\omega||}(\xi_0,\xi_{||})$ specified in eq.(72). The explicit form of the radial electron propagators is given in eqs. (A.14) and (A.15).
It is a straightforward calculation to verify that eq.(B.2) is identical to the following formula:

$$\langle\, \bar{\jmath}_{[\omega_1]}(\xi)\, \bar{\jmath}_{[\omega_2]}(\eta)\, \rangle^0 \;\approx$$

$$-4\,(\frac{k_F}{2\pi})^{d-1}\, \delta_{\omega_1,\omega_2}\, \delta^{d-1}_{\omega_1,k_F}\left(\vec{\xi}_\perp(\vec{\omega}_1)\right) \cdot \left\langle\, \overline{\partial}^{\omega_1}\varphi^{||}_{[\omega_1]}(\xi_0,\xi_{||})\, \overline{\partial}^{\omega_2}\varphi^{||}_{[\omega_2]}(\eta_0,\eta_{||})\, \right\rangle_{S^0(\varphi^{||}_{[\omega_1]})}$$

$$= \left(\frac{-2}{\sqrt{\pi}}(\frac{k_F}{2\pi})^{\frac{d-1}{2}}\right)^2 \left\langle\, \overline{\partial}^{\omega_1}\varphi_{[\omega_1]}(\xi)\, \overline{\partial}^{\omega_2}\varphi_{[\omega_2]}(\eta)\, \right\rangle_{S^0(\{\varphi_{[\omega_1]}\})} \quad , \quad \text{(B.3)}$$

where we use eq.(A.7) for the propagator of the radial bose field $\varphi^{\|}_{[\omega]}(\xi_0, \xi_{\|})$, and eq.(75) to deduce the second part of the equation.
Formula (B.3) reproduces the bosonization identity for the current density $\bar{j}_\omega$, cf. eqs. (46) - (48).
Hence, we have verified our claim thus showing the consistency of the bosonization formulas for the electron fields and the current densities.

Appendix C : Calculation of the Electron Propagator for a System with Long-Range, Transverse Current-Current Interactions

In order to apply eq.(80) to the calculation of the propagator

$$G^{\|\alpha}_{\omega,n}(\xi_0 - \eta_0, \xi_{\|} - \eta_{\|}) \;:=\; -\left\langle\, \psi_{\omega\alpha}(\xi_0, \xi_{\|})\; \psi^*_{\omega\alpha}(\eta_0, \eta_{\|})\,\right\rangle^{\|}_n$$

of the radial electron field $\psi^{\|}_{\omega\alpha}$, we have to determine the effective action $S^T_n(\varphi^{\|}_{[\omega]})$ of the bosonic field $\varphi^{\|}_{[\omega]}$ propagating along the direction $[\omega]$.

The propagator $\langle\, \hat{\varphi}_{[\omega]}(k)\; \hat{\varphi}_{[\omega]}(k')\,\rangle_n$ of the bosonized interacting system is given in eq.(88). It can be written as

$$\langle\, \hat{\varphi}_{[\omega]}(k)\; \hat{\varphi}_{[\omega]}(k')\,\rangle_n \;=\; (2\pi)^{d+1}\,\delta^{(d+1)}(k+k')\;\frac{1}{k_0^2 + (v_{Fn}\vec{\omega}\vec{k})^2}\;\mathcal{K}^n_{[\omega]}(k) \quad , \qquad \text{(C.1)}$$

with

$$\mathcal{K}^n_{[\omega]}(k) \;=$$

$$\frac{1 + \frac{2}{\lambda_n^{d-1}}\frac{1}{2\pi}(\frac{k_F}{2\pi})^{d-1} g_n \hat{V}(|\frac{\vec{k}}{\lambda_n}|)\left[\frac{1}{d-1}\sum\limits_{\omega'\in\mathcal{S}+}\frac{k_0^2}{k_0^2+(v_{Fn}\vec{\omega}'\vec{k})^2}\left((\vec{\omega}')^2 - \frac{(\vec{\omega}'\vec{k})^2}{\vec{k}^2}\right)(1-\delta_{\omega,\omega'})\right]}{1 + \frac{2}{\lambda_n^{d-1}}\frac{1}{2\pi}(\frac{k_F}{2\pi})^{d-1} g_n \hat{V}(|\frac{\vec{k}}{\lambda_n}|)\frac{1}{d-1}\sum\limits_{\omega'\in\mathcal{S}+}\frac{k_0^2}{k_0^2+(v_{Fn}\vec{\omega}'\vec{k})^2}\left((\vec{\omega}')^2 - \frac{(\vec{\omega}'\vec{k})^2}{\vec{k}^2}\right)} \qquad \text{(C.2)}$$

One observes that the function $\mathcal{K}^n_{[\omega]}(k)$ is bounded by $\;0 \le \mathcal{K}^n_{[\omega]}(k) \le 1$, for arbitrary k .
In explicit calculations, expression (C.2) for $\mathcal{K}^n_{[\omega]}(k)$ is not convenient. We replace

it by

$$\tilde{\mathcal{K}}^n_{[\omega]}(k) \;=\; \frac{1+g_n\hat{V}(|\frac{\vec{k}}{\lambda_n}|)\,\mathbf{\Pi}^t_n(k)}{1+g_n\hat{V}(|\frac{\vec{k}}{\lambda_n}|)\left[\mathbf{\Pi}^t_n(k)+\frac{2}{\lambda_n^{d-1}}\frac{1}{2\pi}(\frac{k_F}{2\pi})^{d-1}\frac{k_0^2}{k_0^2+(v_{Fn}\vec{\omega}\vec{k})^2}\left((\vec{\omega})^2-\frac{(\vec{\omega}\vec{k})^2}{\vec{k}^2}\right)\right]} \tag{C.3}$$

which has the same asymptotics for small momenta k as $\mathcal{K}^n_{[\omega]}(k)$. The function $\mathbf{\Pi}^t_n(k)$ has been defined in eq.(45), with $v_F \to v_{Fn}$.
The action $S^T_n(\varphi_{[\omega]})$ of the bosonic field $\hat{\varphi}_{[\omega]}(k)$ which reproduces the propagator (C.1) – but with the function $\mathcal{K}(k)$ replaced by $\tilde{\mathcal{K}}(k)$ – is given by

$$S^T_n(\varphi_{[\omega]}) \;=\; \frac{1}{2}\int_{\mathbf{R}\times\overline{Q}^d} đ^{d+1}k\;\hat{\varphi}_{[\omega]}(-k)\;\left[\left(k_0^2+(v_{Fn}\vec{\omega}\vec{k})^2\right)\;\left(\tilde{\mathcal{K}}^n_{[\omega]}(k)\right)^{-1}\right]\;\hat{\varphi}_{[\omega]}(k) \quad . \tag{C.4}$$

We define

$$\left(\tilde{\mathcal{K}}^n_{[\omega]}(k)\right)^{-1} \;:=\; 1+T^n_{[\omega]}(k) \quad , \tag{C.5}$$

with

$$T^n_{[\omega]}(k) \;:=\; \frac{\frac{2}{\lambda_n^{d-1}}\frac{1}{2\pi}(\frac{k_F}{2\pi})^{d-1}\frac{k_0^2}{k_0^2+(v_{Fn}\vec{\omega}\vec{k})^2}\left((\vec{\omega})^2-\frac{(\vec{\omega}\vec{k})^2}{\vec{k}^2}\right)}{g_n^{-1}\hat{V}^{-1}(|\frac{\vec{k}}{\lambda_n}|)+\mathbf{\Pi}^t_n(k)} \quad .$$

The effective action $S^T_n(\varphi^{||}_{[\omega]})$ of the modes propagating along the direction $[\omega]$ is deduced from the action $S^T_n(\varphi_{[\omega]})$ by averaging over the momenta perpendicular to $[\omega]$. One obtains :

$$S^T_n(\varphi^{||}_{[\omega]}) \;=\; \frac{1}{2}\int_{\mathbf{R}} đk_0\int_{\frac{-k_F}{2}}^{\frac{k_F}{2}} đk_{||}\;\hat{\varphi}_{[\omega]}(-k)\;\left[\left(k_0^2+(v_{Fn}k_{||})^2\right)\left(1+t^n_{[\omega]}(k_0,k_{||})\right)\right]\;\hat{\varphi}_{[\omega]}(k) \quad , \tag{C.6}$$

with

$$t^n_{[\omega]}(k_0,k_{||}) \;:=\; \frac{1}{k_F^{d-1}}\int_{\overline{Q}^{d-1}} đ^{d-1}k_\perp\;T^n_{[\omega]}(k) \quad .$$

For small momenta $(k_0,k_{||})$ and a large scale factor λ_n, the calculation yields the following asymptotic formula for $t^n_{[\omega]}(k_0,k_{||})$:

$$t^n_{[\omega]}(k_0,k_{||}) \quad\sim\quad \tilde{t}^n_{[\omega]}(k_0,k_{||})$$

$$:=\begin{cases} a(d)\,\frac{k_0^2}{k_0^2+(v_{Fn}k_{||})^2}\,\frac{k_F}{|k_0|}\,\frac{1}{\lambda_n^{d-1}} & , \text{ for } 1\gg|c_nk_0|\gg\frac{1}{\lambda_n^\alpha} \\ a(d)\,\frac{k_0^2}{k_0^2+(v_{Fn}k_{||})^2}\,k_Fc_n\,\lambda_n^{\alpha-(d-1)}\,\frac{2}{1+\alpha}\,\cdot & \\ \cdot\,\Gamma(\frac{d}{1+\alpha})\,\Gamma(\frac{1+\alpha-d}{1+\alpha})\left[(\frac{\lambda_n^\alpha c_n|k_0|}{\pi^{d-1}2^{d-2}})^{\frac{d-1-\alpha}{1+\alpha}}-1\right] & , \text{ for } \frac{1}{\lambda_n^\alpha}\gg|c_nk_0| \end{cases} , \tag{C.7}$$

where $d = 2, 3$ is the dimension of space, $a(d)$ is a positive constant of order unity, and the constants $c_n := \frac{v_{Fn}^2}{k_F^{d-1}} g_n > 0$ and α, with $0 \leq \alpha \leq 2$, parametrize the interaction potential : $g_n \hat{V}(|\vec{p}|) = c_n (\frac{k_F}{|\vec{p}|})^\alpha$.
For $1 \gg |c_n k_0| \gg \frac{1}{\lambda_n^\alpha}$, one observes that $\tilde{t}^n$ takes values between $\frac{1}{\lambda_n^{d-1}}$ and $\lambda_n^{\alpha-(d-1)}$.
If $\alpha < d-1$, it follows that the interactions can be neglected in the limit $\lambda_n \to \infty$. If $\alpha > d-1$, the interactions become important as soon as k_0 is of order $\frac{1}{\lambda_n^{d-1}}$ or smaller.
In the range $\frac{1}{\lambda_n^\alpha} \gg |c_n k_0|$, eq.(C.7) can be rewritten as

$$\tilde{t}^n_{[\omega]}(k_0, k_{||}) = c(d) \frac{k_0^2}{k_0^2 + (v_{Fn} k_{||})^2} \cdot$$

$$\cdot \begin{cases} b(d,\alpha) \frac{3}{1+\alpha} \frac{\pi}{\sin(\pi \frac{d}{1+\alpha})} k_F c_n^{\frac{d}{1+\alpha}} \left(\frac{|k_0|}{\lambda_n}\right)^{\frac{d-1-\alpha}{1+\alpha}} & , \text{ for } \alpha > d-1 \\ -\frac{2}{1+\alpha} k_F c_n \ln\left(\frac{\lambda_n^{d-1} g_n |k_0|}{k_F \pi^{d-1} 2^{d-2}}\right) & , \text{ for } \alpha = d-1 \\ b(d,\alpha)\, k_F c_n\, \lambda_n^{\alpha-(d-1)} & , \text{ for } d-1 > \alpha \geq 0 \end{cases} . \tag{C.8}$$

For $\alpha < d-1$, the effect of the interactions is suppressed by a factor $\frac{1}{\lambda_n^{d-1-\alpha}}$. For $\alpha \geq d-1$, one observes a singular k_0 - dependence of the function $\tilde{t}$.

Using the methods presented in appendix A, the calculation of the radial electron propagators is straightforward. Following eq.(80), the large-distance asymptotics of, e.g., the propagator $G^{||1}_{\omega,n}$ is obtained by evaluating the expression

$$\langle\, \psi^*_{\omega 1}(\zeta)\, \psi_{\omega 1}(0)\, \rangle^{||}_n \sim \frac{1}{\sqrt{2\pi}} \Big[e^{-i\sqrt{v_{Fn}\pi}\left(\alpha^{[\omega]}_{\zeta,0}(\zeta) - \alpha^{[\omega]}_{\zeta,0}(0)\right)} \cdot$$
$$\cdot \int \mathcal{D}\varphi^{||}_{[\omega]}\, e^{-\tilde{S}^T_n(\partial_\mu \varphi^{||}_{[\omega]} + \Pi^{\zeta,0}_{[\omega]\mu})} \,:\, e^{-i\sqrt{v_{Fn}\pi}\varphi^{||}_{[\omega]}(\zeta)} \,:\, :\, e^{i\sqrt{v_{Fn}\pi}\varphi^{||}_{[\omega]}(0)} \,:\, \Big] , \tag{C.9}$$

with $\zeta = (\xi_0, \frac{\xi_{||}}{v_{Fn}})$. For each ray $[\omega]$, the vector potential $\Pi^{\zeta,0}_{[\omega]\mu}$ and the function $\alpha^{[\omega]}_{\zeta,0}$ are given in eqs.(A.6) and (A.9) in appendix A. However, in contrast to appendix A, we here display the Fermi velocity v_{Fn} explicitly. Then, on the rhs. of eqs.(A.6) and (A.9), there is an additional factor $\frac{1}{\sqrt{v_{Fn}}}$. Note also the supplementary factor $\sqrt{v_{Fn}}$ in the exponent of the vertex operators.

We again write the action $S^T_n(\varphi^{||}_{[\omega]})$ as a functional, $\tilde{S}^T_n(\partial_\mu \varphi^{||}_{[\omega]})$, of the derivatives $\partial_\mu \varphi^{||}_{[\omega]}$.

One has that

$$\tilde{S}_n^T(\partial_\mu \varphi_{[\omega]}^{\|} + \Pi_{\zeta,0}^{[\omega]\mu}) \;=\; S_n^T(\varphi_{[\omega]}^{\|}) \;+\; \tilde{S}_n^T(\Pi_{\zeta,0}^{[\omega]}) \quad ,$$

where the action $S_n^T(\varphi_{[\omega]}^{\|})$ is defined in eq.(C.7). The contribution $\tilde{S}_n^T(\Pi_{\zeta,0}^{[\omega]})$ of the disorder fields is given by

$$\tilde{S}_n^T(\Pi_{\zeta,0}^{[\omega]}) \;=\; -\pi \int_{\mathbf{R}} đk_0 \int_{\frac{-k_F}{2}}^{\frac{k_F}{2}} đk_{\|} \, [e^{ik\zeta} - 1] \, \frac{v_{Fn}}{k_0^2 + (v_{Fn}k_{\|})^2} \, \left(1 + t_{[\omega]}^n(k)\right) \quad , \text{(C.10)}$$

and the contribution of the vertex-operators in eq.(C.9) amounts to

$$\int \frac{\mathcal{D}\varphi_{[\omega]}^{\|}}{Z_n^T} \, e^{-S_n^T(\partial_\mu \varphi_{[\omega]}^{\|})} \; : e^{-i\sqrt{v_{Fn}\pi}\varphi_{[\omega]}^{\|}(\zeta)} : : e^{i\sqrt{v_{Fn}\pi}\varphi_{[\omega]}^{\|}(0)} : =$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left[\pi \int_{\mathbf{R}\times k_F} đ^2k \, [e^{ik\zeta} - 1] \, \left(\frac{v_{Fn}}{k_0^2 + (v_{Fn}k_{\|})^2} \, \frac{1}{1 + t_{[\omega]}^n(k)}\right)\right] \quad . \text{(C.11)}$$

We display the behaviour of the fermionic Green function $G_{\omega,n}^{\|1}(\zeta)$ in the following two regions :

$$\mathrm{I} \;=\; \left\{ \zeta = (\xi_0, \frac{\xi_{\|}}{v_{Fn}}) \;:\; c_n \lambda_n^\alpha \ll |\zeta| \right\} ,$$

$$\mathrm{II} \;=\; \left\{ \zeta = (\xi_0, \frac{\xi_{\|}}{v_{Fn}}) \;:\; |\zeta| \ll c_n \lambda_n^\alpha \text{ and } |\xi_0| \ll |\frac{\xi_{\|}}{v_{Fn}}| \right\} .$$

In region I , the propagator $G_{\omega,n}^{\|1}(\zeta)$ is obtained by evaluating the integrals in eqs. (C.10) and (C.11) by using the asymptotic form (C.9) :

$$\langle \, \psi_{\omega 1}^*(\zeta) \, \psi_{\omega 1}(0) \, \rangle_n^{\|} \quad \sim \quad (\frac{-i}{2\pi}) \, e^{i \arg(i\xi_0 + \frac{\xi_{\|}}{v_{Fn}})} \; .$$

$$
\cdot \begin{cases}
\exp\Bigg(- \overline{u}_{\rm I} k_F \, c_n^{\frac{d}{1+\alpha}} \lambda_n^{\frac{\alpha-(d-1)}{1+\alpha}} \cdot \\
\quad \cdot \Bigg\{ |\zeta|^{\frac{\alpha-(d-1)}{1+\alpha}} \frac{1+\alpha}{\alpha-(d-1)} \cos\left[\frac{\alpha-(d-1)}{1+\alpha} \arctan(|\frac{v_{Fn}\xi_0}{\xi_\parallel}|)\right] - \\
\qquad - |\frac{\xi_\parallel}{v_{Fn}}| |\zeta|^{\frac{-d}{1+\alpha}} \cos\left[\frac{-d}{1+\alpha} \arctan(|\frac{v_{Fn}\xi_0}{\xi_\parallel}|)\right] \Bigg\}\Bigg) \quad , & 2 \geq \alpha > d-1 \\
\\
\exp\left(-v\, k_F c_n \ln^2 |\zeta|\right) \quad , & \alpha = d-1 \\
\\
\frac{1}{\sqrt{|v_{Fn}\zeta|}} \left[(1 + w \frac{k_F c_n}{\lambda_n^{d-1-\alpha}})\xi_\parallel^2 + (v_{Fn}\xi_0)^2\right]^{-\frac{1}{4}(\sqrt{1+w\frac{k_F c_n}{\lambda_n^{d-1-\alpha}}})^{-1}} \cdot \\
\qquad \cdot \left[\xi_\parallel^2 + (v_{Fn}\xi_0)^2\right]^{-\frac{w}{8}\frac{k_F c_n}{\lambda_n^{d-1-\alpha}}} \quad , & d-1 > \alpha > 0
\end{cases} \quad ,
$$

(C.12)

where

$$
\overline{u}_{\rm I} \;=\; c(d,\alpha)\,\frac{2}{1+\alpha}\,\frac{\pi}{\sin\pi(\frac{d}{1+\alpha})}\,\Gamma(\frac{d}{1+\alpha}) \qquad \text{and} \qquad v \;=\; c(d,\alpha=d-1)\,\frac{1}{d} \quad .
$$

Remark : As it stands, the limit of the propagator for $\alpha \downarrow d-1$ is singular and does not reproduce the formula for $\alpha = d-1$. The reason is that, for $\alpha > d-1$, there is a subleading term, not displayed in eq.(C.12), which becomes leading for $\alpha = d-1$. Idem for $\alpha \uparrow d-1$.

In region II, one finds :

- for $d-1 < \alpha \leq 2$

$$
\langle\, \psi^*_{\omega 1}(\zeta)\, \psi_{\omega 1}(0)\, \rangle_n^{\parallel} \quad \sim \quad (\frac{-i}{2\pi})\, e^{i \arg(i\xi_0 + \frac{\xi_\parallel}{v_{Fn}})} \, .
$$

$$
\cdot \begin{cases}
e^{-u_{\rm II} \frac{k_F}{\lambda_n^{d-1}} \frac{|\xi_\parallel|}{v_{Fn}}} \quad , & \text{for } c_n v_{Fn} \lambda_n^\alpha \gg |\xi_\parallel| \gg \frac{v_{Fn}\lambda_n d-1}{k_F} \quad , \\
\frac{1}{|\xi_\parallel|} \quad , & \text{for } \frac{v_{Fn}\lambda_n^{d-1}}{k_F} \gg |\xi_\parallel| \gg \frac{1}{k_F} \quad .
\end{cases}
$$

(C.13)

- for $0 < \alpha < d-1$

$$\langle\, \psi^*_{\omega 1}(\zeta)\, \psi_{\omega 1}(0)\, \rangle^{\|}_n \quad \sim \quad (\frac{-i}{2\pi})\, e^{i \arg(i\xi_0 + \frac{\xi_\|}{v_{Fn}})}\, \frac{1}{|\xi_\||} \quad , \quad \text{for } |\xi_\|| \gg \frac{1}{k_F} \quad . \tag{C.14}$$

Remark : In eqs.(C.13) and (C.14), we have omitted subleading terms in an expansion in $\frac{1}{\lambda_n}$.

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